

SHOKUROV'S LOCAL INDEX CONJECTURE FOR THREEFOLDS

ABSTRACT. We prove Shokurov's local index conjecture in dimension three.

CONTENTS

1. Introduction	1
2. Notation and conventions	2
3. Original covers and change of field	4
4. Pointed sections and constrained valuations	14
5. Cartier contact and pointed complements	51
6. Original computing orbits	60
7. Terminal pair models and marked exits	70
8. Lattice approximation and natural characters	84
9. Bounded ambient computing divisors	96
10. Complete point-exit surfaces	107
11. Point exits	121
12. Valuations at curve poles	137
13. Equivariant pencils and complete fibres	145
14. Curve exits	161
15. Computing curves above one	164
16. Exact minima greater than one	169
17. The threefold index theorem	181
References	183

1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

Shokurov conjectured that the local Cartier index of a singularity $X \ni x$ is bounded if $\dim X$ and $\text{mld}(X \ni x)$ are fixed. The conjecture was first recorded in a paper of Kawakita [Kaw15, Question 5.2] indicating that Shokurov proposed the conjecture in 2005.

Conjecture 1.1 (Shokurov). *Let d be a positive integer and a a positive real number. Then there exists a positive integer r depending only on d and a satisfying the following.*

Assume that $X \ni x$ is a germ of dimension d and $\text{mld}(X \ni x) = a$. Assume that either $a = 0$ or $X \ni x$ is klt. Then rK_X is Cartier near x .

Note that [Kaw15, Question 5.2] did not require that $a = 0$ or $X \ni x$ is klt, but a recent counterexample [NS26b, Theorem 1.2] disproved it. Thus either we need to assume that $X \ni x$ is klt, or we need to assume that $\text{mld}(X \ni x) = 0$. The $\text{mld}(X \ni x) = 0$ case is a special case of [Liu26, Theorem 1.1] in all dimensions, so the only non-trivial part is the case when $X \ni x$ is klt. Conjecture 1.1 was proven in the following cases:

- (1) When $d = 3$, $a > 1$, and $X \ni x$ is terminal [Kaw93, HLL25].

Date: October 13, 2026.

2020 Mathematics Subject Classification. Primary 14B05; Secondary 14E30.

Key words and phrases. Kawamata log terminal singularities, minimal log discrepancies, Cartier index, threefold pairs, cyclic covers.

- (2) When $d = 3$ and $a = 0$ [Ish00, Fuj01].
- (3) When $X \ni x$ is toric [Amb09, Amb16]
- (4) When $d = 3$ and $a = 1$ [Kaw15].
- (5) When $d \leq 2$ [CH21].
- (6) When $X \ni x$ is a quotient singularity [NS25b].
- (7) When $a = 0$ [Liu26].

The goal of this paper is to prove Conjecture 1.1 in dimension 3. We have the following theorem.

Theorem 1.2. *Let a be a positive real number. Then there exists a positive integer $r = R_3(a)$ depending only on a , such that for any klt threefold germ $X \ni x$ with $\text{mld}(X \ni x) = a$, rK_X is Cartier near x .*

Conjecture 1.1 is also expected to hold for pairs with finite rational coefficients [CH21, Conjecture 6.3]. In this paper we also prove this conjecture in dimension 3 in full generality.

Theorem 1.3. *Let a be a positive real number and let $\Gamma \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer $r = R_3(a, \Gamma)$ depending only on a and Γ , such that for any threefold germ $(X \ni x, B)$ with $\text{mld}(X \ni x, B) = a$ and $B \in \Gamma$, $r(K_X + B)$ is Cartier near x .*

Structure of the paper. Section 2 fixes notation. Section 3 constructs the cyclic covers and proves the field-transfer results. Sections 4–6 establish the section, Cartier contact, and computing-divisor orbit bounds. Sections 7–9 construct terminal pair models and bound the groups fixing computing centres. Sections 10–14 bound the stabilizers for contractions to points and curves when the minimal log discrepancy is at most one. Sections 15 and 16 treat minimal log discrepancies greater than one. Section 17 combines these bounds to prove Theorem 17.1 for klt pairs and Corollary 17.2.

2. NOTATION AND CONVENTIONS

We adopt the standard notation and terminology for the minimal model program from [Sho92, KM98, BCHM10] and use them freely.

Set-up 2.1. All assertions about a germ are understood after restriction to a Zariski neighbourhood of its distinguished closed point. A pair (X, B) consists of a normal variety and an effective rational Weil divisor such that $K_X + B$ is $\mathbb{Q}$ -Cartier. Separate $\mathbb{Q}$ -Cartierness of K_X and B is not part of this convention.

Definition 2.2. A *prime divisor over X* is a prime divisor E on a normal birational model $f : Y \rightarrow X$, with its primitive valuation $\text{ord}_E : k(X)^* \rightarrow \mathbb{Z}$. Its log discrepancy is

$$A_{X,B}(E) = 1 + \text{ord}_E(K_Y - f^*(K_X + B)).$$

We write $A_X(E) = A_{X,0}(E)$ for the log discrepancy with zero boundary. For a closed point x , we set

$$\text{mld}(X \ni x, B) = \inf_{\text{centre}_X(E) = \{x\}} A_{X,B}(E).$$

For a closed subset $Z_0 \subset X$, the notation $\text{mld}_{Z_0}(X, B)$ takes the infimum over all primitive prime divisors whose centres are contained in Z_0 . A *computing divisor* attains the specified infimum. A pair is klt, respectively lc, if all log discrepancies are positive, respectively nonnegative, on the neighbourhood in question. A terminal pair has log discrepancy greater than one for every divisor exceptional over that neighbourhood. The condition $\text{mld}(X \ni x, B) \geq \epsilon$ only tests divisors whose centre is exactly $\{x\}$.

Definition 2.3. For a rational Weil divisor L that is $\mathbb{Q}$ -Cartier near x , the *local Cartier index* $\text{ind}_x(L)$ is the least positive integer r such that rL is an integral Cartier divisor near x . For a

set Γ , we write $B \in \Gamma$ if every coefficient of the prime components of B belongs to Γ . For a finite set $\Gamma \subset \mathbb{Q} \cap [0, 1]$, let

$$d_\Gamma = \text{lcm}\{\text{den}(b) : b \in \Gamma\},$$

where denominators are in lowest terms and the empty least common multiple is one.

Definition 2.4. If a finite group G acts on a function field and F is a primitive divisorial valuation, its *decomposition group* is $H = \text{Stab}_G(F)$. Its *inertia group* $I_G(F)$ is the kernel of $H \rightarrow \text{Aut}(\kappa(F))$. For a uniformizer t_F and $\sigma \in I_G(F)$, the conormal character is $\lambda_F(\sigma) = \overline{\sigma^* t_F / t_F}$. It is a character $I_G(F) \rightarrow k^*$. A rationally scaled valuation $v = c \text{ord}_F$, $c > 0$, has $A_{X,B}(v) = c A_{X,B}(F)$. Restriction to a subfield is always renormalized explicitly when a primitive divisor is required.

Definition 2.5. For a rank-one reflexive sheaf $\mathcal{L}$, write $\mathcal{L}^{[j]} = (\mathcal{L}^{\otimes j})^{**}$, with the usual reflexive dual interpretation for negative j . An action on a log-canonical sheaf is its *natural action* on rational differential tensors with the indicated boundary poles. It is not modified by a choice of an auxiliary character. If mB is integral, this sheaf is $(\omega_X^{[m]} \otimes \mathcal{O}_X(mB))^{**}$.

Definition 2.6. An *lc N -complement* of a pair (X, B) on a germ is an effective rational divisor $C \geq B$ for which (X, C) is lc and $N(K_X + C)$ is integral Cartier and locally linearly trivial. When positivity is required only for divisors centred at x , it will be stated explicitly; positive-dimensional lc centres through x are not thereby excluded.

Definition 2.7. An *lc place* of an lc pair is a primitive divisorial valuation of log discrepancy zero. A *quasi-monomial valuation* is a valuation that is monomial with nonnegative real weights on a simple normal crossings stratum of a log-smooth birational model; its discrepancy is the corresponding weighted sum of the discrepancies of the stratum's boundary divisors. The centre of a valuation is always taken on the model named in the assertion. An ϵ -lc pair has every log discrepancy at least ϵ on the specified neighbourhood, whereas a closed-centred gap is restricted to centre exactly the named point.

Definition 2.8. A projective contraction $f : Y \rightarrow Z$ means a projective morphism with $f_* \mathcal{O}_Y = \mathcal{O}_Z$. It is *of Fano type over Z* if there is an effective rational divisor Δ for which (Y, Δ) is klt and $-(K_Y + \Delta)$ is ample over Z . For a finite group acting on a variety, *equivariant $\mathbb{Q}$ -factoriality* means that every invariant integral Weil divisor is $\mathbb{Q}$ -Cartier; ordinary $\mathbb{Q}$ -factoriality will be specified when it is needed.

Definition 2.9. All actions on functions are pullback actions. Fix a compatible identification of the roots of unity in k with $\mathbb{Q}/\mathbb{Z}$; over $\mathbb{C}$ use $\exp(2\pi it) \leftrightarrow t \bmod \mathbb{Z}$. A character phase is the resulting class of its value; signed and positive rational representatives will always be distinguished. The positive representative of a trivial eigenvalue is one when a closed-centred quotient weight requires a coordinate to be positive. The notation k_χ denotes the one-dimensional representation of character χ .

Definition 2.10. An effective real ideal is a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{c_j}$ of nonzero coherent ideals with $c_j > 0$. Its order is $v(\mathfrak{a}) = \sum_j c_j v(\mathfrak{a}_j)$, and $A_{X,B,\mathfrak{a}}(v) = A_{X,B}(v) - v(\mathfrak{a})$. The corresponding minimal log discrepancy uses the same centre convention as Definition 2.2. A non-lc one-dimensional minimum is assigned $-\infty$. For a valuation v centred at a closed point of an n -dimensional germ, put $\mathfrak{a}_t(v) = \{f : v(f) \geq t\}$ and

$$\text{vol}(v) = \limsup_{t \rightarrow \infty} \frac{n! \text{length}(\mathcal{O}_{X,x}/\mathfrak{a}_t(v))}{t^n}, \quad \widehat{\text{vol}}_{X,B}(v) = A_{X,B}(v)^n \text{vol}(v)$$

when the discrepancy is finite. The associated graded ring is $\text{gr}_v \mathcal{O}_{X,x} = \bigoplus_\lambda \mathfrak{a}_\lambda(v)/\mathfrak{a}_{>\lambda}(v)$.

Definition 2.11. A real Weil b-divisor over X assigns to every normal proper birational model a real Weil divisor, compatibly with pushforward. It is b-Cartier if, on some model, it is Cartier and its traces on all higher models are pullbacks; rational b-Cartier has the corresponding denominator-clearing meaning. It is b-nef if it descends to a nef real Cartier divisor on some

model. Relative nefness is always named explicitly. A generalized subpair consists of a normal X , a real Weil divisor B , and nef $\mathbb{Q}$ -divisor data M , with $K_X + B + M_X$ real Cartier. On a model $f : Y \rightarrow X$ where M descends, define B_Y by

$$K_Y + B_Y + M_Y = f^*(K_X + B + M_X).$$

The generalized log discrepancy of a prime E on Y is $1 - \text{coeff}_E(B_Y)$. Generalized lc and generalized ϵ -lc mean that these numbers are, respectively, nonnegative and at least ϵ on every higher model. A generalized pair has $B \geq 0$; a subpair allows negative coefficients.

Definition 2.12. A log-smooth pair has smooth underlying variety and simple normal crossings boundary support. An lc pair is dlt if it has a log resolution on which every exceptional prime has positive log discrepancy; it is plt if every prime exceptional over the pair has positive log discrepancy. These terms include the requirement that boundary coefficients be at most one. Quasi-étale means finite and étale at every codimension-one point. A collection of projective varieties is bounded if it occurs among the fibres of a projective family over a scheme of finite type; it is bounded up to isomorphism in codimension one if every member is isomorphic in codimension one to a member of a bounded collection. A small birational map is an isomorphism in codimension one.

3. ORIGINAL COVERS AND CHANGE OF FIELD

We use the discrepancy and natural-action conventions of Section 2. We first compare primitive valuations and logarithmic characters under a finite quotient. We then construct the cover associated to an integral multiple of the total log divisor and consider change of the algebraically closed ground field.

Theorem 3.1 (Functorial resolution and principalization). *Over a field of characteristic zero, a reduced variety admits a resolution by blowups in smooth centres which is an isomorphism over its smooth locus. A nonzero coherent ideal on a smooth variety, together with a simple normal crossing divisor, admits a sequence of blowups in smooth centres meeting that divisor normally, after which the transformed ideal is invertible and its support together with the transformed boundary is simple normal crossing. These constructions can be chosen to commute with smooth morphisms and extensions of the ground field [Kol05, Theorems 35–36].*

We use this result to principalize the ideal of a closed subset while retaining simple normal crossings with the boundary.

The following relative results will preserve $\mathbb{Q}$ -factoriality when we change fields. A real pair has the same discrepancy convention as a rational pair, with a real boundary and a real Cartier total log divisor.

Theorem 3.2 (Relative basepoint-free theorem). *Let $\pi : X \rightarrow Z$ be a proper surjective morphism of integral quasi-excellent Noetherian algebraic spaces of equal characteristic zero over a scheme S , with X normal and Z admitting a dualizing complex. Let (X, Δ) be a klt $\mathbb{R}$ -pair, and let $H \in \text{Pic}(X)$ be π -nef. If $aH - (K_X + \Delta)$ is π -big and π -nef for some integer $a > 0$, then mH is π -generated for all sufficiently large integers m [LM22, Theorem 11.1(ii)].*

For a negative extremal contraction, this applies in particular to line bundles numerically trivial on the contracted ray.

Theorem 3.3 ($\mathbb{Q}$ -factoriality after good contractions). *Let $\pi : X \rightarrow Z$ be projective and surjective between integral quasi-excellent Noetherian algebraic spaces over a scheme, with X normal and $\mathbb{Q}$ -factorial. Let $R \subset \text{NE}(X/Z)$ be an extremal ray and $f : X \rightarrow Y$ its contraction. If either $\dim X > \dim Y$, or f is birational with exceptional locus a prime divisor, and if f is good (a rational line bundle numerically trivial on R descends up to a positive multiple), then Y is $\mathbb{Q}$ -factorial [LM22, Lemma 19.2, cases (i)–(ii) of Lemma 19.1].*

We will use the birational divisorial case, retaining the actual exceptional prime under extension.

Theorem 3.4 ($\mathbb{Q}$ -factoriality after flips). *Let $\pi : X \rightarrow Z$ be as in Theorem 3.3. Assume that Z admits a dualizing complex. Let $\Delta \geq 0$ be a rational Weil divisor such that (X, Δ) is klt, and let $f : X \rightarrow Y$ be a good small contraction of an extremal ray on which $K_X + \Delta$ is negative. If X^+ is its flip, then X^+ is $\mathbb{Q}$ -factorial whenever X is [LM22, Lemma 20.3(i)].*

This treats the small steps of the same finite sequence.

Lemma 3.5 (Good scaling divisor criterion). *Let $\pi : X \rightarrow Z$ be projective and surjective between integral quasi-excellent Noetherian algebraic spaces of equal characteristic zero, with X normal and Z admitting a dualizing complex. Let $\Delta \geq 0$ be a rational Weil divisor such that (X, Δ) is klt and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. A $\mathbb{Q}$ -invertible sheaf A is a good scaling divisor if it is π -big, $K_X + \Delta + A$ is π -nef, and étale locally on Z it has an effective rational representative whose addition to Δ preserves klt. In particular, every π -ample $\mathbb{Q}$ -invertible sheaf A for which $K_X + \Delta + A$ is π -nef is good [LM22, Definition 21.2 and Lemma 21.3].*

Theorem 3.6 (Relative MMP with scaling). *Let $\pi : X \rightarrow Z$ be projective and surjective between integral Noetherian algebraic spaces of equal characteristic zero, with X normal and Z quasi-excellent with a dualizing complex. Suppose X is $\mathbb{Q}$ -factorial, Δ is a rational divisor such that (X, Δ) is klt, and A is a good scaling divisor. Then the relative MMP with scaling of A exists. If in addition $\Delta \geq 0$ is a rational Weil divisor, $K_X + \Delta$ is $\mathbb{Q}$ -Cartier and π -pseudoeffective, and $cK_X + \Delta$ is π -big for some rational $c \leq 1$, every such process terminates with a minimal model [LM22, Theorem 21.7; Corollary 21.9(i), using Theorem 21.8].*

The relative bigness hypothesis will be used over a birational base.

For a prime E , the classical discrepancy in the next statement is $a(E, X, \Delta) = A_{X, \Delta}(E) - 1$.

Theorem 3.7 ($\mathbb{Q}$ -factorial extraction). *Let X be an integral normal excellent Noetherian scheme of equal characteristic zero with a dualizing complex and canonical divisor K_X . Let $\Delta \geq 0$ be an $\mathbb{R}$ -Weil divisor such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier and (X, Δ) is klt, and let $g : Y \rightarrow X$ be a projective log resolution. For any subset $\mathcal{E}$ of the g -exceptional prime divisors with classical discrepancy $a(E, X, \Delta) \leq 0$ for every $E \in \mathcal{E}$, there is a projective birational morphism $h : Z \rightarrow X$, with Z $\mathbb{Q}$ -factorial, whose h -exceptional primes are exactly the birational transforms of $\mathcal{E}$. For $\mathcal{E} = \emptyset$, h is small; the proof constructs it by a finite sequence of divisorial contractions and flips [LM22, Theorem 22.1 and proof, pp. 87–88].*

The empty extraction set gives the finite sequence needed for field transfer; it does not impose a projectivity assumption over the ground field.

Lemma 3.8. *Let $m \geq 1$ be an integer. Let a finite group J act faithfully on a normal variety U in characteristic zero, preserving an effective rational divisor B_U . Suppose the finite quotient $\pi : U \rightarrow Q = U/J$ exists as a variety. Suppose mB_U is integral and $m(K_U + B_U)$ has a natural eigen-generator of character χ_U . Let F be a primitive divisorial valuation of $k(U)$ with centre on U , let $I = I_J(F)$, and write F_Q for its primitive restriction to the quotient $Q = U/J$. Define B_Q by $K_U + B_U = \pi^*(K_Q + B_Q)$. Then, with $e = |I|$,*

$$\text{ord}_F|_{k(Q)} = e \text{ord}_{F_Q}, \quad A_{U, B_U}(F) = e A_{Q, B_Q}(F_Q), \quad \chi_U|_I = \lambda_F^{m A_{U, B_U}(F)}.$$

The conormal character λ_F is faithful on I . If $A_{Q, B_Q}(F_Q) = p/q > 0$ in lowest terms and $\delta = |I \cap \ker \chi_U|$, then

$$e = \delta \frac{q}{\gcd(q, m)}.$$

At a codimension-one prime with upstairs boundary coefficient b and ramification index u , the quotient boundary coefficient is $1 - (1 - b)/u$.

The kernel of the natural character measures exactly the additional inertia beyond the denominator of the downstairs discrepancy.

Proof. We first identify the ramification and the crepant quotient boundary, then compute the character on the divisorial local ring.

Step 1. In this step, we identify the primitive restriction, ramification index, and crepant quotient boundary. The extension $k(U)/k(Q)$ is finite Galois with group J . Restriction of ord_F has discrete value group of finite index in $\mathbb{Z}$ and residue field of transcendence degree $\dim U - 1$. It therefore defines a primitive divisorial valuation F_Q , and $\text{ord}_F|_{k(Q)} = e_F \text{ord}_{F_Q}$ for a positive integer e_F . Normalize a model displaying F_Q in $k(U)$. Its generic divisorial local rings give this finite extension of discrete valuation rings. Put $J_F = \text{Stab}_J(F)$ and $f_F = [\kappa(F) : \kappa(F_Q)]$. The group J_F acts on the completed local extension, its residue action has kernel I and image of order f_F , and

$$|J_F| = e_F f_F, \quad |J| = e_F f_F [J : J_F].$$

Indeed the completed normalization is the product of the factors indexed by the extensions of F_Q ; each factor has rank $e_F f_F$, as follows by taking its length modulo a downstairs uniformizer. The group permutes these factors transitively, and the residue extension is Galois. Consequently $e_F = |I| = e$.

Let t be a uniformizer at F . The map $\sigma \mapsto \overline{\sigma^* t}/t$ is independent of t on I , since inertia fixes the residue of every unit. It is faithful. In fact, an element in its kernel acts identically on every graded piece of the valuation filtration. If $D_\sigma = \sigma^* - 1$, then D_σ raises valuation by at least one. For a finite-order element σ of order N , the identity

$$0 = (1 + D_\sigma)^N - 1 = N D_\sigma + \sum_{j=2}^N \binom{N}{j} D_\sigma^j$$

forces $D_\sigma = 0$: on any element with nonzero image under D_σ , the first term has strictly smaller valuation than the others. Here N is a unit because the residue characteristic is zero. The values of this faithful character are roots of unity and hence lie in k^* . In particular, I is cyclic.

Choose compatible canonical divisors by pulling back a rational top form on Q . At a codimension-one point of ramification index u , the differential order increases by $u - 1$. Thus, if the boundary coefficient upstairs is b , the coefficient downstairs must satisfy

$$u \text{coeff}(B_Q) = b + u - 1.$$

This constructs the stated rational boundary. It also shows that its total log divisor is $\mathbb{Q}$ -Cartier. To see this without separate $\mathbb{Q}$ -Cartier assumptions on the summands, work over the semilocal ring of a finite fibre of π . A Cartier multiple of $K_U + B_U$ is principal there. The norm of its rational equation has divisor equal to $\deg(\pi)$ times the corresponding multiple of $K_Q + B_Q$. This is a Cartier multiple on a neighbourhood of the downstairs point.

At the two divisorial local rings, the same differential calculation gives

$$\text{ord}_F(\pi^* \eta) = e \text{ord}_{F_Q}(\eta) + (e - 1)$$

for a rational top form η on Q . One may compute this with a separating residue transcendence basis and a uniformizer: the residue extension is separable, and differentiating a downstairs parameter of order e contributes exactly $e - 1$. Subtracting the coefficient of the pulled-back total log divisor gives $A_{U, B_U}(F) = e A_{Q, B_Q}(F_Q)$.

Step 2. In this step, we compute the natural character and the resulting inertia formula. Let τ be the given eigen-generator. On a model regular at the generic point of F , its order as a rational m -canonical tensor is

$$\ell = m(A_{U, B_U}(F) - 1) \in \mathbb{Z}.$$

Choose lifts $y_1, \dots, y_{\dim U - 1}$ of a separating residue transcendence basis. A local canonical generator is $dt \wedge dy_1 \wedge \dots \wedge dy_{\dim U - 1}$. On its leading term inertia acts by λ_F : it fixes the residue differentials, while the conormal differential contributes λ_F . The leading coefficient of a valuation

unit is fixed. Writing τ as t^ℓ times a unit times the m -th tensor power of this generator therefore gives

$$\chi_U|_I = \lambda_F^{\ell+m} = \lambda_F^{mA_{U,B_U}(F)}.$$

Finally put $d = \gcd(q, m)$, $q = dq_0$, and $m = dm_0$. The integer $mA_{U,B_U}(F) = mep/q$ implies $q_0 \mid e$. Write $e = vq_0$. Since $\gcd(q_0, m_0p) = 1$, faithfulness of λ_F gives

$$\delta = \gcd(e, mA_{U,B_U}(F)) = \gcd(vq_0, vm_0p) = v.$$

Hence $e = \delta q / \gcd(q, m)$, as claimed. $\square$

Theorem 3.9. *Let $(X \ni x, B)$ be a klt germ of dimension $n \geq 1$ over an algebraically closed field of characteristic zero, and let $m \geq 1$ satisfy mB integral. Set $r = \text{ind}_x(K_X + B)$ and $s = r / \gcd(r, m)$. There is a connected cyclic cover $\rho : (V \ni o) \rightarrow (X \ni x)$ of degree s , etale in codimension one, with a single underlying point over x . If B_V is the finite Weil pullback, then*

$$K_V + B_V = \rho^*(K_X + B), \quad m(K_V + B_V) \sim 0$$

with the second divisor integral Cartier. Its natural m -log-canonical line has a faithful character χ of $G = \text{Gal}(V/X)$. If E is a primitive divisor centred at x with $A_{X,B}(E) = p/q$ in lowest terms and F is any primitive extension to $k(V)$, then

$$e = |I_G(F)| = \frac{q}{\gcd(q, m)}, \quad A_{V,B_V}(F) = \frac{p}{\gcd(q, m)}, \quad |G| = e[\kappa(F) : \kappa(E)][G : \text{Stab}_G(F)].$$

The conormal character is faithful and

$$\chi|_{I_G(F)} = \lambda_F^{mA_{V,B_V}(F)}.$$

The closed-point mld is positive, rational and attained.

The integer m clears the boundary before the cyclic cover is taken. The degree and the natural character are those of this original cover.

Proof. We construct the cyclic algebra and prove that its closed fibre has one underlying point. Its natural tensor then permits the use of Lemma 3.8 for every primitive lift.

Step 1. In this step, we construct the connected integral cyclic cover and determine its degree. Put $L = K_X + B$, $D = mL$, $A = \mathcal{O}_{X,x}$, and $K = k(X)$. The integers j for which jL is integral Cartier at x form the subgroup $r\mathbb{Z}$. Thus the index of the integral Weil divisor D is $s = r / \gcd(r, m)$. Choose a rational top form η defining K_X and a rational function $g \in K^*$ with $\text{div}(g) = sD$ on $\text{Spec } A$. After shrinking X about x , this identity holds on X . Let $z^s = g$, put $K' = K(z)$, and normalize X in K' . This normalization is finite.

The extension K'/K is Galois because K contains all s -th roots of unity. Its group is a subgroup of μ_s , acting by $z \mapsto \zeta z$. If its order is j , then $j \mid s$ and $z^j \in K$. The identity $(z^j)^{s/j} = g$ gives $\text{div}(z^j) = jD$ on $\text{Spec } A$. Minimality of the index s forces $j = s$. Hence the normalization is integral, the cover is connected, and its Galois group is $G = \mu_s$.

At a height-one prime of A , let t_P be a parameter and let $d_P \in \mathbb{Z}$ be the coefficient of D . Write $g = t_P^{sd_P} u_P$ with u_P a unit. The algebra

$$A_P[w]/(w^s - u_P), \quad w = z/t_P^{d_P},$$

is finite etale: w and sw^{s-1} are units. It is normal and has fraction field K' , so it is the integral closure of A_P . Thus ρ is etale in codimension one.

Step 2. In this step, we prove that the closed fibre has one underlying point. Let C be the integral closure of A in K' . Then $C^G = A$. Since C is finite over A , its reduced closed fibre is a product $T = k^b$ for an integer $b \geq 1$. Taking invariants is exact, by averaging over G , and therefore $T^G = k$. The action on the b points is transitive.

Suppose $b > 1$, and let $J \subsetneq G$ stabilize one of them. Choose a nontrivial character of G/J , viewed as the character $\zeta \mapsto \zeta^i$ of G , with $0 < i < s$. On the transitive set G/J , it has an

eigenfunction that is nonzero at every point. Lift this function through $C \rightarrow T$ and project onto its character eigenspace. The resulting element $a_0 \in C$ is a unit, since it is nonzero at every maximal ideal of C . In K' the same eigenspace is Kz^i , so $a_0 = vz^i$ for some $v \in K^*$. Now $a_0^s = v^s g^i \in A^*$, whence

$$\operatorname{div}(v) + iD = 0 \quad \text{on } \operatorname{Spec} A.$$

This contradicts the index s . Hence $b = 1$, and C is local with residue field k . The algebraic cover consequently has a single underlying point o above x .

Step 3. In this step, we identify the faithful natural log character and compare the primitive valuations. Codimension-one etaleness shows that the finite Weil pullback B_V has the coefficients of B , and that $K_V = \operatorname{div}(\rho^*\eta)$ is compatible with K_X . The equation $z^s = g$ then gives equalities of Weil divisors

$$m(K_V + B_V) = \rho^*D = \operatorname{div}(z).$$

In particular, the total log divisor upstairs is $\mathbb{Q}$ -Cartier. The rational tensor

$$\tau = \frac{(\rho^*\eta)^{\otimes m}}{z}$$

has order $-mB_V$ at every codimension-one prime. Multiplication by τ identifies the reflexive log-canonical sheaf with $\mathcal{O}_V$ in codimension one and hence everywhere. Under $\sigma_\zeta^*z = \zeta z$, it transforms as $\sigma_\zeta^*\tau = \zeta^{-1}\tau$. Its natural character χ is therefore faithful. The point o is fixed by G , so multiplying this generator by a unit does not change its character on the fibre at o .

Every extension of a valuation centred at x has centre at o : its valuation ring contains the integral closure C , and its centre lies over x . Apply Lemma 3.8 to $V \rightarrow X$. Its character kernel is trivial, so $\delta = 1$. The asserted inertia order, discrepancy and conormal identity follow. The decomposition of the finite extension of divisorial local rings in that lemma also gives

$$|G| = |I_G(F)|[\kappa(F) : \kappa(E)][G : \operatorname{Stab}_G(F)].$$

In particular no residue-degree factor is discarded. Applying the discrepancy comparison to every prime upstairs also proves that (V, B_V) is klt.

Step 4. In this step, we prove positivity, rationality, and attainment of the closed-point minimum. Choose $g_r \in K^*$ with $\operatorname{div}(g_r) = rL$ near x . For a primitive prime E' centred at x , on a model displaying it we have

$$A_{X,B}(E') = 1 + \operatorname{ord}_{E'}(\eta) - \frac{\operatorname{ord}_{E'}(g_r)}{r} \in \frac{1}{r}\mathbb{Z}_{>0}.$$

Here the order of η is its differential order, an integer, and positivity is the klt hypothesis. There is at least one such prime. For $n = 1$, it is the closed point of the normal curve. For $n \geq 2$, a prime component of the exceptional divisor of the normalized blowup of the maximal ideal of x has centre x . The nonempty set of discrepancies is therefore a subset of $r^{-1}\mathbb{Z}_{>0}$ and has a least element. This proves positivity, rationality and attainment. $\square$

An integer clearing a rational boundary means a positive integer whose product with that boundary is integral.

Definition 3.10. Fix a klt threefold germ $(X \ni x, B)$ with $a = \operatorname{mld}(X \ni x, B) = p/q > 0$, and an integer m clearing B and divisible by q . *Original cover data* consist of the cover, group and natural character of Theorem 3.9, a primitive lift F of a computing divisor, and $H = \operatorname{Stab}_G(F)$. In these data, $I_G(F) = 1$, $A_{V,B_V}(F) = a$, the action of H on $\kappa(F)$ is faithful, and

$$\operatorname{ind}_x(K_X + B) \leq m[G : H]|H|.$$

Indeed $q \mid m$ makes the inertia order in Theorem 3.9 equal to one. The residue action then has trivial kernel by Definition 2.4, and

$$r = \gcd(r, m)|G| \leq m[G : H]|H|.$$

Thus the remaining group factors are the orbit size and the full stabilizer of the prescribed original lift.

Lemma 3.11. *Let (U, B) be an effective rational klt pair over an algebraically closed field of characteristic zero, and let $Z_0 \subsetneq U$ be nonempty and closed. The infimum of log discrepancies of primitive primes whose centres are contained in Z_0 is unchanged by algebraically closed field extension. Every integral Cartier multiple of $K_U + B$, and its exact local index at a marked closed point, are preserved. If a finite group J acts faithfully, preserves the pair and Z_0 , and has quotient $\pi : U \rightarrow Q$, with finite crepant boundary B_Q , then*

$$\mathrm{mld}_{\pi(Z_0)}(Q, B_Q) = \inf_{\mathrm{centre}_U(F) \subset Z_0} \frac{A_{U,B}(F)}{|I_J(F)|}.$$

This equality, including its infimum over all primes, is preserved under the extension.

A prescribed finite algebraic equivariant diagram with its marked divisors, finite covers, coherent ideals, natural linearizations, normalizations and Stein factorizations admits a common realization over an algebraically closed field of finite transcendence degree over $\mathbb{Q}$, embedded in the original field and in $\mathbb{C}$. For an existing marked crepant terminal-model diagram, the terminal pair properties, equivariant $\mathbb{Q}$ -factoriality, relative ampleness, full fibre schemes and the displayed all-prime comparisons can be retained on that realization.

The quotient comparison controls every prime after extension, including primes that are not scalar extensions of individual primes over the original field.

Proof. We use Theorem 3.1 to compare all divisorial discrepancies, then preserve the log-canonical sheaves and descend the marked diagram. A finite sequence from Theorem 3.7, together with Theorems 3.2, 3.3 and 3.4, will preserve equivariant $\mathbb{Q}$ -factoriality.

Step 1. In this step, we compute the infimum on a finite resolution and preserve it under field extension. Apply Theorem 3.1 first to resolve U . On the resulting smooth variety, principalize the pullback of the ideal of $\mathrm{Sing}(U) \cup \mathrm{Supp} B$, starting with the empty simple normal crossing divisor. Then principalize the pullback of $\mathcal{I}_{Z_0}$, compatibly with the resulting simple normal crossing support. This gives a projective log resolution $f : W \rightarrow U$ on which $\mathcal{I}_{Z_0}$ is invertible. Write

$$K_W + \sum_i (1 - \alpha_i) D_i = f^*(K_U + B), \quad \mathcal{I}_{Z_0} \mathcal{O}_W = \mathcal{O}_W \left(- \sum_i n_i D_i \right).$$

The D_i include the exceptional and boundary divisors and the support of the principalized ideal; their union has simple normal crossings. We allow zero coefficients in either sum. All $\alpha_i > 0$, and at least one n_i is positive.

Let P be any primitive prime with centre in Z_0 , and let Z_P be its centre on W . Properness gives this centre, and it lies in the support of the principalized ideal. At its generic point, choose regular parameters $z_1, \dots, z_c$, with $c = \mathrm{codim}_W Z_P$, so that the boundary components through Z_P are $D_{i_j} = (z_j = 0)$, $1 \leq j \leq t$. Put $w_j = \mathrm{ord}_P(z_j) \in \mathbb{Z}_{>0}$. Complete these parameters by lifts $y_1, \dots, y_{\dim U - c}$ of a separating residue transcendence basis.

On a model regular at P , write $z_j = u^{w_j} v_j$ with u a uniformizer and v_j units. In the pullback of

$$\frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_c}{z_c} \wedge dy_1 \wedge \dots \wedge dy_{\dim U - c},$$

at most one factor du/u occurs in a nonzero term; all remaining factors are regular at the divisorial local ring. Its pole order is therefore at most one. It follows that $A_W(P) \geq \sum_j w_j$, and hence

$$(3.1) \quad A_{U,B}(P) \geq \sum_{j=1}^t \alpha_{i_j} w_j + \sum_{j=t+1}^c w_j.$$

Some component through Z_P has positive n_i . Positivity of all terms gives the lower bound below, and every D_i with $n_i > 0$ is itself an allowed primitive prime. Thus

$$(3.2) \quad \text{mld}_{Z_0}(U, B) = \min_{n_i > 0} \alpha_i.$$

For an algebraically closed extension k'/k , the displayed components remain integral, and smoothness, simple normal crossings and both divisor identities persist. Formula (3.2), applied over k' , proves preservation of the entire infimum. The same inequality proves the klt criterion $\alpha_i > 0$ and the lc criterion $\alpha_i \geq 0$, so these properties are preserved as well.

Step 2. In this step, we preserve Cartier multiples, exact indices, and natural linearizations. For a positive integer d , integrality of $d(K_U + B)$ is equivalent to integrality of dB , and its rational coefficients do not change under the extension. When it is integral, on the smooth locus $j : U_{\text{sm}} \hookrightarrow U$ its log-canonical sheaf is the line

$$(\bigwedge^{\dim U} \Omega_{U_{\text{sm}}/k})^{\otimes d} \otimes \mathcal{O}_{U_{\text{sm}}}(dB).$$

Its extension by j_* is the reflexive sheaf $\mathcal{O}_U(d(K_U + B))$. This extension commutes with scalar extension. On an affine open of U , a finite affine cover of its intersection with the smooth locus computes sections by a kernel of the two restriction maps. Flat tensor product preserves this kernel; the canonical differentials also commute with base change. Geometric normality identifies the resulting sheaf with the same reflexive log-canonical sheaf on $U_{k'}$.

Invertibility of a finitely presented module at a marked point ascends and descends through the faithfully flat local map at that point. This proves preservation of each Cartier multiple separately and hence of the exact index. It applies to any specified integral Weil divisor. Natural linearizations commute with the differential and sheaf maps just used. In particular an equality $\sigma^* \tau = \chi(\sigma) \tau$ for a generator remains the same equality after extension, with the same order of $\chi(\sigma)$.

Step 3. In this step, we establish and preserve the all-prime discrepancy comparison on the actual quotient. The invariant algebra commutes with field extension because its projection is $|J|^{-1} \sum_{\sigma \in J} \sigma^*$. Thus the quotient after extension is still the actual quotient by J . Lemma 3.8 applies locally over each finite fibre: first choose an integral Cartier multiple of the total log divisor and a generator over its semilocal preimage. The tensor product of all its J -translates is an invariant natural generator of a further multiple, on a J -stable neighbourhood. The lemma therefore gives the crepant quotient boundary and the finite discrepancy comparison for every prime. Its boundary coefficients lie in $[0, 1)$, and every prime downstairs has an extension upstairs. The quotient pair is consequently effective and klt.

The equality $\pi^{-1} \pi(Z_0) = Z_0$ holds set-theoretically because Z_0 is J -stable and the fibres of a finite quotient are group orbits. Extension and restriction of divisorial valuations therefore preserve the required centre conditions. The discrepancy formula of Lemma 3.8 gives the displayed infimum in the statement. Apply (3.2) to (Q, B_Q) and $\pi(Z_0)$ to obtain its preservation under extension. This argument tests all primes through the quotient minimum.

Step 4. In this step, we preserve terminality using finitely many resolution inequalities. There is also a finite resolution criterion for terminality of an effective klt pair. On a log resolution, every already exceptional prime must have $\alpha_i > 1$. A further prime whose centre lies in such a component then has discrepancy greater than one by the preceding differential inequality. Off the exceptional locus, a prime exceptional over the original variety has centre of codimension $c \geq 2$. If no strict boundary component contains its centre, its discrepancy is at least c . If exactly one does, it is at least $c - 1 + \alpha_i > 1$. If at least two do, the inequalities $\alpha_i + \alpha_j > 1$ for the intersecting strict boundary components suffice. These inequalities are necessary by blowing up their generic codimension-two intersections outside the exceptional locus. Thus terminality is equivalent to these finitely many inequalities, together with $\alpha_i > 1$ for the exceptional components. Their coefficients and the nonemptiness of their relevant intersections are unchanged by algebraically

closed extension. This proves the asserted preservation of terminal pair properties, given log resolution.

Step 5. In this step, we realize the entire marked diagram over a common algebraically closed field. Choose finite presentations for every scheme, morphism, coherent ideal, divisor, sheaf and specified identity. Include the group automorphisms, the multiplication maps of each cover algebra, the marked points, the supporting ample line bundles and the logarithmic linearizations. Display prescribed divisorial valuations on models and include these models and their primes. Include the finitely many log resolutions needed above. All coefficients belong to a field $K_0 \subset k$ finitely generated over $\mathbb{Q}$, enlarged to contain the finitely many character values. Let k_0 be its algebraic closure inside k . It is algebraically closed of finite transcendence degree over $\mathbb{Q}$, and it embeds into $\mathbb{C}$ by choosing an embedding of a transcendence basis and extending it over the algebraic extension. The diagram has realizations over k_0 , k and $\mathbb{C}$.

Integrality and normality descend faithfully flatly and are geometric over an algebraically closed field of characteristic zero. The marked prime divisors therefore remain prime, with primitive divisorial local rings. Their discrepancies are unchanged. Equality of two marked prime ideals is detected faithfully flatly, which preserves their stabilizers. Within a stabilizer, triviality on the residue function field is tested on finitely many field generators, so inertia is preserved. The orders of natural characters are unchanged as well. The exact-index argument above retains the degree of the original cover in Theorem 3.9. The original finite algebra and its multiplication maps are retained, so its closed fibre is the base change of the entire original fibre scheme.

Every ideal marking is retained as its coherent ideal. Pullbacks, restrictions and the kernels and cokernels of the specified maps commute with flat scalar extension. In particular this retains the scheme-theoretic fibres, including their multiplicities. A finite normal birational morphism remains finite, normal and birational, so a specified normalization remains the normalization. If its conductor is marked, that ideal also commutes with extension: locally it is the kernel of the action on the finitely presented quotient of the normalization algebra by the original algebra.

For a proper morphism $p : S \rightarrow T$, a finite affine cover of $p^{-1}(T')$, for each affine $T' \subset T$, computes $p_*\mathcal{O}_S$ by a kernel. Flatness identifies its scalar extension with the corresponding pushforward after extension. Consequently $\mathrm{Spec}_T(p_*\mathcal{O}_S)$ gives the same Stein factorization. Projectivity and ampleness of each specified line bundle ascend and descend faithfully flatly. The canonical and logarithmic sheaf identities retain crepancy. The preceding finite criteria retain terminality and the all-prime comparisons.

Step 6. In this step, we preserve equivariant $\mathbb{Q}$ -factoriality through a finite MMP on the quotient. For an invariant effective klt pair on a normal variety Y , put $Q_Y = Y/J$, assuming this finite quotient is part of the diagram. Then J - $\mathbb{Q}$ -factoriality of Y is equivalent to $\mathbb{Q}$ -factoriality of Q_Y . Indeed, a downstairs Weil divisor pulls back to an invariant Weil divisor, and the norm argument in Lemma 3.8 descends a Cartier multiple. Conversely, an invariant Weil divisor has constant coefficient along each orbit of primes. Dividing those coefficients by the ramification indices expresses it as the rational pullback of a Weil divisor on Q_Y .

The quotient is klt by the finite comparison already proved. Descent of its $\mathbb{Q}$ -factoriality follows by testing each downstairs Weil divisor with faithful flatness. We prove ascent over an affine open of Q_Y . Every variety and contraction used below is of finite type over a characteristic-zero field, hence is Noetherian and excellent; its base admits a dualizing complex. Models projective over this affine open are quasi-projective. These are the scheme and base hypotheses of the relative theorems stated above.

First, the relative numerical divisor space of a projective morphism between quasi-projective varieties is preserved by an algebraically closed extension. To prove surjectivity, spread a line bundle over the larger field to the product with an integral finite-type parameter scheme over the smaller field. After shrinking, it is a line bundle on that product. Choose a point of the parameter scheme over the smaller algebraically closed field. For each proper curve over the larger field, pull the family back to the product with that curve. Degree is constant on the

connected parameter scheme, so the chosen specialization has the same degree on every such curve. Thus it has the same numerical class as the original line bundle. For injectivity, spread a proper curve in a fibre to a flat projective family of curves, including its map to the target point in the spreading data. Its specialization over the smaller algebraically closed field is an effective fibre cycle of the same degree against any fixed line bundle. Numerical triviality is therefore preserved. These arguments also preserve relative Picard ranks.

Write (Q_Y, Δ) for the effective rational klt quotient pair and suppose that Q_Y is $\mathbb{Q}$ -factorial. Theorem 3.1 supplies a projective log resolution $g : W_0 \rightarrow Q_Y$. Apply Theorem 3.7 with the empty extraction set; its discrepancy requirement on selected primes is then vacuous. Its finite sequence consists of divisorial contractions and flips and ends in a projective small $\mathbb{Q}$ -factorialization $Q^+ \rightarrow Q_Y$.

We record why the relative MMP hypotheses hold in this construction. Write

$$K_{W_0} + \Delta_0 = g^*(K_{Q_Y} + \Delta) + \Gamma_0, \quad \Delta_0 \geq 0, \quad \Gamma_0 \geq 0,$$

where Δ_0 and Γ_0 are the positive and negative parts of the crepant boundary. They have disjoint support, Γ_0 is exceptional, and every coefficient of Δ_0 is less than one. Let F_0 be the reduced sum of the g -exceptional primes outside $\text{Supp } \Gamma_0$, and take a rational $\varepsilon > 0$ small enough that $\Theta_0 = \Delta_0 + \varepsilon F_0$ still has all coefficients less than one. This is an effective rational simple normal crossing boundary on the smooth, hence $\mathbb{Q}$ -factorial, variety W_0 . It is klt by the differential inequality already proved. Moreover,

$$K_{W_0} + \Theta_0 = g^*(K_{Q_Y} + \Delta) + \Gamma_0 + \varepsilon F_0$$

is g -pseudoeffective. Since g is birational, its generic fibre is a point, so Θ_0 is g -big; the bigness condition in Theorem 3.6 holds with $c = 0$. The construction in Theorem 3.7 chooses a relatively ample scaling divisor whose sum with the adjoint is nef. By Lemma 3.5, it is good and has local effective rational representatives keeping the augmented pair klt. We retain this divisor and these representatives. Thus all the existence and termination hypotheses of Theorem 3.6 hold, giving the finite sequence used here.

The final map $Q^+ \rightarrow Q_Y$ is an isomorphism. Indeed, push forward a relatively ample Cartier divisor on Q^+ . Its pushforward is $\mathbb{Q}$ -Cartier on Q_Y , and its pullback equals the original divisor since there are no exceptional primes. It is therefore numerically trivial on every contracted curve. Relative ampleness excludes a positive-dimensional fibre, so the map is finite and birational to a normal variety and hence an isomorphism.

After scalar extension, the same finite sequence starts with a smooth variety. The effective rational boundaries at every stage remain klt by the resolution criterion. Their total log divisors remain $\mathbb{Q}$ -Cartier by the sheaf argument above. Each contraction retains relative Picard rank one, its connected-fibre identity $f_*\mathcal{O} = \mathcal{O}$, and the relative ampleness of the negative log divisor. The exceptional prime of a divisorial step remains prime. A small step and its flip remain small and birational, and the positive log divisor on the flip remains relatively ample. Thus the same two maps still form a flip.

We verify goodness before applying either $\mathbb{Q}$ -factoriality theorem. Write $f' : W' \rightarrow T'$ for one of the extended negative contractions and Θ' for its boundary. Relative Picard rank one makes its contracted cone a single extremal ray. Let $\mathcal{M}$ be a rational line bundle numerically trivial on this ray, and clear a denominator to obtain a line bundle $\mathcal{L}$. It is f' -nef, and $\mathcal{L} - (K_{W'} + \Theta')$ has the numerical class of $-(K_{W'} + \Theta')$, which is f' -ample. In particular it is f' -big and f' -nef. All the hypotheses of Theorem 3.2 hold for f' , $\mathcal{L}$, and the integer $a = 1$. Hence a positive power of $\mathcal{L}$ is f' -generated.

Locally on T' , the generated sections give a morphism to projective space over T' . This morphism contracts every curve in a fibre of f' , because $\mathcal{L}$ has degree zero on each such curve. It has zero-dimensional image on every fibre: otherwise a curve in that image is dominated by a curve in the projective source fibre and has positive degree against $\mathcal{O}(1)$, a contradiction. Taking the Stein factorization of this morphism gives a normal scheme finite and birational over

T' , and therefore equal to T' . The pushforward of the generated power of $\mathcal{L}$ is therefore a line bundle, and its evaluation pullback is that power. These assertions are local on T' , so they give descent on all of T' . This proves goodness for every rational line bundle numerically trivial on the contracted ray, including new line bundles after extension.

Induct on the finite sequence. Its initial smooth variety is $\mathbb{Q}$ -factorial. At a divisorial step, Theorem 3.3 applies to the good contraction with its prime exceptional locus. At a flip, Theorem 3.4 applies to the good small extremal contraction of the effective rational klt pair, with negative adjoint and the actual flip just retained. Each next variety is therefore $\mathbb{Q}$ -factorial. The final variety is the scalar extension of Q_Y , so this proves ascent, including for Weil divisors that first appear over the larger field.

The quotient equivalence now gives preservation of equivariant $\mathbb{Q}$ -factoriality. The auxiliary finite MMP sequences may be included among the finite diagram data when choosing K_0 . The prescribed crepant models and maps themselves are retained through their graphs, boundary identities and ample line bundles; they are not replaced by newly chosen models. Together with the previous comparisons this proves all assertions, with the equivariant $\mathbb{Q}$ -factorial assertion supplied by the finite MMP argument. $\square$

Lemma 3.12 (Field transfer for the discrepancy of a computing divisor). *Let $(U \ni x, B)$ be an effective rational klt germ of positive dimension over an algebraically closed field of characteristic zero, and suppose K_U is $\mathbb{Q}$ -Cartier near x . Among primitive divisors centred at x that compute $\text{mld}(U \ni x, B)$, the least value of $A_U(E)$ is unchanged by algebraically closed field extension. A divisor attaining this least value remains a computing divisor with the same pair and bare discrepancies after the extension, and the corresponding assertion holds on descent between the original field and an algebraically closed subfield of finite transcendence degree over $\mathbb{Q}$.*

For a fixed klt pair, one resolution will display every divisor computing its closed-point minimum. This allows a second discrepancy to be minimized on the same finite list.

Proof. We identify the computing divisors on a fixed resolution and then transfer their discrepancies between fields.

Step 1. In this step, we identify all computing divisors and their least bare discrepancy on the fixed resolution. By Theorem 3.1, take a projective log resolution $f : W \rightarrow U$ of the pair and the maximal ideal $\mathfrak{m}_x$. Include all exceptional primes, strict boundary components and components of the principalized ideal in a simple normal crossing divisor $\sum_i D_i$. Write

$$\begin{aligned} K_W + \sum_i (1 - \alpha_i) D_i &= f^*(K_U + B), \\ K_W + \sum_i (1 - \beta_i) D_i &= f^* K_U, \\ \mathfrak{m}_x \mathcal{O}_W &= \mathcal{O}_W \left(- \sum_i n_i D_i \right). \end{aligned}$$

The second equality is available because K_U is $\mathbb{Q}$ -Cartier. Put $J_x = \{i : n_i > 0\}$. This is a nonempty finite set, and $\alpha_i > 0$ for all i . By (3.2),

$$a := \text{mld}(U \ni x, B) = \min_{i \in J_x} \alpha_i.$$

Let E be any primitive computing divisor and consider its centre on W . If this centre has codimension one, its local ring is a discrete valuation ring and E is exactly the corresponding D_i , for some $i \in J_x$. Otherwise the centre has codimension at least two and lies in some D_i with $i \in J_x$. Inequality (3.1) then contains the term $\alpha_i \text{ord}_E(D_i) \geq a$ and at least one other strictly positive term: either another boundary term or a positive order of a parameter transverse to that component. This would give $A_{U,B}(E) > a$. Thus the full set of primitive computing divisors is the finite set

$$\{D_i : i \in J_x, \alpha_i = a\},$$

and the least bare discrepancy in question is

$$\min_{\substack{i \in J_x \\ \alpha_i = a}} \beta_i.$$

Step 2. In this step, we preserve the computing divisors and both discrepancies under extension and descent. Under algebraically closed field extension, each D_i remains prime, the simple normal crossing and ideal identities persist, and both lists of discrepancy coefficients remain unchanged by Lemma 3.11. The same differential inequality excludes every additional prime from the computing list. Hence the two finite minima, as well as every minimizing divisor and both its discrepancies, are preserved.

For descent, retain f , its marked divisors, the principalized ideal and both canonical pullback identities in the finite diagram of Lemma 3.11. Its realization over an algebraically closed subfield of finite transcendence degree has the same two lists of coefficients. The same finite-minimum argument applies there and after either scalar extension. $\square$

4. POINTED SECTIONS AND CONSTRAINED VALUATIONS

We retain the valuation and character conventions of Section 2. Our aim is to find a section whose order is smaller than its character degree. We use constrained volume minimization, then compare integral section counts while retaining the original invariant field.

The optimization and pinning method follows [Ope26b, Appendices E–H], in the complement framework of [Ope26a]. We use the versions dated 25 September 2026. We prove below the additional assertions needed for a gap only on the original closed-centred lc places.

4.1. Preliminary results.

Theorem 4.1 (Izumi estimate). *Let $X \ni x$ be a $\mathbb{Q}$ -Gorenstein klt germ with maximal ideal $\mathfrak{m}_x$, and put $\text{ord}_x(f) = \max\{j : f \in \mathfrak{m}_x^j\}$ for nonzero $f \in \mathcal{O}_{X,x}$. There is $c = c(X, x) > 0$ such that every real valuation v centred at x , with finite log discrepancy, and every such f satisfy*

$$v(\mathfrak{m}_x) \text{ord}_x(f) \leq v(f) \leq cA_X(v) \text{ord}_x(f)$$

[Li18, Theorem 3.1].

Lemma 4.2 (Monomial retraction). *Let X be a smooth variety and $\pi : Y \rightarrow X$ a proper birational morphism, with Y smooth and D a reduced simple normal crossing divisor such that π is an isomorphism off D . Let $r_{Y,D}$ be the monomial retraction of real valuations of $\mathbb{C}(X)$ centred on X . Then*

$$A_X(r_{Y,D}(v)) \leq A_X(v),$$

with equality precisely when v is monomial on (Y, D) . A valuation in this category is Abhyankar if and only if it is quasi-monomial [JM12, Proposition 3.7, Lemma 5.3 and Corollary 5.4].

Theorem 4.3 (Summation for multiplier ideals). *Let X be a normal $\mathbb{Q}$ -Gorenstein variety, $\mathfrak{a}_1, \dots, \mathfrak{a}_r, \mathfrak{b}$ nonzero coherent ideals, and $\alpha, \beta > 0$. Then*

$$\mathcal{J}(X, (\mathfrak{a}_1 + \dots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta) = \sum_{\substack{\lambda_i \geq 0 \\ \sum_i \lambda_i = \alpha}} \mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta).$$

Only finitely many distinct ideals occur on the right [JM08, Theorem 1 and Corollary 2].

Theorem 4.4 (Stable degeneration). *Let $(X \ni x, \Delta)$ be a klt germ with $X = \text{Spec } R$ over an algebraically closed field of characteristic zero, where Δ is an effective $\mathbb{Q}$ -divisor and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Then the following hold.*

- (i) *The normalized volume on real valuations centred at x has a unique minimizer up to positive scaling. A minimizer v is quasi-monomial and $\text{gr}_v R$ is finitely generated. The induced pair $(X_0 = \text{Spec } \text{gr}_v R, \Delta_0)$ is klt, and $(X_0, \Delta_0; \xi_v)$ is a K -semistable log Fano cone for the Reeb vector ξ_v supplied by the valuation grading.*

- (ii) If v is a quasi-monomial valuation centred at x , $\mathrm{gr}_v R$ is finitely generated, the induced pair $(X_0 = \mathrm{Spec} \mathrm{gr}_v R, \Delta_0)$ is klt, and $(X_0, \Delta_0; \xi_v)$ is a K -semistable log Fano cone for that induced Reeb vector, then v minimizes normalized volume.

[XZ25, Theorems 1.1–1.2].

Theorem 4.5 (Inversion of adjunction for log canonicity). *Let X be a normal variety over an algebraically closed field of characteristic zero. Let S be a reduced divisor and $B \geq 0$ an $\mathbb{R}$ -divisor with no component in common with S , and assume that $K_X + S + B$ is $\mathbb{R}$ -Cartier. Let $\nu : S^\nu \rightarrow S$ be the normalization, and define the full different B_{S^ν} by divisorial adjunction. Then $(X, S + B)$ is lc near S if and only if (S^ν, B_{S^ν}) is lc [Kaw07, Theorem].*

Theorem 4.6 (Existence of minimal models and Mori fibre spaces). *Let $\pi : X \rightarrow U$ be projective between normal quasi-projective varieties and let (X, Δ) be a klt pair with effective real boundary. Then the following hold.*

- (i) *If Δ is π -big and $K_X + \Delta$ is π -pseudo-effective, or if $K_X + \Delta$ is π -big, then $K_X + \Delta$ has a log terminal model over U . In the latter case it has a log canonical model.*
- (ii) *If X is $\mathbb{Q}$ -factorial and $K_X + \Delta$ is not π -pseudo-effective, there exists a $(K_X + \Delta)$ -MMP over U terminating in a Mori fibre space.*

[BCHM10, Theorem 1.2 and Corollary 1.3.3].

Theorem 4.7 (Divisor MMP on a log Fano model). *Let $X \rightarrow U$ be projective, with U normal affine. Suppose that X is $\mathbb{Q}$ -factorial, (X, Δ) is dlt with $\Delta \geq 0$, and $-(K_X + \Delta)$ is ample over U . Then X is a Mori dream space over U . Every divisor MMP over U terminates in a nef model or a contraction on which the negative of its transform is relatively ample [BCHM10, Corollary 1.3.2 and following discussion].*

Theorem 4.8 (Prescribed extraction for a klt pair). *Let (X, Δ) be a quasi-projective klt pair, let $f : W \rightarrow X$ be a log resolution, and let $\mathcal{E}$ be a subset of its exceptional prime divisors satisfying $A_{X, \Delta}(E) \leq 1$. There is a projective birational morphism $Y \rightarrow X$ with Y $\mathbb{Q}$ -factorial whose exceptional prime divisors are exactly the transforms of the members of $\mathcal{E}$. In particular the empty choice gives a small $\mathbb{Q}$ -factorialization [BCHM10, Corollary 1.4.3].*

Lemma 4.9 (Negativity). *Let $\pi : Y \rightarrow X$ be projective birational between normal quasi-projective varieties. Suppose $\pi^* L \equiv M + G + E$, where L is $\mathbb{R}$ -Cartier, M is π -nef and $\mathbb{R}$ -Cartier, $G \geq 0$, E is π -exceptional, and G, E have no common components. Then $E \geq 0$ [BCHM10, Lemma 3.6.2(2)].*

Theorem 4.10 (Semiample and ample models). *Let $X \rightarrow U$ be projective between normal quasi-projective varieties and let (X, Δ) be klt with effective real Δ big over U . If $K_X + \Delta$ is nef over U , it is semiample over U . If $\phi : X \dashrightarrow Y$ is a weak log canonical model over U , then it is a semiample model, the ample model $X \dashrightarrow Z$ exists, and the map factors through a contraction $Y \rightarrow Z$ [BCHM10, Corollary 3.9.2 and Lemma 3.9.3].*

Theorem 4.11 (Boundedness of weak log Fano varieties). *Fix an integer $d \geq 1$ and $\epsilon > 0$. The projective varieties X of dimension d admitting an effective boundary B such that (X, B) is ϵ -lc and $-(K_X + B)$ is nef and big form a bounded family [Bir16, Theorem 1.1].*

Theorem 4.12 (Bounded monotone lc complements). *Fix an integer $d \geq 1$ and a finite set $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$, and set $\Phi(\mathcal{R}) = \{1 - r/j : r \in \mathcal{R}, j \in \mathbb{Z}_{>0}\}$. There is an integer $n = n(d, \mathcal{R}) > 0$ with the following property. Let $f : X \rightarrow Z$ be a projective contraction of normal quasi-projective varieties over an algebraically closed field of characteristic zero, with $\dim X = d$. Suppose that (X, B) is lc, its coefficients lie in $\Phi(\mathcal{R})$, X is of Fano type over Z , and $-(K_X + B)$ is nef over Z . Near each point of Z there is $B^+ \geq B$ such that (X, B^+) is lc, $n(K_X + B^+)$ is integral Cartier and linearly trivial, and*

$$nB^+ \geq n[B] + \lfloor (n+1)\{B\} \rfloor.$$

For Z a point the same conclusion holds globally [Bir19, Theorems 1.7–1.8].

Theorem 4.13 (Rational connectedness of log Fano varieties). *Let (X, Δ) be a projective klt pair, with $\Delta \geq 0$. If $-(K_X + \Delta)$ is nef and big, then X is rationally connected [HM07, Corollaries 1.3(1) and 1.5(2)].*

Lemma 4.14 (Images of Fano-type varieties). *Let X be a projective variety of Fano type in characteristic zero. If $X \rightarrow Y$ is a contraction, then Y is of Fano type. A flip of a projective Fano-type variety is again of Fano type [PS09, Lemma 2.8(i)–(ii)].*

Theorem 4.15 (Weak toroidalization). *Let $f : X \rightarrow B$ be a dominant morphism of varieties over a field of characteristic zero, and let $Z \subsetneq X$ be closed. There are projective birational modifications $X' \rightarrow X$, $B' \rightarrow B$, and a morphism $f' : X' \rightarrow B'$ over f , such that X', B' are smooth quasi-projective, admit strict toroidal embeddings with simple normal crossing boundaries, and f' is toroidal. The inverse image of Z is a divisor contained in the source boundary, and the source modification is an open embedding off that boundary [ADK13, Theorem 1.1].*

Theorem 4.16 (Chart criterion for log smoothness). *A morphism $f : (X, M) \rightarrow (Y, N)$ of fine log schemes is log smooth if and only if, étale locally and after choosing a chart $Q \rightarrow N$, it has a chart $Q \rightarrow P$ by fine monoids such that the kernel and torsion part of the cokernel of $Q^{\text{gp}} \rightarrow P^{\text{gp}}$ are finite of orders invertible on X , and the induced morphism*

$$X \longrightarrow Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$$

is smooth. The chart can be chosen with injective group map [Kat96, Theorem 4.1 and Remark 4.2].

Theorem 4.17 (Generalized canonical bundle formula). *Let $\rho : X \rightarrow X'$ be a projective birational morphism of normal projective varieties, let M be a $\mathbb{Q}$ -Cartier divisor on X nef over a point, and put $M' = \rho_* M$. Let $(X', B' + M')$ be a generalized subpair with rational B' and $\mathbb{Q}$ -Cartier $K_{X'} + B' + M'$. Let $f : X' \rightarrow Z'$ be a contraction to a normal projective variety. Assume that $K_{X'} + B' + M' \sim_{\mathbb{Q}, f} 0$, that B' is effective over the generic point of Z' , and that the generalized pair is generalized lc over that generic point. Then the moduli b -divisor constructed from the generalized log canonical threshold discriminants is $\mathbb{Q}$ - b -Cartier and b -nef [Fil20, Theorem 1.4].*

Theorem 4.18 (Bounded generalized Calabi–Yau models). *Fix an integer $d \geq 1$ and $\epsilon > 0$. Let X range over projective rationally connected varieties admitting a generalized ϵ -lc pair $(X, B + M)$ of dimension d , with $B \geq 0$ and b -divisor data M descending to a real Cartier divisor nef over a point on a projective birational model, for which $K_X + B + M_X \sim_{\mathbb{R}} 0$. These X form a bounded family up to isomorphism in codimension one [Bir23b, Theorem 1.7].*

Theorem 4.19 (Effective birationality for integral divisors). *Fix an integer $d \geq 1$ and $\epsilon > 0$. There is an integer $m = m(d, \epsilon) > 0$ such that, if X is a projective ϵ -lc d -fold and N is a nef and big integral Weil divisor with $N - K_X$ pseudo-effective, both $|m'N + L|$ and $|K_X + m'N + L|$ define birational maps for every integer $m' \geq m$ and every integral pseudo-effective divisor L [Bir23a, Theorem 1.1].*

Theorem 4.20 (Seifert bundle construction). *Let X be a normal variety in characteristic zero, $\mathcal{L}$ a rank-one reflexive sheaf, D_i distinct prime divisors, and $0 < s_i < 1$ rational numbers. Set*

$$\mathcal{A} = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}^{[j]} \left(\sum_i \lfloor js_i \rfloor D_i \right),$$

with the natural multiplication. Then $\text{Spec}_X \mathcal{A} \rightarrow X$, with its grading action, is a Seifert $\mathbb{G}_m$ -bundle if there is an integer $M > 0$ with all $M s_i$ integral and $\mathcal{L}^{[M]}(\sum_i M s_i D_i)$ locally free. Every Seifert $\mathbb{G}_m$ -bundle arises uniquely in this way. Gluing the spectra of the two half-graded algebras gives its proper compactification with zero and infinity sections [Kol04, Theorem 7 and Section 14].

Theorem 4.21 (Monomial multiplier ideals). *Let $\mathfrak{a}$ be a nonzero monomial ideal in $\mathbb{C}[x_1, \dots, x_n]$, let P be its Newton polyhedron, and let $c > 0$ be rational. The multiplier ideal $\mathcal{J}(\mathbb{A}^n, \mathfrak{a}^c)$ is generated by the monomials x^u with $u + (1, \dots, 1) \in \text{Int}(cP)$ [How01, Main Theorem].*

Theorem 4.22 (Relative vanishing for integral Weil divisors). *Let (X, Δ) be an analytic lc pair and $\pi : X \rightarrow Y$ a projective morphism. Let L be a $\mathbb{Q}$ -Cartier integral Weil divisor. If $L - (K_X + \Delta)$ is nef and big over Y , and its restriction to each lc centre C is big over $\pi(C)$, then $R^q \pi_* \mathcal{O}_X(L) = 0$ for every $q > 0$ [Fuj22, Theorem 5.8].*

Theorem 4.23 (Smooth relative Kawamata–Viehweg vanishing). *Let $\pi : X \rightarrow S$ be a proper surjective morphism of characteristic-zero varieties with X smooth. Let D be a $\mathbb{Q}$ -divisor which is nef and big over S , and suppose that the support of its fractional part is simple normal crossing. Then*

$$R^i \pi_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0 \quad (i > 0)$$

[Fuj09c, Theorem 0.1].

Lemma 4.24 (Smooth loci and coefficient fields). *For a reduced scheme locally of finite type over a perfect field, the regular locus equals the smooth locus and is dense. Let A be a complete Noetherian regular local k -algebra of dimension d , with residue field κ separable over k . Then A is isomorphic as a k -algebra to $\kappa[[t_1, \dots, t_d]]$ [Sta26, Lemma 33.25.8 and Lemma 15.39.4, Tags 04QM and 0C35].*

4.2. Invariant sections and the original quotient.

Definition 4.25. Let R be a local integral $\mathbb{C}$ -algebra with an action of a finite cyclic group G , let $\chi : G \rightarrow \mathbb{C}^*$ be a character, and let $0 \neq h \in R$ satisfy $\sigma^* h = \chi(\sigma)h$ for every $\sigma \in G$. For an integer $j \geq 0$, set

$$R_j = \{s \in R : \sigma^* s = \chi(\sigma)^j s \text{ for every } \sigma \in G\}.$$

A nonzero $Y \in \text{Frac}(R)^G$ has *cost at most j* if $h^j Y \in R$. For a real valuation w nonnegative on R , the *initial form* $\text{in}_w(f)$ of a nonzero $f \in R$ is its class in the degree- $w(f)$ summand of $\text{gr}_w R$.

The degree j records the character on the whole group. It also records the power of h that clears the denominator.

Lemma 4.26. *Let R be a local integral $\mathbb{C}$ -algebra with an action of a finite cyclic group G . Let $\chi : G \rightarrow \mathbb{C}^*$ be a character, let $0 \neq h \in R$ have character χ , and use the spaces R_j and costs of Definition 4.25. Let v be a real valuation nonnegative on R with $v(h) = 1$. Then the following hold.*

- (i) *For every integer $j \geq 0$, the map $s \mapsto s/h^j$ is a bijection from $R_j \setminus \{0\}$ to the nonzero invariant rational functions of cost at most j . For all integers $j, k \geq 0$, products of functions of costs at most j and k have cost at most $j + k$.*
- (ii) *For every integer $j \geq 1$, the condition $v(s) \geq j$ for every nonzero $s \in R_j$ is equivalent to $v(Y) \geq 0$ for every invariant rational function of cost at most j .*
- (iii) *Suppose w is a G -invariant real valuation nonnegative on R , and write $\bar{h} = \text{in}_w(h)$. Let $j \geq 0$ be an integer and let $\bar{s} \in \text{gr}_w R$ be nonzero and homogeneous. If $\bar{s}/\bar{h}^j$ is G -invariant, there is an $s \in R_j$ with $\text{in}_w(s) = \bar{s}$. Thus this fraction is the initial form of an invariant rational function of cost at most j .*

This permits each homogeneous ratio to be lifted with its original integer cost and its full character.

Proof. For $s \in R_j \setminus \{0\}$, numerator and denominator in s/h^j have the same character, so their quotient is invariant. Conversely, if Y is invariant and $h^j Y$ is regular, then $h^j Y$ has character χ^j . These constructions are inverse. The product assertion follows from

$$h^{j+k} YZ = (h^j Y)(h^k Z).$$

The equality $v(s/h^j) = v(s) - j$ proves (ii).

For (iii), invariance of w makes its filtration G -stable. Since $\bar{h}$ has character χ , invariance of the given fraction implies that $\bar{s}$ has character χ^j . Choose a lift $s_0 \in R$ of $\bar{s}$ in its filtration degree and put

$$s = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)^{-j} \sigma^* s_0.$$

This is the projection to R_j . The same projection fixes $\bar{s}$ in the associated graded ring. Consequently $s \neq 0$ and $\text{in}_w(s) = \bar{s}$. Multiplicativity of initial forms now gives the assertion for s/h^j . $\square$

The discrepancy hypothesis will be used only after primitive normalization on the original quotient. The next lemma isolates that calculation.

Lemma 4.27. *Let $S \ni o$ be a Gorenstein klt algebraic germ, and let a finite cyclic group G act faithfully on S , fixing o . Suppose that $\pi : (S \ni o) \rightarrow (X \ni x) = (S/G \ni \pi(o))$ is quasi-etale and $X \ni x$ is $\mathbb{Q}$ -Gorenstein klt. Let ω be a divisor-free local canonical generator of character χ , and let $0 \neq h \in \mathcal{O}_{S,o}$ have character χ , with $(S, \text{div}_S h)$ lc. Set*

$$D_X = \frac{1}{|G|} \text{div}_X(h^{|G|}).$$

Fix $\epsilon > 0$, and assume that every primitive prime E' over X centred exactly at x with $A_{X,D_X}(E') = 0$ satisfies $A_X(E') \geq \epsilon$. Let F be a primitive prime over S centred exactly at o , let $c > 0$, and suppose $w = c \text{ord}_F$ satisfies $A_S(w) = w(h) = 1$. Let E be the primitive restriction of F to $\mathbb{C}(X)$, put $e = |I_G(F)|$, and set $\gamma = ce$. Then

$$w|_{\mathbb{C}(X)} = \gamma \text{ord}_E, \quad A_X(E) = E(D_X) = \frac{1}{\gamma}, \quad A_{X,D_X}(E) = 0, \quad \gamma \leq \frac{1}{\epsilon}.$$

The bound concerns the value group of a valuation on $\mathbb{C}(X)$, with its primitive generator unchanged.

Proof. The quasi-etale hypothesis identifies the quotient boundary of the bare germ with zero. Lemma 3.8, applied with $m = 1$ and the generator ω , gives

$$\text{ord}_F|_{\mathbb{C}(X)} = e \text{ord}_E, \quad A_S(F) = e A_X(E).$$

Thus $w|_{\mathbb{C}(X)} = \gamma \text{ord}_E$ and $1 = A_S(w) = \gamma A_X(E)$. The invariant function $h^{|G|}$ also gives

$$1 = w(h) = \frac{w(h^{|G|})}{|G|} = \gamma E(D_X).$$

It follows that $A_{X,D_X}(E) = 0$. Every function in $\mathfrak{m}_{X,x}$ pulls back into $\mathfrak{m}_{S,o}$, so its value under w is positive. Hence E is centred exactly at x . The assumed gap applies to this primitive prime and gives $1/\gamma \geq \epsilon$. $\square$

4.3. Optimization on a common lc locus.

Definition 4.28. Let $S \ni o$ be a $\mathbb{Q}$ -Gorenstein klt germ, let $0 \neq h \in R = \mathcal{O}_{S,o}$, and suppose $(S, \text{div } h)$ is lc. For nonzero $F_j \in R$ and positive integers m_j , the *common lc slice* is

$$\mathcal{Q} = \{q : A_S(q) = q(h) = 1, \ q(F_j/h^{m_j}) \geq 0 \text{ for all } j\}.$$

Here q ranges over nontrivial quasi-monomial valuations nonnegative on R ; centres whose closures contain o are included. A test is *exact* if its value is zero everywhere on $\mathcal{Q}$. An *allowed height* is a function

$$b(q) = b_0 q(h) - \sum_{\alpha} b_{\alpha} q(g_{\alpha}), \quad b_{\alpha} \geq 0, \quad 0 \neq g_{\alpha} \in R,$$

which is strictly positive on the entire slice. Its coefficients may be real; a rational height has rational coefficients.

Including the faces with larger centres is essential for the positivity requirement on a height.

Lemma 4.29. *Let $S \ni o$, h , $\mathcal{Q}$, and b be as in Definition 4.28, with $n = \dim S \geq 1$, rational b , and a closed-centred point in $\mathcal{Q}$. Then $b(q) \operatorname{vol}(q)^{1/n}$ has a unique minimizing ray on the closed-centred part of $\mathcal{Q}$. This ray is the ordinary normalized-volume minimizing ray of an effective rational klt pair on S . Its associated graded ring is finitely generated and normal, and its underlying cone is klt. The initial equation of h defines an lc pair on that cone. The induced height is positive on the entire common central slice, and the grading minimizes its height-volume functional there. Every automorphism preserving the slice and the height fixes the representative normalized by $q(h) = 1$.*

The rational klt boundary in this lemma depends on the germ and the finite tests. No uniform perturbation parameter is asserted.

Proof. We encode the inequalities by an lc boundary, control the error of retraction to its lc locus, and then choose one positive penalty parameter.

Step 1. In this step, we bound the errors in orders and volumes under monomial retraction. Use Theorem 3.1 to resolve the equations and the maximal ideal. On a fixed smooth model P with reduced simple normal crossing divisor D , let $r_D(v)$ denote monomial retraction. For a polynomial g of degree at most k in fixed affine generators,

$$(*) \quad 0 \leq v(g) - r_D(v)(g) \leq C(k+1)A_{P,D}(v).$$

We give the uniformity argument, including weights approaching faces. Choose normal coordinates z and tangential coordinates y . Multiplication by the affine generators in a fixed basis of the finite extension $\mathbb{C}(P)/\mathbb{C}(z, y)$ gives a polynomial relation $\sum_{j=0}^e p_j g^j = 0$, with e fixed and $\deg p_j \leq C(k+1)$. For independent positive coordinate weights, two least terms in this relation tie, so the leading exponent of g has total degree at most $C(k+1)$. Perturbing normal weights and then giving the tangential coordinates small positive weights gives this bound on every vertex of the least normal face, hence on that face.

Choose a term $z^\alpha y^\beta$ on this face. Products of $z_i \partial_{z_i} - j$, for the finitely many integers $0 \leq j \leq C(k+1)$ other than α_i , followed by ∂_y^β , isolate z^α times a unit modulo the ideal of larger normal weight. This ideal is finitely generated; faithful flatness of completion makes the same assertion in the local ring. Each logarithmic differential operator lowers value by at most $A_{P,D}(v)$. Indeed, wedging its contraction of $d \log z \wedge dy$ with $d \log f$ shows that the pulled-back logarithmic form has order at least -1 at a prime; rational monomial approximation gives the same inequality for quasi-monomial valuations. There are $O(k+1)$ operators. The isolated function has value $r_D(v)(g)$, proving (*).

Let Ψ be a fixed effective lc rational sum of principal divisors, write $K_P + B_P = f^*(K_S + \Psi)$, and retract to the coefficient-one components of B_P . Put $e(v) = A_{S,\Psi}(v)$ and denote this retraction by $r(v)$. The log-smooth discrepancy formula, including the finitely many positive component discrepancies, gives

$$A_{P,B_P^{-1}}(v) \leq Ce(v), \quad |A_S(v) - A_S(r(v))| \leq Ce(v).$$

The same error bound holds for every fixed resolved equation and for the maximal ideal. If bare discrepancies are bounded and both maximal-ideal values are bounded below, (*) and Theorem 4.1 give, on every $f \in R$,

$$(**) \quad r(v)(f) \leq v(f) \leq (1 + Ce(v))r(v)(f), \quad \operatorname{vol}(v)^{1/n} \geq \frac{\operatorname{vol}(r(v))^{1/n}}{1 + Ce(v)}.$$

For completeness, if $k = \operatorname{ord}_o(f)$, represent f modulo $\mathfrak{m}_o^{[Lk]}$ by a polynomial of degree $[Lk]$. One fixed large L , chosen using the two lower maximal-ideal bounds and Izumi, makes the error have larger value for both valuations. Apply (*) to that polynomial and use $r(v)(f) \geq ck$. The volume inequality follows from inclusion of valuation ideals.

Step 2. In this step, we encode the common lc slice by an effective lc boundary and construct its klt perturbations. By Lemma 4.2, with the relative canonical correction retained, the normalized lc locus of $(S, \text{div } h)$ is the finite coefficient-one monomial skeleton on a log resolution. Its normalized cones are compact simplices, since their bare discrepancies are positive. The tests cut closed polyhedra. Thus $\mathcal{Q}$ is compact, including its noncentred faces, and $0 < \beta \leq b \leq B$ on it.

Choose small positive rational θ_j with $\sum \theta_j < 1$, and set $\mathbf{b}_j = (h^{m_j}, F_j)$. The discrepancy of the real ideal with orders $(1 - \sum \theta_j)q(h) + \sum (\theta_j/m_j)q(\mathbf{b}_j)$ is

$$A_S(q) - q(h) + \sum_j \theta_j (q(h) - \min\{q(h), q(F_j)/m_j\}).$$

It is nonnegative and vanishes exactly on the rays of $\mathcal{Q}$. On a common principalization take general $P_j = h^{m_j} + a_j F_j$. Their free residual divisors meet the resolved boundary transversely and contain none of its strata. Their coefficients $\theta_j/m_j < 1$ introduce no new lc places. Hence

$$\Psi = (1 - \sum \theta_j) \text{div } h + \sum_j (\theta_j/m_j) \text{div } P_j$$

is effective lc with precisely those zero-discrepancy rays. On these rays $q(P_j) = m_j q(h)$. Resolve also all g_α . For every sufficiently small positive rational λ ,

$$\Psi_\lambda = \Psi - \lambda b_0 \text{div } h + \lambda \sum_\alpha b_\alpha \text{div } g_\alpha$$

is effective and klt. Effectivity uses the reserved coefficient of h ; former zero discrepancies become $\lambda b > 0$, and the finitely many other resolved discrepancies stay positive.

Step 3. In this step, we choose a single penalty for which the ordinary minimizer lies exactly on the common slice. Normalize a centred competitor by $A_S(v) = 1$. On this fixed germ Izumi implies

$$\text{vol}(v) \geq c_1, \quad \text{vol}(v) \geq c_2/v(\mathbf{m}_o).$$

For the second inequality choose a generator z of minimal value and filter $R/\mathfrak{a}_t(v)$ by powers of z . For $j \leq t/(2v(z))$, its successive quotient has length at least $\text{length } R/(\mathbf{m}_o^{\lceil t/(2C) \rceil}, z)$, which is bounded below by a positive constant times t^{n-1} . Sum these quotients. The generator comes from one fixed finite list, so the constants are uniform in v .

Fix a centred point of $\mathcal{Q}$, with objective M . By Theorem 4.4, an ordinary minimizer v for Ψ_λ exists and is quasi-monomial. It satisfies $(e(v) + \lambda b(v)) \text{vol}(v)^{1/n} \leq \lambda M$. Izumi bounds $|b(v)|$, so $e(v) = O(\lambda)$. Retraction gives $A_S(r(v)) = 1 + O(e(v))$ and $b(v) = b(r(v)) + O(e(v))$. Positivity on the full lc slice then bounds both heights below. The sublevel inequality bounds volume above, hence bounds $v(\mathbf{m}_o)$ below. The resolved ideal comparison does the same for $r(v)$. Thus $r(v)$ is centred and $(**)$ applies.

The objective at v minus that at $r(v)$ is at least

$$\frac{e(v)(1 - \lambda C' - \lambda C b(r(v)))}{1 + C e(v)} \text{vol}(r(v))^{1/n}.$$

Choose λ below a fixed threshold for this finite data. This is positive if $e(v) > 0$, contrary to minimality. Hence the ordinary minimizer lies exactly on $\mathcal{Q}$. Conversely every constrained minimizer is an ordinary minimizer for this same λ . Uniqueness in Theorem 4.4 proves uniqueness here, and gives invariance under the stated automorphisms.

Step 4. In this step, we establish positivity and constrained minimality on the central slice. The same theorem gives finite generation and the central klt pair. Each principal equation specializes to its principal initial equation: multiplicativity gives $\text{in}_w((f)) = (\text{in}_w f)$. Thus the central boundary is rational Cartier and removing it leaves a $\mathbb{Q}$ -Gorenstein klt cone. Let the encoding coefficients and valid penalties tend to zero. Uniqueness keeps the constrained valuation and its graded ring fixed. The central coefficient of $\text{div}(\text{in}_w h)$ tends to one, proving lc of that pair by

testing each prime. On a central common valuation q_0 , the initial pencil equations have value at least $m_j q_0(h_0)$. Consequently

$$0 < A_{S_0, \Psi_{\lambda, 0}}(q_0) \leq \lambda b_{\text{central}}(q_0).$$

Equality holds at the grading. Central ordinary minimality now gives positivity and constrained minimality on the whole central slice, as asserted. $\square$

Lemma 4.30. *Let $\mathcal{Q}$ be a nonempty common lc slice on an n -dimensional klt germ, as in Definition 4.28. Assume that $\mathcal{Q}$ contains a closed-centred valuation.*

- (i) *Given $q_i \in \mathcal{Q}$ and $p_i \geq 0$ with $\sum p_i = 1$, there is $u \in \mathcal{Q}$ such that $u(f) \geq \sum p_i q_i(f)$ for every nonzero regular f . If some q_i with $p_i > 0$ is centred, so is u . Opposite finite-cost ratios interpolate exactly, and every allowed height satisfies $b(u) \leq \sum p_i b(q_i)$.*
- (ii) *If centred w, u and nonnegative v satisfy $u \geq tw + (1 - t)v$ on regular functions, with $0 < t < 1$, then, for $w(f) > 0$,*

$$\text{vol}(u) \leq t^{-n} \frac{tw(f)}{tw(f) + (1 - t)v(f)} \text{vol}(w).$$

If v is centred, $t = 1/2$, and $v(f) \geq (1 + \eta)w(f) > 0$, with $0 < \eta \leq 1$, then

$$\text{vol}(w) + \text{vol}(v) - 2\text{vol}(u) \geq c(n, \eta) \text{vol}(w)$$

for some $c(n, \eta) > 0$.

- (iii) *On the rays normalized by $b = 1$, two valuations whose volumes approach the minimum agree multiplicatively on all regular functions, with an error depending only on dimension and relative volume error. A real allowed height has a unique centred minimizing ray, continuous in its coefficients for this uniform multiplicative topology. If w is its minimizer, every $q \in \mathcal{Q}$ satisfies*

$$q(f) \leq n \frac{b(q)}{b(w)} w(f).$$

The last estimate includes noncentred q . It supplies the balance bound on every central slice used below.

Proof. We construct interpolants, compare their volumes, and use these comparisons to control the minimizing rays.

Step 1. In this step, we interpolate finite function values and pass to all regular functions. For a finite list of functions g_j , put $s_j = \sum p_i q_i(g_j)$ and choose positive rational targets $s'_j \leq s_j$. Include h with target one and every test numerator F_j with target m_j . Choose N with all N/s'_j integral. Theorem 4.3 applied to $\sum (g_j^{N/s'_j})$ says that a unit multiplier ideal would give a klt product with exponents $t_j \geq 0$, $\sum s'_j t_j = 1$. But then $\sum t_j q_i(g_j) < 1$ for every i with $p_i > 0$; averaging contradicts $\sum t_j s_j \geq 1$. A prime witnessing non-klt, divided by its positive bare discrepancy, therefore meets all lower targets. Lc of h forces its h -value to be exactly one. Compactness of the slice first lets the rational targets increase to the desired values, then gives the finite-intersection property over every regular function. Positivity on generators of the maximal ideal preserves a closed centre. For opposite ratios the two lower inequalities add to zero, so each is equality. This proves (i), also homogeneously on the cone of rays.

Step 2. In this step, we prove the volume comparisons in (ii). If the graded weights of a centred valuation are a_k , with multiplicities, integration of its colength function gives

$$\lim_{L \rightarrow \infty} L^{-n} \sum_k e^{-a_k/L} = \text{vol}(v).$$

Polynomial colength bounds control the tails. A homogeneous basis of $\text{gr}_w R/(\text{in}_w f)$, multiplied by all powers of $\text{in}_w f$, is a basis of $\text{gr}_w R$: reduction modulo that nonzerodivisor proves independence

and spanning. Lift it to $a_i f^j$ in R . Its values under u are at least $tw(a_i) + j(tw(f) + (1-t)v(f))$. The two geometric series in the exponential sums give the first inequality in (ii).

For the strict inequality choose a basis adapted to both finite valuation flags, truncated at TL . If its two weights are W_i, V_i , the deficit in twice the interpolated exponential sum is at least $\sum_i (e^{-W_i/(2L)} - e^{-V_i/(2L)})^2$. Multiply a complement to $\mathfrak{a}_{\eta L/4}(w)$ by $f^{\lfloor L/w(f) \rfloor}$. For large L , every nonzero element of the resulting space has w -value below $(1 + \eta/4)L$ and v -value at least $(1 + 2\eta/3)L$. Its dimension is asymptotic to $\text{vol}(w)(\eta L/4)^n/n!$. Thus it supplies this many separated entries in the two adapted flags. Divide the deficit by L^n , let $L \rightarrow \infty$, then $T \rightarrow \infty$. For example one may take

$$c(n, \eta) = \frac{(\eta/4)^n}{2n!} (e^{-(1+\eta/4)/2} - e^{-(1+2\eta/3)/2})^2.$$

Step 3. In this step, we control normalized near-minimizers and derive the final variational inequality. Let $V_b > 0$ be the infimum of volume on $b = 1$; positivity follows from Izumi and the positive lower bound of b on the compact slice. An interpolant of two normalized competitors has height at most one and volume at least V_b . The strict inequality rules out a factor $1 + \eta$ difference in the value of any function when both volumes are below $(1 + \delta)V_b$ and $2\delta < c(n, \eta)$. This is one bound for every function. Approximate a real height by rational heights. Their minimizers are uniformly multiplicatively Cauchy by this estimate. A fixed late member bounds their maximal-ideal values below, so a limit in the compact slice is still centred. Inclusions of valuation ideals give convergence of volumes. This proves existence, uniqueness, and the stated continuity; finite generation is used only for rational heights.

Finally interpolate $(1-t)w + tq$ and apply the first inequality in (ii). Minimality gives

$$1 \leq [1 + t(b(q)/b(w) - 1)]^n (1-t)^{1-n} \frac{w(f)}{(1-t)w(f) + tq(f)}.$$

Its right derivative at zero is nonnegative, giving $q(f)/w(f) \leq nb(q)/b(w)$. Units have both values zero. $\square$

4.4. Equivariant degeneration and exact signs.

Lemma 4.31. *Let $S \ni o$ be a Gorenstein klt germ with a faithful cyclic action G , fixing o , and quasi-etale quotient. Let a canonical generator ω and a regular lc equation h have the same full character χ . Let w be an invariant closed-centred quasi-monomial valuation with $A_S(w) = w(h) = 1$ and finitely generated normal klt graded ring R_0 . Then the following hold.*

- (i) *The cone $S_0 = \text{Spec } R_0$ is Gorenstein. Its canonical generator has character χ and fine weight $w(h)$. The whole action stays faithful and quasi-etale, and $(S_0, \text{div } h_0)$ is lc. The canonical-weight assertion also holds for an invariant closed-centred quasi-monomial valuation with normal finitely generated klt graded ring without the equality $A_S(w) = w(h)$: its canonical weight is $A_S(w)$. Any additional commuting torus character of the generator is retained.*
- (ii) *Suppose a common slice $\mathcal{Q}$ is defined by invariant ratios F_j/h^{m_j} of w -value zero, and let $\mathcal{Q}_0$ be its initial slice. For every invariant finite-cost ratio $g = U/h^b$ with $w(g) = 0$,*

$$q(g) \leq 0 \ (q \in \mathcal{Q}) \iff q_0(\text{in}_w g) \leq 0 \ (q_0 \in \mathcal{Q}_0).$$

There is no restriction on b . Exact defining tests remain exact on the cone, and all nonzero pure weights belong to its normalized common slice.

- (iii) *Taking invariants commutes with the filtration: $\text{gr}(R^G) = R_0^G$. The lattice generated by invariant homogeneous weights is exactly $w(\mathbb{C}(S)^{G*})$. If it is $\gamma\mathbb{Z}$, then $w|_{\mathbb{C}(S)^G/\gamma}$ is an actual primitive divisor centred at x .*

In particular, neither the group nor the invariant lattice is replaced in passing to the cone.

Proof. We construct the degeneration on the original local ring, compute its canonical form, and use lc perturbations to transfer signs.

Step 1. In this step, we construct the degeneration and identify its canonical weight and character. Choose eigenfunctions x_i whose initials generate R_0 , including any prescribed finite list of equations, and put $\alpha_i = w(x_i) > 0$. Subduction raises the remainder value to infinity. Izumi makes this convergence maximal-ideal adic. Consequently

$$\widehat{R} = \mathbb{C}[[X_1, \dots, X_l]]/I, \quad R_0 = \mathbb{C}[X_1, \dots, X_l]/\text{in}_\alpha I.$$

The actual valuation ideals are the images of the monomial threshold ideals: those ideals are finitely generated, so the completed assertion descends by faithful flatness. Finitely many initial relations show that nearby rational regradings of the fine lattice have the same initial ideal. Indeed the old initial ideal is contained in the new one, and both quotients have dimension $\dim S$, by comparison with maximal-ideal powers. The old quotient is a domain, so the inclusion is equality. The regradings are monomial on the fixed chart of w , and retain the selected initials.

Scale a rational regrading to integral weights a_i and form the extended Rees algebra $A[t, x_1 t^{-a_1}, \dots, x_l t^{-a_l}]$ on a small affine representative. Its central fibre E is normal Cartier and its general fibre is S . The total space is normal: it is normal off E , satisfies S_2 along E , and is regular at its generic point. Its central valuation is the Gauss extension with t -value one. The logarithmic-form calculation gives

$$\text{div}(\omega \wedge dt/t) = (A_S(v) - 1)E.$$

The residue of $t^{-A_S(v)}\omega \wedge dt/t$ generates the canonical sheaf of E . It has exactly character χ . Nearby rational regradings identify its fine weight with $A_S(w)$, hence with the weight of h_0 . There are no nonconstant units in a positive graded domain with degree-zero part $\mathbb{C}$, so these generators differ only by scalars. This calculation also preserves any additional commuting torus character, and does not require an lc equality when computing the canonical weight.

Step 2. In this step, we preserve the faithful quasi-etale action and establish log canonicity of the initial boundary. If a nonidentity element acquired a fixed divisor or became trivial on the central fibre, its fixed locus at a general smooth point would be smooth over the parameter: averaging a tangent lift of ∂_t gives an invariant lift. The same fixed locus would occur in nearby nonzero fibres, contradicting the original faithful quasi-etale action. For an effective principal boundary Δ with $A_S(w) = w(\Delta)$, homogenize its equations. The only relative canonical coefficient over the zero parameter is $A_S(v) - v(\Delta) = 0$. Hence the Rees pair with its central fibre is crepant to the lc product pair. Adjunction gives the lc initial boundary. At codimension-one points of the normal central fibre the total space is smooth over the parameter, so its different is precisely that boundary.

Step 3. In this step, we transfer signs on the common slices and retain the original invariant value group. Encode $\mathcal{Q}$ by the boundary Ψ used in Lemma 4.29, choosing the pencils general on both the original and central resolutions. If $q(g) \leq 0$ everywhere on $\mathcal{Q}$, then $\Psi + \delta(\text{div } U - b \text{div } h)$ is effective lc for sufficiently small rational $\delta > 0$. On the zero-discrepancy components this is exactly the assumed sign; every other resolved discrepancy stays nonnegative. Specialization, just proved, gives the central lc boundary. Its discrepancy on $q_0 \in \mathcal{Q}_0$ is at most $-\delta q_0(\text{in}_w g)$, proving the forward sign. Conversely, start from the central sign and construct that lc perturbation there. Theorem 4.5 on the normal Rees space makes its total pair lc near the central vertex. This neighbourhood meets the section through the original point over a nonzero parameter, giving the original lc perturbation and the reverse sign. The argument is for each arbitrary b ; it imposes no degree cutoff. The defining lower inequalities then transfer exactness. Pure weights have lc discrepancy zero by the canonical weight calculation and vanish on every weight-zero test.

Finally the Reynolds operator makes invariants exact in every filtration degree. Every invariant symbol lifts with exactly its value, and differences give the full value group on $\text{Frac}(R^G) = \mathbb{C}(S)^G$. A finite field extension preserves rational rank and residue transcendence degree. Thus the

restriction is Abhyankar; in rational rank one it is a multiple of a primitive divisor. Its maximal-ideal values are positive, so its centre is the original closed point. $\square$

Lemma 4.32. *Consider a sequence of the data in Lemma 4.31, in dimension at most d , with exact tests. Suppose that on the common central slices a fixed C satisfies $q_0(s) \leq Cw_0(s)$ for all homogeneous regular s . After a subsequence the valuations can be replaced, on their original fields, by invariant valuations w' satisfying all the same conditions, uniformly multiplicatively approaching w on regular functions, for which*

$$q_0(s) \leq (1 + \eta_i)m, \quad \eta_i \longrightarrow 0,$$

simultaneously for every positive integer m , every regular symbol of fine weight $m \operatorname{wt}(h_0)$ and full character χ^m , and every common q_0 . The central balance estimate persists with a fixed new constant.

This refinement changes neither the common actual slice nor any of its exact equations.

Proof. We refine the valuations by compression and bound the number of refinements by rational rank.

Step 1. We first describe compression of two gradings. On the first cone take a closed-centred valuation v in its common normalized lc slice, invariant under G and the fine torus, with normal finitely generated klt graded ring. Lift simultaneous homogeneous generators, with first and second weights $\alpha_i, \beta_i > 0$, to the original ring. For small positive δ ,

$$\operatorname{in}_{\alpha+\delta\beta} I = \operatorname{in}_\beta \operatorname{in}_\alpha I.$$

To verify this, lift finitely many generators of the right side to relations. Positivity leaves only finitely many terms that could compete with their leading faces, so it gives one inclusion for all small δ . The dimension/domain argument in Lemma 4.31 gives equality. The prime initial ideal defines an actual valuation on the original ring. Outside a countable set of parameters, $(a, b) \mapsto a + \delta b$ is injective on the joint lattice. Its rational rank plus residue transcendence degree is then $\dim S$; Lemma 4.2 identifies it as quasi-monomial on a smooth model. Canonical transport, retaining the old torus, gives joint canonical weight $(1, 1)$. Applying the canonical-weight calculation to the actual compressed valuation, without assuming its lc equality, therefore proves that its discrepancy and h -value are both $1 + \delta$. For any $\zeta > 0$, decrease δ until

$$(1 - \zeta)\alpha_i \leq \alpha_i + \delta\beta_i \leq (1 + \zeta)\alpha_i$$

for every generator. These inequalities hold for every monomial. The descriptions of the actual threshold ideals, obtained by subduction and faithful-flat intersection, give the same inequalities for the values of every regular function. They control cancellations because they compare entire ideals, rather than one chosen expression of a function. Normalization by $1 + \delta$ therefore approaches w multiplicatively on all functions at once. Every retained zero-weight equation stays exact. A nonzero second value on an old weight-zero rational function gives a nonzero kernel in the joint-to-old lattice map, and hence a strict increase of rational rank.

Step 2. In this step, we use persistent defects to increase rational rank and force the defects to tend to zero. Suppose now that the claimed defects do not tend to zero. Choose a fixed rational $c > 0$ and s, m, q_0 with $q_0(s)/m - 1 \geq 2c$. On the central common slice set $g(q) = q(s)/m - q(h_0)$ and $M = \max g$. The heights

$$b_\mu = q(h_0) + \mu(cq(h_0) - g(q)), \quad 0 \leq \mu < (M - c)^{-1},$$

are positive allowed heights. Interpolate a valuation maximizing g , possibly noncentred, with a fixed centred point of weight t . As μ approaches its upper endpoint within t , the height is $O(t)$ and Lemma 4.30 with $f = h_0$ bounds volume by $O(t^{1-n})$. The minimum objective tends to zero. A fixed-germ volume lower bound forces the optimizer eventually to have $g > c$. Continuity in μ therefore supplies a rational parameter with $0 < g(v) < c$, unless $\mu = 0$ already works. At this parameter $b_\mu(v) \geq 1$, whereas $b_\mu \leq 2 + 1/c$ on both actual and central slices.

Its rational optimizer is invariant under the full group and old torus, and is nonzero on the neutral ratio s/h_0^m . Compression increases rational rank and can be chosen arbitrarily close on every function. On a joint homogeneous symbol of degree (a, b) , old balance gives $b \leq Ca$. The second central balance is at most $d(2+1/c)b$, while the compressed grading is $(a+\delta b)/(1+\delta)$. For $\delta < 1$ this retains a uniform balance constant. Iterate along subsequences with fixed dimensions and ranks. There can be at most d rank increases. Therefore the supremum of all defects tends to zero, with its quantifier over all degrees intact. Choose the finitely many compression errors to tend to zero; their product gives the required comparison. $\square$

Lemma 4.33. *In the setting of Lemma 4.31, assume the tests are exact. Let finite-cost invariant ratios Y_j^+, Y_j^-, Z_α have opposite pair values and zero Z_α -values throughout $\mathcal{Q}$. Write their initials as A_j, B_j, C_α . Suppose a neutral subfield K_* contains every $A_j B_j, C_\alpha$, every common central valuation is trivial on K_* , and the monomials associated with a basis of the integral kernel of $\mathbb{Z}^{\{j\}} \rightarrow \text{wt}(A_j)$ are algebraic over K_* . Then every polynomial in the positive ratios and the Z_α , whose least initial sum is nonzero, has its predicted value wherever its terms tie on $\mathcal{Q}$. Algebraic independence of their initials is preserved at every later valuation in $\mathcal{Q}$, and the same noncancellation holds on its common cone when that cone exists.*

The kernel condition is finite and the conclusion holds for all polynomial degrees at once.

Proof. Every kernel monomial is algebraic over K_* , by expressing its exponent in the chosen integral basis. For a polynomial P , multiply by the negative partners of one least term, obtaining an actual finite-cost ratio g with $w(g) = 0$. Its nonzero initial is a sum of kernel monomials times products $A_j B_j$ and C_α ; hence it is algebraic over K_* . A valuation trivial on a field is trivial on its algebraic elements: positive or negative value would give a unique least term in an equation with nonzero constant and leading coefficients. Reverse sign transfer therefore gives $q(g) \leq 0$ everywhere on $\mathcal{Q}$. At a tie its terms have value zero, so the reverse inequality holds too. This proves the assertion for P . A putative relation among later initials can be separated into tied homogeneous groups; the assertion just proved rules out each nonzero group. Finally transfer the same upper bound for the same actual g to the later cone and use its tied-term lower bound. No reciprocal was assumed to have finite cost, and no new degree-dependent cutoff is used. $\square$

4.5. Angular divisors and projective comparison. For a positive fine-graded cone as in Lemma 4.31, put $R = R_0^G$. Let M be the lattice generated by the weights occurring in R , let $\mathcal{C} \subset M_{\mathbb{R}}$ be their cone, and let $\sigma = \mathcal{C}^\vee$. The effective torus has character lattice M . Its invariant field is denoted by K . Choose homogeneous rational functions z^m , multiplicatively in $m \in M$, so that $\text{Frac } R = K(z^M)$. Let $u \in M_{\mathbb{Q}}$ be the weight of h_0 , and write

$$h_0^{|G|} = f_h z^{|G|u}, \quad \theta = \omega_0/h_0 = \theta_0 \wedge d \log z_1 \wedge \cdots \wedge d \log z_r, \quad \gamma = \frac{\text{div } f_h}{|G|}.$$

The form θ is invariant under both groups. For a primitive prime P of K , define

$$\begin{aligned} a(P) &= 1 + \text{ord}_P(\theta_0), \\ \Delta_P &= \{y : P(f) + \langle y, m \rangle \geq 0 \text{ for every } fz^m \in R\}, \\ L_\mu(P) &= \min_{y \in \Delta_P} \langle y, \mu \rangle, \quad D(P) = \gamma(P) + L_u(P). \end{aligned}$$

For a rational Cartier b-divisor A on this field, write

$$S(A) = \{0\} \cup \{f \in K^* : \text{div } f + A \geq 0\}.$$

The inequality here is tested on every prime on every model.

Proposition 4.34. *For the cone data just defined, the following hold.*

- (i) *One has $u \in \text{int } \mathcal{C}$. Every interior rational ray μ has a normal projective quotient V_μ with field K , and L_μ is the Cartier b-pullback of its ample ray divisor. The function a is an integral lc discrepancy, represented on each V_μ by an effective boundary with trivial adjoint.*

The divisor D is effective, b -nef and big, and $a + D$ is the discrepancy of an effective generalized klt Calabi–Yau pair. Each V_μ is of Fano type.

- (ii) Rational Gauss valuations (P, y) , $y \in \Delta_P$, have lc-pair discrepancy $a(P)$; pure weights in σ have discrepancy zero. They exhaust the divisorial lc tests, up to scaling, and lift to the original finite cover. At an actual prime P of V_μ , a minimizing y and an integer $e > 0$ satisfy

$$ey \in \text{Hom}(M, \mathbb{Z}), \quad a(P) + (P, y)(h_0) = 1/e.$$

Thus outside $a = 0$ one has $a(P) = e = 1$ and $(P, y)(h_0) = D(P) = 0$.

- (iii) For $\xi = fz^k$, $m \geq 0$ integral, and $k + mu \in \text{int } \mathcal{C}$, set $J_m = \text{div } f + m\gamma + L_{k+mu}$. For every integer $q > 0$ and $g \in K$,

$$g \in S(qJ_m) \iff g\xi^q h_0^{qm} \in R_0.$$

In particular $S(mD) = \{g \in K : gh_0^m \in R_0\}$.

The integral discrepancy a , rather than an index bound for D , will supply the discrete information in the proof.

Proof. We construct the ray quotients, compute their divisorial orders, and then identify the section spaces.

Step 1. In this step, we construct the Gauss valuations and projective ray quotients in the actual lattice. The homogeneous fraction group splits over the free lattice M , giving $K(z^M)$. Finitely many homogeneous algebra generators define Δ_P . Adding a large positive cocharacter makes their inequalities hold, so this polyhedron is nonempty. The coefficient-adjusted monomial minimum defines a Gauss valuation. For rational y it is discrete and has residue transcendence degree $\dim S - 1$, so it is divisorial up to scaling. Write locally $\theta_0 = t^{a(P)} d \log t \wedge dy_1 \wedge \cdots$. Adjust a finite-index lattice of torus monomials by powers of t to obtain units. The leading logarithmic wedge has order -1 at a primitive divisor. Thus the lc discrepancy of (P, y) is $a(P)$, and that of a pure weight is zero. Lc makes a nonnegative; it is integral by its definition from a rational form. Klt of the bare cone gives $\langle y, u \rangle > 0$ on every nonzero pure ray, proving that u is interior.

The ray algebra is generated by the integral points in a rational polyhedral cone of monomials in the fixed generators. It is normal by the normality of the ambient graded algebra. Its Proj is therefore normal projective. Every neutral rational function can be written as a ratio on that ray: shift a common weight to a large multiple of the interior ray by an occurring monomial. Thus its field is K . Linear programming gives

$$L_\mu(P) = \max_{\lambda_j \geq 0, \sum \lambda_j m_j = \mu} - \sum_j \lambda_j P(f_j),$$

where $f_j z^{m_j}$ are the generators. Clearing the denominators of the finitely many vertices gives the free tautological system on a Veronese ray algebra. The formula computes its maximal pole order on every higher model, which is exactly Cartier b-pullback. This proves the assertion for L_μ and for D ; regularity of h_0 gives effectivity of D .

Step 2. In this step, we identify the quotient discrepancies and the divisorial lc tests. At a prime of the ray quotient localize an affine ray chart over its generic DVR. A height-one component of the pulled-back uniformizer is torus invariant and dominates this prime. Its primitive order is $e(P, y)$ with e integral. The ray section is invertible there, so y minimizes the ray height. This prime lies on the cone itself and has bare discrepancy one, proving the displayed identity. It follows that the boundary $-K_{V_\mu}(\theta_0)$ is effective lc, and subtracting D from that boundary, with nef b-part D , gives effective generalized klt data. Strict positivity on higher primes follows from klt of the cone at a minimizing Gauss extension. Replace a semiample determination of D by a general divided member; this gives an ordinary effective klt Calabi–Yau boundary. Bigness permits a small ample summand in a second effective representative. Mix sufficiently little of

that representative into the first and subtract its ample summand. The remaining boundary is klt with ample negative adjoint, proving Fano type.

For an arbitrary divisorial lc test, nontrivial restriction to K is an integral multiple of a prime: the residue transcendence inequality forces equality for that restriction. The same logarithmic wedge gives discrepancy at least $ea(P)$, with strict inequality if adjusted torus-unit residues are dependent. Zero discrepancy therefore forces independence and no cancellation among tied monomials. This is precisely the Gauss rule. A trivial restriction similarly gives a nonzero pure weight. Finite discrepancy comparison is Lemma 3.8; the positive grading makes the vertex the unique point over the quotient vertex, so the centres have the required closures.

Step 3. In this step, we characterize sections by the Gauss-order inequalities. Finally J_m is the minimum Gauss order of ξh_0^m . If the displayed inequalities hold, an invariant power of $g\xi^q h_0^{qm}$ is nonnegative at every Gauss prime and every pure weight. A pole of a homogeneous rational function would have a torus-invariant prime component, whose order is Gauss. Normality rules out this pole, and normality upstairs then puts the original semi-invariant in R_0 . The converse is evaluation of a regular function. This proves (iii). $\square$

Definition 4.35. For the cone of Proposition 4.34, let M be its actual invariant weight lattice, σ the dual cone, and $u \in M_{\mathbb{Q}}$ the weight of h_0 . The *width* of $k \in M_{\mathbb{R}}$ is

$$\|k\| = \text{width}(k) = \max_{\substack{y \in \sigma \\ \langle y, u \rangle = 1}} |\langle y, k \rangle|.$$

The grading valuation is normalized by $w(h_0) = 1$. Along a sequence of cones, a width bound $O(L)$ means $\|k\| \leq CL$ with C independent of the cone.

Since u is interior to the full-dimensional pointed weight cone, the slice in this definition is compact and spans the dual vector space. The width is therefore a norm. The normalized grading is a point of that slice, so an invariant homogeneous ratio of weight k has absolute grading value at most $\|k\|$.

In particular, Lemma 4.32 implies the following angular bound on every floor prime P , meaning $a(P) = 0$, for which the angular tests are nonnegative:

$$(4.1) \quad P(\Gamma) \leq (1 + \eta_i)D(P) \quad \text{for every effective } \Gamma \sim_{\mathbb{Q}} D.$$

Indeed, write $\Gamma = D + \text{div } \phi/m$. Its regular symbol is $h_0^m \phi$, of full character χ^m . Normalize a height-minimizing Gauss lift by its positive h_0 -value $D(P)$ and apply that lemma.

All projective comparisons below take place on projective total spaces and projective bases over $\mathbb{C}$. Their rational Cartier b-parts are nef over a point on projective determinations. By relative generalized Calabi–Yau data over $f : X \rightarrow Z$ we mean effective trace B , such a b-part M , and $K_X + B + M_X \sim_{\mathbb{Q}} f^*L$. We use their discrepancies homogeneously on multiples of prime orders. Sequential assertions below are required on every subsequence.

Lemma 4.36. *Fix $d \geq 1$ and $\delta > 0$. Let $X \rightarrow W \rightarrow Z$ be projective morphisms of projective varieties, with $X \rightarrow W$ a K_X -negative Mori fibre space in dimension at most d . Assume X is $\mathbb{Q}$ -factorial, ordinarily δ -lc, and of Fano type over Z . All generalized Calabi–Yau data in this lemma are over Z , have effective trace, and have rational Cartier b-parts nef over a point. Suppose these data include a klt member. A chosen horizontal lc place of a generically lc member, or a chosen place computing a vertical discriminant of a generically klt member, admits, after a bounded field extension, a monomial chart of the following form:*

$$x \geq 0, \quad e_* \geq 0, \quad M_{\text{ch}}x = r_{\text{ct}}e_*.$$

The matrices belong to a fixed finite list; the integral contact vectors may be unbounded. The original uniformizer has order pe_* , with bounded positive integer ρ . Integral matches restrict to positive integral multiples of primitive original-field orders. On each fixed e_* -slice, all effective generalized Calabi–Yau discrepancies have a uniform angular Lipschitz constant. Horizontal charts are orthants with the same homogeneous assertion.

The generic fibre has rational logarithmic coordinates of bounded pole degree, and its chart valuations are Gauss on their field. A function of pole degree $m \geq 1$ has angular Lipschitz constant $O(m)$. The same assertions hold on a bounded normal model isomorphic in codimension one to the generic fibre. If a b -order has effective trace of degree $O(\xi)$ and is b -linearly equivalent to a difference of nef b -parts of trace degrees $O(\xi)$, its horizontal constant is $O(\xi)$, also for $\xi \rightarrow 0$.

The contact vector and the nef indices do not enter these constants.

Proof. We retain the chosen place in a bounded ordinary complement, toroidalize the resulting bounded marked families, and control orders by generic-fibre degrees.

Step 1. In this step, we retain the chosen places in bounded ordinary complements. Over a generic base DVR subtract the discriminant discrepancy times its fibre. The resulting generalized subpair is lc and the chosen minimizing components have coefficient one. On a common resolution perturb slightly towards klt data with an ample b -part, provided by a log Fano boundary. Replace that ample part by divided general members, keeping a big ordinary boundary and positive coefficients at the finitely many chosen components. Increase every other exceptional horizontal coefficient and every unwanted vertical coefficient to a number below one. The result is an effective big klt boundary Δ with $K + \Delta \sim_{\mathbb{Q}, W} E$, where E is effective and supported exactly on the unwanted components. A section of a divisible multiple of E is constant on the normal projective generic fibre; its order at a retained component forces it to be a regular DVR function. Its fixed part is therefore that multiple of E . Theorems 4.6 and 4.10 contract precisely this fixed part. The retained big klt boundary makes the resulting model of Fano type, and the selected places survive. The horizontal case uses Theorem 4.8 after the same small ordinary perturbation.

On a nef determination, Lemma 4.9 gives $h^*M_Y - M_Z \geq 0$. Thus ordinary discrepancies of the effective boundary dominate generalized discrepancies. In particular the ordinary pair consisting of the selected coefficient-one divisor is lc. Run its negative adjoint MMP using Theorem 4.7, and apply Theorem 4.12 with seed coefficients zero and one. Crepant pullback of the complement dominates this divisor by MMP negativity and retains it as an lc place. The complement index is bounded only by dimension. Several vertical minimizers can be kept together. On a common resolution their coefficient-one locus is connected: if $H = B^=1$ and $E_0 = [-B^{<1}]$, Theorem 4.23 gives $R^1h_*\mathcal{O}(E_0 - H) = 0$, hence a surjection from $\mathcal{O}_Y$ to $h_*\mathcal{O}_H(E_0)$ through $h_*\mathcal{O}_H$. Its reduced image is the connected special fibre. Adjacent coefficient-one strata consequently give continuous monomial interpolation of all normalized Cartier orders between minimizing places.

Step 2. In this step, we construct a finite list of monomial charts while retaining the original uniformizer orders. Theorem 4.11 bounds the ordinary generic Fano fibres. For a bounded very ample polarization A , a general smooth curve gives $-K_F A^{f-1} \leq (f-1)A^f + 2$. A bounded complement index therefore bounds the degree of its reduced support. The embedded marked pairs lie in finitely many finite-type families. A bounded extension suffices to define the embedding: Theorem 4.13 makes the Picard group of a smooth resolution a torsion-free lattice of bounded rank; its finite Galois image injects modulo three. The invariant polarization then defines a projective-space form of bounded dimension, split by a maximal subfield of its central simple algebra, of bounded degree. Apply Theorem 4.15 to these fixed marked families, and stratify the complements by noetherian induction. A trait lifts to their proper modifications even when its special point is on the boundary. There are finitely many exponent matrices. Units are retained in their full character lattices: for equations $y_j = u_j x^{\alpha_j}$, split the kernel of the exponent map and use its unit characters as chart coordinates. After tame unit roots, the group map is injective and the remaining map is smooth. Theorem 4.16 gives the claimed log-smooth charts. Saturated base change to the trait yields exactly $M_{\text{ch}}x = r_{\text{ct}}e_*$. Saturation occurs in the same groupification, so integral matches remain integral. Restricting through a finite or etale field extension never divides their orders. The bounded initial extension contributes only ρ . The selected lc place is monomial because its crepant boundary is bounded above by the reduced toroidal boundary and it has zero discrepancy for both.

Step 3. In this step, we bound angular order variation by generic-fibre degrees. On the finite chart list choose rational functions z_j whose logarithmic differentials form a relative basis. Their pole degrees and integral slopes are bounded. At a vertical monomial prime the adjusted units $z_j^{e_j} t_*^{-v(z_j)}$ have independent residue differentials; this gives the Gauss rule. For a rational function g of pole degree m , the image of $(z_1, \dots, z_f, g)$ satisfies $\sum_{j=0}^q c_j(z) g^j = 0$, with q bounded and $\deg c_j = O(m)$. Intersecting general coordinate level divisors proves these degree bounds. The Gauss value of each coefficient has slopes $O(m)$. Two least terms tie, so the value of g lies among $(v(c_i) - v(c_j))/(j - i)$. Continuity on the finitely many pieces proves the same Lipschitz bound on the whole slice.

To treat a nef b-part at two prescribed valuations, replace it on a common determination by the difference of general divided members of $M + \eta A$ and ηA , avoiding both centres. Choose η after the determination so its pushed degree is arbitrarily small. The resulting ordinary boundary has positive and negative horizontal degrees bounded by $-K_F A^{f-1} + o(1)$. A clearing pluriform divided by the chosen logarithmic volume has pole degree proportional to that clearing integer. The preceding order estimate divided by that integer gives a uniform discrepancy bound; vertical coefficients contribute constants on the slice. The same replacement on bounded models in codimension one proves that variant. For the last assertion choose both ample errors $o(\xi)$. The representing rational function has pole degree, divided by its denominator, $O(\xi)$. If its integral pole degree is below one it is constant on the normal projective fibre. Hence no additive constant remains as $\xi \rightarrow 0$. $\square$

Lemma 4.37. *Consider sequences of the projective relative generalized Calabi–Yau data described above, with total dimensions bounded and total spaces of Fano type over their bases. For a generically klt member the discriminant discrepancy is*

$$B(P) = \min_{T|_{C(Z)} = e_T \text{ ord}_P} A(T)/e_T.$$

The minimum is attained and the moduli b-part is rational Cartier and nef over a point. The following assertions hold.

- (i) *Let A_t be finite rational piecewise-affine continuous families, concave at each valuation, with A_0 klt and parameter ranges tending to infinity. If A_t is nonincreasing and $A_t(T_i)/A_0(T_i) \rightarrow 1$ whenever $A_0(T_i) \rightarrow 0$, for every fixed $t \geq 0$, the same condition holds for discriminants. On signed ranges the same conclusion holds if the ratio assumption is made whenever $A_0(T_i) < \delta$, for a fixed $\delta > 0$; the discriminant cutoff may be smaller. The families are eventually generically klt on each fixed parameter interval.*
- (ii) *Let A_i be the discrepancies of klt effective generalized Calabi–Yau data over projective contractions $f_i : X_i \rightarrow Z_i$ as above. Let H_i be rational $\mathbb{Q}$ -Cartier divisors on Z_i , and fix an integer $m > 0$. Suppose that for every sequence of primitive orders T_i with $A_i(T_i) \rightarrow 0$, the phases of $\text{ord}_{T_i}(mf_i^* H_i)$ have real lifts in $[-o(A_i(T_i)), O(A_i(T_i))]$. Then, for one fixed multiple m' of m , the phases of $\text{ord}_{P_i}(m' H_i)$ have the same property relative to the discriminant discrepancies $B_i(P_i) \rightarrow 0$.*
- (iii) *Suppose $a \geq 0$ takes values in a fixed rational grid on primitive primes, $D \geq 0$, and $a + tD$ is effective and klt for $0 < t \leq t_0$, with affine total base class. On a uniform smaller interval its discriminant is $a' + tD'$, the successive lexicographic minima, and a' has bounded denominator on every model. If also $a + \lambda J + sC$ is effective on $0 < \lambda \leq \lambda_0$, $0 < s \leq \lambda\theta_0$, with affine base classes, J in a fixed rational grid, $C \geq 0$, and on $a = 0$*

$$D \leq C \leq KD, \quad -\eta_i C \leq J \leq KC, \quad \eta_i \rightarrow 0,$$

then its discriminant is exactly $a' + \lambda J' + sC'$ on a uniform smaller sector. Both a', J' have bounded denominator. Eventually no horizontal prime has $a = 0, J < 0$. These endpoint data have effective traces and rational Cartier nef b-parts.

All primitive orders in this statement belong to the indicated original function fields.

Proof. We give the lattice and minimization arguments, since they are what make the conclusions uniform in contact orders and indices.

Step 1. In this step, we reduce the projective comparisons to monomial charts along a finite Mori induction. Extract the finitely many primes of a chosen klt member with discrepancy below a small fixed $\delta_* < 1$. For (ii) this member is A_i itself; it is klt on the whole total space, including over the boundary of the base. Finiteness follows on a nef determination from the log-smooth formula: below one, a prime must be monomial in the resolved support, and positivity bounds its integral weights. The construction of an ordinary klt boundary in Lemma 4.36 permits Theorem 4.8 to retain exactly these primes. The nef trace-pullback error is effective by Lemma 4.9; hence ordinary discrepancies on the extracted model are at least δ_* . Run a K -MMP over the base. If the map is not birational, it gives a Mori fibre space; all relative Calabi–Yau data remain crepant. Their traces stay effective whenever their values at the extracted primes are at most one. Monotonicity or the signed ratio assumption, after trimming the growing range, ensures this in (i). In (iii) choose δ_* below the grid spacing: extracted primes have $a = 0$, and the stated inequalities ensure effectivity of all the data. Mori bases stay of Fano type by the same relative log Fano construction. Each step lowers dimension.

On a generic DVR a common resolution gives finitely many vertical components computing the minimum: their ratios $A(T)/e_T$ are averaged by monomial weights, while horizontal discrepancies are positive. This proves attainment. Theorem 4.17 applies on the projective models, with their effective generic traces and absolutely nef determining divisors. It supplies the moduli data. Effectivity downstairs follows by testing a prime on the total space covering a base prime, whose ratio is at most one. Minima compose through the successive fields.

Step 2. In this step, we construct integral approximations in the original chart lattices. For a fixed chart matrix write $Mx_i = r_i$, $x_i \geq 0$, and choose $x_{0,i}$ with $lx_{0,i}$ integral, using a fixed maximal minor. The integer l is bounded. Put $U_i = x_i - x_{0,i}$ in $H = \ker M$, with lattice $\Lambda = H \cap \mathbb{Z}^r$. Two elementary approximation forms will be used:

$$(***) \quad \begin{array}{ll} \alpha_i \rightarrow 0 : & Mn_i = E_i r_i, \quad |n_i - E_i x_i| = O(E_i \alpha_i), \quad E_i \alpha_i \rightarrow 0; \\ \gamma > 0 : & Mn_i = E_i r_i, \quad |n_i - E_i x_i| \leq \gamma, \quad 1 \leq E_i \leq J(\gamma). \end{array}$$

Both have $n_i \geq 0$ integral. The second is pigeonhole approximation of lU_i in H/Λ : subdivide a fundamental torus into Q^k cubes and subtract two of $0, lU_i, \dots, Q^k lU_i$. The finite matrix list makes $J(\gamma)$ uniform in r_i . For the first, diagonalize the characters whose distance of $\ell(lU_i)$ to $\mathbb{Z}$, divided by $l\alpha_i$, is bounded. Their signed limits define a vector u_* . Sampling integers j with $j\alpha_i$ in a fixed positive interval, the points $jl(U_i - \alpha_i u_*)$ converge in distribution to Haar measure on the annihilator of those characters. For the other characters the geometric-series bound is $O(\alpha_i / \text{dist}(\ell(lU_i), \mathbb{Z})) \rightarrow 0$. Let the sampling intervals shrink slowly to zero and target neighbourhoods of radius $o(j\alpha_i)$. Set $E_i = jl$ and add the approximating lattice point to $E_i x_{0,i}$. This proves the first assertion. Errors below one force nonnegative integral coordinates. Horizontally the same Fourier argument, sampling real scalars, gives $n_i = s_i u_i + O(s_i)$, $s_i \rightarrow 0$, $s_i |u_i| \rightarrow \infty$, whenever $u_i \geq 0$ and $|u_i| \rightarrow \infty$.

Step 3. In this step, we transfer the long-interval discrepancy and signed-phase estimates to discriminants. A first bounded horizontal lc threshold for A_t , normalized by A_0 , gives vectors u_i on which concavity forces $A_L(u_i) \leq 1 - L/C$. Uniform angular Lipschitz constants force $|u_i| \rightarrow \infty$. The horizontal approximation then gives integral orders with $A_0 \rightarrow 0$ and negative A_L for a fixed large L , contradicting the ratio assumption. Hence bounded parameters are generically klt. If a discriminant ratio fails, follow its finitely many minimizers until a half-drop. If a minimizer with small A_0/e is lost at a transition, the connected minimizing locus in Lemma 4.36 interpolates one at any intervening small cutoff. In its vertical chart put $\alpha = \rho A_0/e$. If $\alpha \rightarrow 0$, the first approximation in $(***)$ gives the same negative fixed-parameter contradiction. Otherwise use the second form, first choosing its error small, then the discriminant cutoff so that $J(\gamma)\rho A_0/e < \delta/2$.

Concavity and a large fixed parameter again give a negative value below the original cutoff. This proves (i) through the Mori induction.

For (ii), augment H/Λ by a circle recording $\rho H(P_i)$, in the first approximation. If the limiting annihilator had a Lie direction with nonzero last coordinate, choose its negative sign and target a displacement ξ_i with sampling scale $s_i = o(\xi_i)$. The resulting integral order has $A = O(\xi_i)$ and a negative phase of size comparable to ξ_i , contrary to the hypothesis. The circle projection is thus finite. A fixed multiple kills it and gives a phase of size $O(B(P_i))$. Targeting zero shows its limiting signed ratio is nonnegative. These assertions hold for integral multiples of primes: multiply their small primitive lifts; they remain small, so no wrapping occurs. To obtain one multiplier for every sequence, otherwise choose violations $\text{dist}(i!H(P_i), \mathbb{Z}) > i!B(P_i)$. A good fixed subsequence multiplier divides $i!$ eventually, contradicting this inequality. A common multiple similarly excludes a negative limiting ratio. This proves (ii).

Step 4. In this step, we obtain uniform parameter chambers and bounded denominators for their endpoint discrepancies. On integral chart vectors the values of a, J lie in fixed rational grids. Bounded angular Lipschitz constants therefore give finitely many possible linear slopes. Vertically their intercepts also lie in a fixed grid after multiplication by the bounded l ; no bound on their magnitudes is needed. Common linearity cells consequently have normals in a fixed finite list and grid offsets. Their vertices have bounded denominator. They satisfy a uniform distance-to-face estimate: for the closest point y in a nonempty polyhedron, express $x - y$ as a positive combination of independent active normals. Their finitely many possible inverses bound the sum of the coefficients by $C|x - y|$. Taking scalar products gives distance at most C times the largest violated inequality.

On a horizontal $a = 0$ face, a negative J would occur on a bounded primitive extremal generator. Its negative value is separated from zero by the grid, while C is bounded there. This contradicts $J \geq -\eta_i C$. The distance-to-face bound then proves generic klt on a uniform smaller parameter sector. Vertically, optimize first a , then D , or first a , then J , then C . The vertex denominators bound those of a', J' . If minimizers left the a -minimum at parameters tending to zero, interpolate at a first tie to get an arbitrarily small positive a -excess e . Grid spacing ensures that its cell meets the minimum face. Move distance at most Ce to that face. The objective drops by at least $e - tC'e > 0$, contradicting minimality for uniformly small t . Apply the same argument to J inside the a -minimum to get the uniform sector in (iii). Testing at an $a/e, D/e$ -minimizer gives downstairs $D' \leq C' \leq K'D'$ and $-\eta_i C' \leq J' \leq K'D'$, so the induction continues. Finally the moduli b-divisors are affine within each chamber. Their endpoints are determined by finitely many rational positive parameters on a common projective model, and are nef as limits there. Effective traces persist under this limit. $\square$

4.6. Cutoff fields and integral section counts.

Definition 4.38. Projective angular data consist of a projective Fano type variety V , an effective generalized rational lc Calabi–Yau discrepancy a integral on primitive primes, and an effective nef big rational Cartier divisor D , such that $a + D$ is effective generalized klt Calabi–Yau data. Divisor orders always mean Cartier b-pullbacks. A *floor prime* is a primitive prime T with $a(T) = 0$.

The ray model V_u in Proposition 4.34 provides these data.

Lemma 4.39. *Let H be a rational Cartier divisor on a normal projective variety V , and put $R(H) = \sum_P (1 - 1/r_P)P$, where r_P is the reduced coefficient denominator. There is a normal projective compactified Seifert bundle $p : Y \rightarrow V$, with field $\mathbb{C}(V)(z)$ and disjoint rational Cartier sections E_0, E_∞ , satisfying*

$$\text{div } z = p^*H + E_0 - E_\infty, \quad K_Y + E_0 + E_\infty = p^*(K_V + R(H)).$$

The canonical identity is read at generic primes; when $K_V + R(H)$ is $\mathbb{Q}$ -Cartier it is its usual pullback. If effective generalized klt data on V have boundary $B \geq R(H)$ and discrepancy a , the

data on Y with boundary $p^*(B - R(H))$ have, at a primitive order v with nontrivial restriction $w = e \operatorname{ord}_T$,

$$A(v) = a(w) + |v(E_0) - v(E_\infty)| + \operatorname{def}(v), \quad v(E_0) - v(E_\infty) \equiv -w(H) \pmod{\mathbb{Z}}.$$

Here e is a positive integer and the defect is a nonnegative integer, zero precisely for Gauss valuations. Orders trivial on the base have discrepancy one. If V is of Fano type and the base data have trivial total adjoint, a dimension-bounded scaling of H makes the bundle of Fano type. If H is nef and big, then $J = E_\infty + p^*H$ is nef and big, and a section of ℓJ , $\ell > 0$ integral, supplies a section of $\lfloor mH \rfloor$ for some integer $\ell \leq m \leq 2\ell$.

This construction retains both the signed phase and the actual integer degree of a section.

Proof. We construct the compactified bundle and compute its discrepancies before controlling Fano type and section degrees.

Step 1. In this step, we construct the bundle and compute its divisor identities and finite-differential discrepancies. Glue the positive and negative halves of $\bigoplus_{j \in \mathbb{Z}} \mathcal{O}_V(\lfloor jH \rfloor)z^j$, as in Theorem 4.20. A local Cartier multiple gives a finite cyclic quotient of a line chart, so the spaces are normal, the sections rational Cartier, and the compactification projective. At a prime with coefficient b/r , its reduced vertical order takes values r, b on a base uniformizer and z . This proves the first identity; a base top form wedged with $d \log z$ proves the second, including the ramification coefficient $r - 1$. The finite fibre-power map to an ordinary projective bundle proves nefness of E_∞ when H is nef. The intersection $E_\infty(p^*H)^{\dim V} = H^{\dim V} > 0$ proves bigness of J .

For a primitive v , the residue transcendence inequality makes its nontrivial base restriction $e \operatorname{ord}_T$, with e integral. In $d \log x \wedge dy_1 \wedge \cdots \wedge d \log z$ there is at most one logarithmic uniformizer pole. Its excess order is a nonnegative integer. It is zero exactly when the residue of the adjusted fibre monomial is transcendental over $\kappa(T)$, which is the Gauss condition. The canonical identity then gives the formula with both sections present. Removing them adds the sum of their orders, equal to the absolute difference because they are disjoint. The first divisor identity gives the congruence. Trivial restriction is an ordinary prime of the generic $\mathbb{P}^1$.

Step 2. In this step, we obtain Fano type after bounded scaling and identify the integral degrees supplied by sections. The ordinary pair $(V, R(H))$ is klt by nef-part negativity. Run its negative adjoint MMP on a small factorialization and apply Theorem 4.12 to the hyperstandard coefficients $1 - 1/r$. Pulling back gives an ordinary lc boundary $\Delta \geq R(H)$ with bounded index n . Let L be the least common multiple of all these bounded indices. Every coefficient denominator exceeding their bound forces coefficient one in Δ ; the smaller denominators disappear from LH . Thus $\operatorname{Supp} R(LH) \subset \lfloor \Delta \rfloor$. Mix Δ with a small amount of a log Fano boundary to obtain klt log Fano $\Theta \geq R(LH)$. On its bundle use $p^*(\Theta - R(LH)) + (1 - \rho)(E_0 + E_\infty)$. The discrepancy formula proves klt, and for small $\rho > 0$ its negative adjoint is $p^*(-K_V - \Theta - \rho LH) + 2\rho E_\infty$, which is ample. No original index is bounded in this construction.

A torus eigenvector in $H^0(Y, \ell J)$ is gz^j . Its divisor inequalities are exactly $0 \leq j \leq \ell$ and $\operatorname{div} g + (j + \ell)H \geq 0$. Put $m = j + \ell$. $\square$

Lemma 4.40. *Let (X, Δ) be a projective rational klt pair with X $\mathbb{Q}$ -factorial, $p = \dim X > 0$, and $K_X + \Delta \sim_{\mathbb{Q}} 0$. Let N be nef, big and integral Weil, with $|N|$ birational. There is $v > 0$ such that, for every $n \geq 1$,*

$$h^0(X, nN) \leq (n + p)^p v, \quad \dim \operatorname{im}(\operatorname{Sym}^n H^0(X, tN) \rightarrow H^0(X, ntN)) \geq n^p v$$

for every fixed integer $t > 1 + 2p$.

The same volume occurs in both bounds, without making a bounded multiple of N Cartier.

Proof. We construct a Taylor body, prove an interior-point section criterion, and compare both counts using its volume.

Step 1. In this step, we construct the bounded Taylor body and its positive volume. At a general smooth point outside $\text{Supp } N \cup \text{Supp } \Delta$, choose regular parameters and order Taylor exponents by total degree, then lexicographically. Let K be the closed convex hull of $\nu(s)/j$ for sections of jN , $j > 0$. Birationality gives first jets $0, e_1, \dots, e_p$, so K contains their simplex Σ . Intersecting with a fixed general complete-intersection curve through the point bounds $|\nu(s)| \leq Cj$. Hence K is bounded, with volume $v > 0$.

Step 2. In this step, we prove the interior-point section criterion by vanishing and adjunction. We prove the interior-point assertion

$$(\text{****}) \quad \alpha + \mathbf{1} \in \text{int}(tK) \implies \nu(s) = \alpha \text{ for some } s \in H^0(X, tN).$$

Choose rational $t_0 < t$ and finitely many actual section vectors whose cone contains $(t_0, \alpha + \mathbf{1})$ in its interior. Choose primitive positive integral weights u agreeing with the Taylor order on this finite data and all monomials preceding α . Positivity close to equal weights excludes higher Taylor terms. On the plane $u \cdot \beta = u \cdot (\alpha + \mathbf{1})$, take a spanning rational simplex around $\alpha + \mathbf{1}$. Clearing denominators and taking products gives sections of lt_0N with initial monomials z^{β_a} , all of weight lU , where $U = u \cdot (\alpha + \mathbf{1})$. The integer l may be increased so this particular divisor is Cartier; it is not required to be bounded.

Take a projective log resolution $f : Y \rightarrow X$ that is an isomorphism near the chosen point, and put

$$N_Y = f^*N, \quad K_Y + \Delta_Y = f^*(K_X + \Delta).$$

The supports of N_Y and Δ_Y avoid that point. Let $\pi : Z \rightarrow Y$ be the smooth weighted-blowup stack with weights u and exceptional Cartier divisor E . The coordinate differential formula gives

$$K_Z = \pi^*K_Y + \left(\sum_j u_j - 1 \right) E.$$

Set $\Delta_Z = \pi^*\Delta_Y$; this divisor is zero near E . Put $p_0 = u \cdot \alpha$,

$$P_0 = t\pi^*N_Y - p_0E, \quad H_0 = lt_0\pi^*N_Y - lUE.$$

The displayed subsystem of H_0 has base ideal $\mathfrak{b}$, whose restriction to E is generated by the initial monomials. It does not vanish at the generic point of E . Principalize it on a further smooth model $\sigma : W \rightarrow Z$, writing $\mathfrak{b}\mathcal{O}_W = \mathcal{O}_W(-G)$. Take the resolution compatible with E , with smooth strict transform E' and restriction $\tau : E' \rightarrow E$. Then G has no E' -component and $\sigma^*H_0 - G$ is globally generated. Functoriality in Theorem 3.1 permits this on the finite quotient charts. With

$$Q = P_0 - K_Z - \Delta_Z - E, \quad R = \sigma^*Q - G/l, \quad A = K_W + \lceil R \rceil,$$

the identities $K_Y + \Delta_Y \sim_{\mathbb{Q}} 0$ and $U = p_0 + \sum_j u_j$ give

$$\begin{aligned} Q &\sim_{\mathbb{Q}} t\pi^*N_Y - UE, \\ R &\sim_{\mathbb{Q}} (t - t_0)\sigma^*\pi^*N_Y + (\sigma^*H_0 - G)/l. \end{aligned}$$

Thus R is nef and big. After resolving its support, $\Theta = \lceil R \rceil - R$ is an effective simple normal crossing boundary with coefficients less than one, and $A - (K_W + \Theta) = R$. Vanishing gives $H^1(W, \mathcal{O}_W(A)) = 0$, hence surjectivity from sections of $A + E'$ to its restriction to E' . Here vanishing on the stack follows from Theorem 4.22 on its coarse space: if an integral stack coefficient is $en + s$, $0 \leq s < e$, and its fractional boundary coefficient is $\theta < 1$, the coarse boundary coefficient is $(e - 1 + \theta - s)/e \in [0, 1)$. It is klt by finite pullback; tame invariants are exact. This proves the needed vanishing for integral Weil coarse sheaves.

Near E' , the divisors P_0 and $K_Z + E$ are integral and $\Delta_Z = 0$. Rounding and adjunction therefore give

$$(A + E')|_{E'} = \tau^*(P_0|_E) + K_{E'} - \tau^*K_E - \lfloor G|_{E'}/l \rfloor.$$

Rounding commutes with restriction because the total support is simple normal crossing and distinct components of G restrict to distinct primes on E' . Since $\mathcal{O}_E(P_0|_E) = \mathcal{O}_E(p_0)$, pushing along τ identifies the restriction sheaf with

$$\mathcal{O}_E(p_0) \otimes \mathcal{J}_E((\mathfrak{b}|_E)^{1/l}).$$

On each smooth quotient chart $z_k = 1$, the projected vector $\alpha + \mathbf{1}$ is interior to the convex hull of β_a/l . Theorem 4.21 therefore puts z^α in this multiplier ideal. It lifts by the preceding surjectivity. Near E' , the nonexceptional coefficients of $A + E' - \sigma^*P_0$ are nonpositive, so the lift has precisely that initial monomial. At an original prime of X the allowed pole coefficient is $n_B + \lceil -\delta_B - g_B/l \rceil \leq n_B$, with n_B the integral coefficient of tN , $\delta_B, g_B \geq 0$. Thus it descends globally to tN by normality. This proves (****), including when exceptional crepant boundary coefficients on the resolution were negative.

Step 3. In this step, we obtain the upper section count and lower product count from the Taylor body. Distinct Taylor initials count the dimension of a section space by elimination. The unit boxes based at initials in nK are disjoint and lie in $nK + [0, 1]^p \subset (n + p)K$, giving the upper bound. For $y \in nK$, write $x = y/n$. Distribute each fractional coordinate of x among n integer vectors α_r so that $\sum \alpha_r = \lfloor y \rfloor$ and every coordinate of $\alpha_r + \mathbf{1} - x$ lies strictly between zero and two. Since $t > 1 + 2p$, all $\alpha_r + \mathbf{1}$ lie in $\text{int}(tK)$. By (****) their products realize every initial $\lfloor y \rfloor$. The unit boxes based at these initials cover nK , so there are at least $n^p v$ independent products. $\square$

Proposition 4.41. *Consider projective angular data of bounded dimension along a sequence. Let $L \geq 1$, $L_0 \geq 1$ with $L_0 = O(L)$, and finitely many rational functions f_j . Suppose their integral pole b-divisor $J(T) = \max(0, \max_j(-T(f_j)))$ satisfies $J \leq CL_0D$. Let $\mathcal{L}$ consist of the floor primes with $J = 0$, and assume (4.1) on $\mathcal{L}$. On subsequences with $L_0 \rightarrow \infty$, assume in addition that every such prime is eventually trivial on the field generated by $S(L_0D)$. Then, after a subsequence, there are relatively algebraically closed fields $K_* \subset \mathbb{C}(V)$ of fixed transcendence degree p , constants $B, C' > 0$, and numbers $v_i > 0$, such that:*

- (i) *For every fixed C_1 , the whole space $S(C_1LD)$ eventually lies in K_* ; $S(BLD)$ generates K_* . Every prime in $\mathcal{L}$ nonnegative on that space is trivial on its entire field.*
- (ii) *For every fixed integer $N \geq 1$, eventually*

$$\dim S(NLD) \leq C'v_iN^p, \quad \dim \text{Span}\{g_1 \cdots g_N : g_j \in S(BLD)\} \geq v_iN^p.$$

The constants are chosen before N .

The inclusion in (i) is simultaneous for a whole space. It thus controls every rational expression in a varying finite basis.

Proof. We use cutoff fields to compare the section spaces and their product growth.

Step 1. We first construct cutoff fields and their bounded generating systems. Extract all exceptional floor primes with $D(T) \leq 1$, using the klt data $a + D/2$ and ordinary extraction as in Lemma 4.36. On the factorial Fano type model U , D_U is supported on floor primes with coefficients at most one. For $0 < \beta \leq 1$ put

$$I_\beta = \{T : a(T) = 0, D(T) \leq \beta\}, \quad P_\beta = D_U - \sum_{T \in I_\beta} D(T)T.$$

Its divisor MMP ends with $P'_\beta = p^*D_W$, D_W effective ample, and connected fibres. Indeed a vertical principal difference from a semiample pullback comes from the base function field and can be absorbed into D_W . Negativity gives $0 \leq P \leq P_\beta^* \leq D$ for its nef b-pullback. Every removed prime survives: its coefficient in P_β is zero, so it cannot carry a positive contraction error against the effective transform. The fields $K_\beta = \mathbb{C}(W)$ are relatively algebraically closed, nested decreasing, with $K_{0+} = \mathbb{C}(V)$, $K_1 = \mathbb{C}$, and at most $\dim V$ strict drops. Equal transcendence degree would force equality.

Step 2. In this step, we bound the degree of field-generating sections at a cutoff. Set $d = (D - P)/\beta + P$. The data $a + td$ are effective and klt for $0 < t \leq 1$; their total base class is $t(1/\beta - 1)D_W$. Lemma 4.37 gives $a' + td'$ with bounded denominator for a' . If $a'(E_i) = 0, d'(E_i) \rightarrow 0$, the bounded approximation (***) along the finite Mori chain supplies original primitive lifts with $a(T_i) = 0, d(T_i) \rightarrow 0$, and with $e_i k_i \rightarrow 0$ for any prescribed $k_i \rightarrow 0$. To see the last control, fix an approximation error first; its multiplier is bounded. Then decrease the error slowly, also requiring that multiplier times k_i tend to zero. Quantization forces the resulting nonnegative a -values to vanish. Primitive restriction only decreases the error and the multiplier.

There cannot therefore be primes with both $(a' + td')(E_i) \rightarrow 0$ and $0 < D_W(E_i)/\beta \rightarrow 0$. Such lifts would satisfy $D(T_i)/\beta \leq d(T_i) + e_i D_W(E_i)/\beta \rightarrow 0$, hence belong to I_β , where $P(T_i) = 0$, although $P(T_i) = e_i D_W(E_i) > 0$. Original primes with positive D_W/β tending to zero are excluded in the same way by a covering prime: it has $a' = 0, d' \leq D_W \leq 1$. Consequently a small-discrepancy extraction on W is ordinarily δ -lc, and the nonzero coefficients of the pullback $E_0 = D_W/\beta$ are at least $\delta > 0$, after a subsequence.

Choose a fixed large m , and round mE_0 down with errors in $[1, 2]$, obtaining an integral big Weil divisor $N = mE_0 - Q$. Run its MMP. If e_E is the nef trace-pullback error, negativity gives $me_E \leq (Q^+)^* - Q^*$. The effective boundary of a fixed small member $a' + td'$ has coefficient at least $1/2$ on E_0 . Thus on the nef model

$$A_{\text{ordinary}}^+ \geq A + (1/4 - t/m)(Q^+)^*.$$

A small-discrepancy prime on the support has $(Q^+)^* \geq 1$, even if contracted. One outside it cannot be contracted, since its coefficient before the MMP is zero. This proves a uniform ordinary gap on the nef model. Apply Theorem 4.19 to its nef integral Weil divisor, whose difference from K is pseudo-effective. Pullback gives a field-generating space $S(CD_W/\beta)$.

Choose independent ϕ_j in this space and define $H(E) = a'(E) - 1 - \text{ord}_E(d\phi_1 \wedge \cdots \wedge d\phi_p)$. It is the nef moduli b-order after a rational principal adjustment and has bounded denominator. Its pullback satisfies on floor primes

$$H(T) \leq CD(T)/\beta; \quad H(T) \leq -1 \text{ if } T|_{K_\beta} \neq 0 \text{ and all } T(\phi_j) \geq 0.$$

If $D(Q) = 0, a(Q) \leq 1$, the same differential estimate gives $H(Q) \leq 0$. To obtain the reverse bound under (4.1), take an effective representative of $H + \lambda D_W/\beta$. Its rational principal difference has poles bounded by CD/β . Apply the maximal-order inequality to that difference and let $\lambda \downarrow 0$. The result is $H(T) \geq -C\eta_i D(T)/\beta$. In particular all I_β -primes satisfying that inequality are eventually trivial on K_β .

Step 3. In this step, we choose a cutoff plateau and prove exact triviality on its whole field. The pole divisor J is integral and globally generated on a principalization. Thus $LD(T)$ sufficiently small forces $J(T) = 0$ on a floor prime. Choose a plateau

$$K_{\beta_0} = K_{\beta_1} = K_*, \quad L\beta_0 \rightarrow 0, \quad 0 < c' \leq L\beta_1 \leq c,$$

by diagonalizing the finitely many scaled field-drop locations. Primes in I_{β_1} are common and hence trivial on K_* . Therefore $S(mP_{\beta_0}) = S(mP_{\beta_1})$ for every rational $m > 0$. Integral pole orders show that every element of $S(C_1LD)$ has no pole on I_{β_0} eventually. This proves the whole-space inclusion. The generating bound at β_1 gives a fixed generating space $S(B_0LD)$.

If no fixed enlargement forced exact triviality, choose a common prime nonnegative on $S(B_iLD)$, $B_i \rightarrow \infty$, but nontrivial on K_* . At the first later field drop making it trivial, use the preceding quantized H from just before the drop. Its generating functions have cost $O(L)$, so $H(T) \leq -1$. If $L_0\beta \rightarrow \infty$, that field is already generated inside $S(L_0D)$, a contradiction. Otherwise $L_0\beta$ is bounded. On the geometric generic fibre after the drop apply the extra-sector part of Lemma 4.37 to $a + \kappa J$, with $C = D/\beta$ and direction H . The trace inequalities just established give effectivity on fixed small ranges, bounded denominators, and $-C_1\eta_i D/\beta \leq H \leq C_1D/\beta$ on $a = J = 0$. The nonfloor traces have $D = J = 0, H \leq 0$. The no-negative-horizontal conclusion contradicts $H(T) < 0$. This proves exact triviality, on the whole field at one index.

Step 4. In this step, we obtain the upper section count and lower product count from one Taylor volume. On the cutoff extraction the positive coefficients of LD_W have a fixed lower bound. For a fixed large M , $N = \lfloor MLD_W \rfloor$ therefore dominates both LD_W and the generating divisor CD_W/β_1 . Its system is birational. Make N nef by its divisor MMP. For individual rational functions, negativity identifies the section spaces on both sides of every MMP step, even at rational degrees. Consequently, for each fixed N_1 , $S(N_1LD) \subset S(N_1N)$ eventually, and sections of jN pull back into $S(jMLD)$. Choose an ordinary effective klt Calabi–Yau boundary on the Fano type model and apply Lemma 4.40. Its single Taylor volume gives the upper bound with $C' = (p+1)^p$ and the lower product bound after enlarging B to tM . For $p = 0$, take $v_i = 1$. Triviality extends to rational quasi-monomial subcones by resolving any one additional function and using linearity and density of rational divisorial rays. $\square$

4.7. Phases and sections in a prescribed character.

Lemma 4.42. *Let V_i be projective varieties of Fano type of bounded dimension and H_i nef and big rational Cartier divisors. Suppose that, for rational $0 \leq t \leq L_i$, where $L_i \rightarrow \infty$, there are effective generalized pairs*

$$B_i(t) \geq R(H_i), \quad K_{V_i} + B_i(t) + M_i(t)_{V_i} \sim_{\mathbb{Q}} tH_i,$$

whose rational Cartier nef parts are nef over the point. Write $a_i(t)$ for their discrepancies. Suppose $a_i(0)$ is klt and $a_i(t, T)$ is nondecreasing in t for each divisorial order T . Assume that, for every sequence of rational $t_i \rightarrow \infty$ in these intervals and every subsequence, a further subsequence admits a fixed positive integer q with the following property: whenever $a_i(t_i, T_i) \rightarrow 0$ for primitive T_i , the phases of $qH_i(T_i)$ have real representatives satisfying

$$-o(a_i(t_i, T_i)) \leq \theta_i(T_i) \leq O(a_i(t_i, T_i)).$$

Then a bounded positive multiple of H_i has a section. The multiples may be chosen to have one common degree.

Two auxiliary assertions hold in bounded dimension. If $(F_i, R(G_i))$ is klt, G_i is ample, and $-K_{F_i} - R(G_i) \sim_{\mathbb{Q}} I_i G_i$ with $I_i \rightarrow \infty$, then, on the compactified Seifert cone C_i with infinity divisor D_i , it is impossible that $v(D_i) = o(A_{C_i}(v))$ for every primitive sequence with $A_{C_i}(v) \rightarrow 0$. If $P_i \sim_{\mathbb{Q}} 0$ on Fano type varieties with effective generalized klt Calabi–Yau data dominating $R(P_i)$, the same signed phase hypothesis bounds the least positive integer making P_i principal.

Proof. All orders in this proof are orders of the actual Cartier b-pullbacks.

Step 1. We first establish the two auxiliary assertions. Contract the zero section of the Seifert bundle of G_i . The resulting cone is klt and

$$-K_{C_i} \sim_{\mathbb{Q}} (I_i + 1)D_i.$$

Its generalized Calabi–Yau discrepancy family is $A_{C_i} - tD_i$, with nef part $(I_i + 1 - t)D_i$. If the asserted little-oh estimate held, the long-interval part of Lemma 4.37, over a point, would make each fixed member klt. The member at $t = 1$ has discrepancy zero at D_i , a contradiction.

For the second assertion, let r_i be the least principalizing integer and take the root cover $z^{r_i} = \phi_i$, where $\text{div } \phi_i = r_i P_i$. It has degree r_i : a smaller order e of its cyclic action would put z^e in the ground function field and make eP_i principal. The round boundary condition makes the pulled-back generalized pair effective and klt. Take the diagonal quotient

$$(\tilde{F}_i \times \mathbb{P}_y^1) / \mu_{r_i} \longrightarrow \mathbb{P}_{y^{r_i}}^1,$$

using inverse actions on the two factors. This is a quasi-etale Fano type quotient. Above zero, a primitive order restricts integrally to the original field of F_i ; with $u = v(y^{r_i})/r_i > 0$, the Seifert formula gives

$$A(v) \geq a(w) + u, \quad u \equiv -w(P_i) \pmod{\mathbb{Z}}.$$

After the fixed scaling allowed in the hypothesis, its lower phase bound implies $u = o(A(v))$ whenever $A(v) \rightarrow 0$. Apply long-interval comparison to $A - tf^*[0]/r_i$. The initial base discrepancy

at zero is at most $1/r_i$, whereas the discrepancy of the member at $t = 1$ is nonpositive. The comparison forbids $r_i \rightarrow \infty$. This proves the second assertion.

Step 2. We prove nonvanishing by induction on dimension. Suppose, to the contrary, that no mH_i , $1 \leq m \leq N_i$, has a section, where $N_i \rightarrow \infty$. The data at zero permit the dimension-bounded scaling in Lemma 4.39, so that the compactified bundles used below are of Fano type. Choose $k_i \rightarrow \infty$ sufficiently slowly, apply the phase assumption at k_i , and absorb both fixed scalings into H_i , reparametrizing the family. All subsequent choices of growing parameters are slower than N_i . For a small fixed $\eta > 0$, extract the primes with $a_i(k_i) < \eta$ and positive small phase $f_T(H_i)$, including those already on the model. Put

$$\Theta_i = \sum_T f_T(H_i)T, \quad P_i = H_i^* - \Theta_i.$$

The phase hypothesis bounds the coefficients of Θ_i by $C\eta$. We claim that P_i is eventually not big.

Step 3. In this step, we exclude the big and non-pseudo-effective alternatives for P_i . Suppose it is big. On the Seifert bundle use the data at k_i , omitting its zero and infinity sections, and denote its discrepancy by A_i . Extract the primes with $A_i < \delta$, where $\delta > 0$ is fixed and small, and let S_i be their reduced sum, including the original ones. Such orders are Gauss orders: the non-Gauss defect is at least one. Their restrictions to the base are nonzero. The divisor $J_i = E_{\infty,i} + p^*H_i$ has integral trace. Let τ_i be the pseudo-effective threshold of $J_i - \tau_i S_i$. We have

$$S_i \neq 0, \quad k_i \tau_i \rightarrow 0.$$

Indeed, otherwise choose $k_i, \tau_i^{-1} \ll m_i \ll N_i$, round $m_i J_i$ with errors between one and two along S_i , and run the MMP of this big integral divisor. Write $e_H, e_\infty \geq 0$ for the nef pullback errors. Negativity gives $m_i(e_H + e_\infty) \leq (Q_i^+)^* - Q_i^*$, with $Q_i^+ \leq 2S_i^+$. Hence the ordinary discrepancy on the resulting model satisfies

$$A_{\text{ord}}^+ \geq A_i + (1 - \delta)(S_i^+)^* - (k_i + 1)e_H \geq A_i + \frac{1}{2}(S_i^+)^*.$$

A contracted small-discrepancy prime has $Q_i^* \geq 1$ and therefore $(S_i^+)^* \geq 1/2$. All other primes either have $A_i \geq \delta$ or occur on this support. This gives a uniform ordinary gap. Integral effective birationality and Lemma 4.39 give a section of $O(m_i)H_i$, contrary to the choice of N_i .

The threshold divisor has a semiample nonbig model over some Z_i . For $0 < c < 1 - \delta$, subtract cS_i^+ from its generalized boundary and use nef part

$$M_i(k_i) + (2 + c/\tau_i)E_{\infty,i}^+ + (c/\tau_i - k_i - 1)p^*H_i^+.$$

Both displayed coefficients are nonnegative. The resulting discrepancy is

$$A_i^\# = A_i + c((S_i^+)^* - e_J/\tau_i) \geq A_i + cS_i^*,$$

so this is uniformly generalized klt. A geometric generic fibre is rationally connected and is bounded in codimension one by Theorem 4.18. On that bounded fibre, S_i^+ has bounded degree and $J_i^+ \sim_{\mathbb{Q}} \tau_i S_i^+$; thus both $p^*H_i^+$ and $E_{\infty,i}^+$ have degree $O(\tau_i)$.

Bigness of P_i supplies an effective representative $\Sigma_i \sim_{\mathbb{Q}} H_i$ with $\Sigma_i(w) > v(E_\infty) - v(E_0)$ at every member of S_i . When the right side is positive, it is the integral restriction multiplicity times the small positive phase used in Θ_i . A normalized homogeneous section of $J_i - \tau_i S_i$ has divisor

$$(1 - b)E_\infty + bE_0 + p^*\Xi, \quad 0 \leq b \leq 1, \quad \Xi \sim_{\mathbb{Q}} (1 + b)H_i.$$

If $b < 1$, adding a small multiple of $E_0 - E_\infty + p^*\Sigma_i$ increases its orders at every member of S_i , contradicting the threshold. Thus $b = 1$ for all such sections. The threshold field is contained in $K(V_i)$, and E_0 is horizontal over Z_i .

On the bounded fibre, E_0 is an lc place of the zero-parameter pair with both sections included. Use $A_i^\#$ as the klt comparison data. The scaled chart estimate in Lemma 4.36 gives Lipschitz constant $O(\tau_i)$ for E_0 , and a bounded constant for A_i : the difference $A_i^\# - A_i$ is the pullback

of $(c/\tau_i)(J_i - (J_i^+ - \tau_i S_i^+)^*)$. Normalize the order of E_0 by A_i . Its chart norm tends to infinity. Horizontal integral approximation from Lemma 4.37 then gives primitive orders n_i with

$$A_i(n_i) = O(s_i) \rightarrow 0, \quad E_0(n_i) \geq s_i/2.$$

The lower signed phase bound instead gives $E_0(n_i) = o(A_i(n_i))$. This contradiction proves the claim that P_i is not big.

Now let $\eta_i \downarrow 0$ slowly and run the MMP of P_i . On the resulting model, negativity gives

$$0 \leq \Theta_i^* \leq E_i := H_i^* - (P_i^+)^* \leq (\Theta_i^+)^*, \quad \Theta_i^+ \leq O(\eta_i)T_i^+,$$

where T_i^+ is reduced. Choose further parameters tending to infinity but at most k_i and $o(\eta_i^{-1})$. If P_i is not pseudo-effective, take a Mori fibre and put $G_i = -P_i^+|_{F_i}$. The divisor P_i^+ is integral along T_i^+ . The round boundary comparison gives

$$A_{F_i, R(G_i)} \geq a_i(s_i) + \frac{1}{2}(T_i^+)^*, \quad E_i \leq O(\eta_i)A_{F_i, R(G_i)}.$$

Relative Picard rank one and $G_i + H_i^+ = \Theta_i^+$ give $-K_{F_i} - R(G_i) \sim_{\mathbb{Q}} I_i G_i$, $I_i \rightarrow \infty$. Here numerical equivalence becomes rational linear equivalence by the Fano type Mori dream property. The cone height is congruent to $-w(H_i) + E_i(w)$. At a small-discrepancy prime the phase hypothesis and the last estimate make that height little-oh of the cone discrepancy. The first auxiliary assertion excludes this case.

Step 4. In this step, we descend the semiample nonbig case to a base of lower dimension. It remains that P_i^+ is semiample and nonbig, giving $g_i : U_i^+ \rightarrow W_i$. The data

$$\tilde{B}_i(t) = B_i(t/2)^+ - t\Theta_i^+, \quad \tilde{M}_i(t) = M_i(t/2) + (t/2)H_i^+$$

have total class tP_i^+ , dominate $R(P_i^+)$, and have nondecreasing klt discrepancies $\tilde{a}_i(t) = a_i(t/2) + tE_i$. At their small-discrepancy primes, $E_i = o(\tilde{a}_i)$, so $P_i^+ = H_i^+ - E_i$ retains the signed phase property. On the geometric generic fibre, $P_i^+ \sim_{\mathbb{Q}} 0$; the second auxiliary assertion supplies a bounded principalizing integer. Triviality descends to the original generic field, since a line bundle and its inverse with a nonzero geometric section descend by flat base change. The semiample base class is represented by a rational $\mathbb{Q}$ -Cartier divisor H_{W_i} ; after a rational principal adjustment, $qP_i^+ = g_i^*H_{W_i}$.

The base is Fano type by Lemma 4.14. The canonical bundle formula gives effective generalized data of total class tH_{W_i} . They dominate $R(H_{W_i})$: if r_P is a base denominator, the fibre multiplicity e and source denominator r give $r_P \mid er$, while the threshold is at most $1/(er)$. For every growing parameter sequence t_i , the discrepancy $\tilde{a}_i(t_i) = a_i(t_i/2) + t_iE_i$ is klt on the whole source: $a_i(t_i/2) \geq a_i(0) > 0$ and $E_i \geq 0$. The rational $\mathbb{Q}$ -Cartier base divisor H_{W_i} has pullback qP_i^+ , whose signed phases satisfy the source hypothesis after the fixed multiples already chosen. Thus Lemma 4.37(ii) applies to these klt data and gives the required signed phases on the base after one further fixed multiple. Its dimension has decreased. Induction gives a section on W_i , and pullback gives one for a bounded multiple of H_i , since $H_i^* \geq (P_i^+)^*$. A point base is immediate. The zero-dimensional initial case is also immediate. Finally, take a common multiple of the finitely many possible degrees, and absorb discarded finite indices. $\square$

Definition 4.43. For rational divisors A, B on a projective variety, write $A \leq_{\text{eff}} B$ if $B - A$ is rationally linearly equivalent to an effective rational divisor. The notation $B \geq_{\text{eff}} A$ denotes the same relation. Inequalities $A(T) \leq B(T)$ between Cartier b-orders mean inequalities of their actual pullback orders at the specified valuations.

Lemma 4.44. Let (V_i, a_i, D_i) be projective data as in Definition 4.38, and let $L_i \geq 1$. Suppose (4.1) holds along every sequence of floor primes T_i with $L_i D_i(T_i) \rightarrow 0$. Let ℓ_i be rational principal orders and $G_i \geq 0$, $G_i \sim_{\mathbb{Q}} c_i L_i D_i$, $0 \leq c_i \leq C$. Suppose that for each nonfloor prime Q on V_i an integer p_Q satisfies

$$|p_Q - \ell_i(Q)| \leq G_i(Q).$$

Then a fixed positive integer q satisfies

$$\text{dist}(q\ell_i(T_i), \mathbb{Z}) = O(L_i D_i(T_i))$$

on all the indicated floor sequences, after every subsequence.

Proof. We give the scale argument, including the primes at an endpoint of a cutoff interval.

Step 1. In this step, we choose a failure sequence and the first cutoff with maximal field transcendence degree. Suppose the assertion fails. Diagonalize against factorial multiples. Among failing floor sequences choose one for which the cutoff field K_β , $\beta = D(T)$, has largest transcendence degree. The order T is horizontal over that field, by the maximal-order argument in Proposition 4.41. Let τ be the first cutoff having that same field. Then $L\tau \rightarrow 0$, and maximality, after a fixed scaling, has already proved the assertion for $D(Q) < \tau$.

Step 2. We first prove it at $D(Q) = \tau$. If a prime S there still fails, work on the cutoff extraction and set

$$\Pi = S + \sum_{Q \neq S} \gamma \min(1, D(Q)/\tau) Q, \quad 0 < \gamma < 1/4.$$

The threshold t of $D - t\Pi$ satisfies $\tau \leq t \leq (1 + \eta_i)\tau$, by maximality of the order at S . Run the threshold MMP to its semiample nonbig model. On its generic fibre,

$$D^+ \sim_{\mathbb{Q}} t\Pi^+, \quad \sum_{Q \neq S} D(Q)Q^+ \leq_{\text{eff}} 2\eta_i \tau S^+.$$

Here the second comparison is in the sense of Definition 4.43. Indeed the S^+ -coefficient of $D^+ - t\Pi^+$ is at least $-\eta_i \tau$, and every other surviving coefficient is at least $D(Q)/2$, since $t \leq (1 + \eta_i)\tau$ and $\gamma < 1/4$. Thus

$$(D^+ - t\Pi^+) + \eta_i \tau S^+ - \frac{1}{2} \sum_{Q \neq S} D(Q)Q^+ \geq 0$$

coefficientwise. Restricting its rationally trivial threshold term to the generic fibre proves the stated class comparison. Since D^+ restricts to a big divisor, S^+ is horizontal and big on that fibre. Independently, nef-part and MMP negativity give the actual Cartier b-order comparison, on a common determination,

$$0 \leq e_D := (D^+)^* - D \leq t((\Pi^+)^* - \Pi^*).$$

This last inequality remains valid through subsequent MMP steps over the threshold base. Choose signed integers m_i growing slowly so that

$$\sigma_i = \{m_i \ell_i(S)\} \rightarrow 0^+, \quad \eta_i + |m_i| L_i \tau_i = o(\sigma_i).$$

For an irrational limiting phase, take successively better fixed powers and diagonalize. For a rational limit, clear its denominator and take positive residual powers: failure means that the residual divided by $L_i \tau_i$ is unbounded, so the same inequalities follow. This also explains why both signs of m_i are allowed.

Round $m_i \ell_i^+$ to an integral divisor $N_i = m_i \ell_i^+ + n_i$. At S set $n_i(S) = -\sigma_i$. At floor primes below τ , use nearest integral rounding with error at most $C|m_i|LD(Q)$. At the remaining floor primes round upwards, with error in $[0, 1)$. At nonfloor primes use the given integers, so the error is at most $|m_i|G_i$. The positive error on the threshold fibre has class $O(|m_i|L_i \tau_i + \eta_i)S^+$. Thus N_i is not pseudo-effective. On a Mori fibre its negative H_i is integral and ample, with

$$H_i \leq_{\text{eff}} (\sigma_i + o(\sigma_i))S^+, \quad -K_{F_i} \geq_{\text{eff}} S^+.$$

Hence $-K_{F_i} \sim_{\mathbb{Q}} I_i H_i$, $I_i \rightarrow \infty$.

At an ordinary prime with discrepancy tending to zero, the reduced support comparison gives $a_i(w) \rightarrow 0$ and $(\Pi^+)^*(w) \leq A_{F_i}(w)$. Its original restriction is primitive on the cutoff model; it is neither S nor a floor prime above τ . Thus the assertion already proved below τ applies, and

$$D(w) \leq C\tau A_{F_i}(w), \quad m_i \ell_i(w) \equiv o(A_{F_i}(w)) \pmod{\mathbb{Z}}.$$

For the transformed effective divisor, $(G_i^+)^* = G_i + c_i L_i e_D$; its contribution after multiplication by $|m_i|$ is little-oh as well. The negative rounding error is bounded by $\sigma_i S^* + C|m_i|L_i D + |m_i|G_i$; upward unit errors are carried by the reduced boundary. Consequently

$$-o(A_{F_i}(w)) \leq n_i^*(w) \leq O(A_{F_i}(w)).$$

On the cone the positive height is congruent to $-m_i \ell_i(w) - n_i^*(w)$, and is little-oh of its discrepancy at every small-discrepancy primitive prime. The first auxiliary assertion of Lemma 4.42 is a contradiction. This proves the endpoint case.

Step 3. In this step, we prove rational triviality on the generic fibre and bound a principalizing multiple. Now round ℓ_i , on the generic fibre of the cutoff, to an integral divisor $N_i = \ell_i + n_i$, with $|n_i| \leq CL_i D_i^+ + G_i^+$. If either sign of N_i is not pseudo-effective, its Mori fibre has integral ample negative of class at most $O(L_i \tau_i) R_i$, where R_i is the reduced cutoff support, and $-K_{F_i} \geq_{\text{eff}} R_i$. The preceding cone argument applies again: at small ordinary discrepancy, $D(w) \leq \tau_i A_{F_i}(w)$, and the endpoint result makes both the phase and rounding error little-oh. Thus both signs are pseudo-effective. On a Fano type variety the effective cone is rational polyhedral; effective representatives of both signs must sum to zero. Therefore $N_i \sim_{\mathbb{Q}} 0$.

Make the reduced support nef by its MMP and use klt Calabi–Yau discrepancy $a_i + c_0 R_i^*$, for a fixed small $c_0 > 0$. Its small primes satisfy $D(w) \leq \tau_i(a_i + c_0 R_i^*)(w)/c_0$. The endpoint phase estimate gives the signed hypothesis for N_i . The second auxiliary assertion of Lemma 4.42 therefore bounds its principalizing integer q . Triviality descends to the original generic field as in that lemma. Subtract the resulting principal function from $q\ell_i$, and call the difference h_i . Its horizontal trace satisfies

$$|h_i^+| \leq CL_i D_i^+ + qG_i^+,$$

but the chosen failing prime has $|h_i(T_i)|/(L_i D_i(T_i)) \rightarrow \infty$.

Step 4. In this step, we contradict the remaining phase failure with a two-sided discrepancy family. On the cutoff generic fibre start with $a_i^0 = a_i + c_0 D_i/\tau_i$ and consider the concave two-sided family

$$a_i^s = a_i^0 + \frac{sh_i - q|s|G_i}{J_i L_i \tau_i}.$$

Choose $J_i \rightarrow \infty$ more slowly than the last unbounded ratio, and allow $|s| \leq R_i' \rightarrow \infty$ with $R_i'/J_i \rightarrow 0$. The trace inequalities make this family effective, with nef part having positive D_i -coefficient $(c_0 - q|s|c_i/J_i)/\tau_i$. A primitive prime with $a_i^0 < \min(1, c_0)/2$ is an original floor prime below $\tau_i/2$. At all these primes the endpoint result gives $|h_i| + qG_i = O(L_i D_i)$, so $a_i^s/a_i^0 \rightarrow 1$ for each fixed s . Strong signed comparison over a point in Lemma 4.37 makes both fixed members $s = \pm 1$ klt. At T_i , one of these members is negative by the choice of J_i . This final contradiction proves the lemma, with its asserted subsequence uniformity. $\square$

Proposition 4.45. *Use the actual lattice and angular data of Proposition 4.34. Let $L_i \geq 1$. The following assertions are uniform along sequences and their subsequences.*

- (1) *For an invariant eigenfunction $\xi = fz^k$, set $J_m = \text{div}(f) + m\gamma + L_{k+mu}$. Suppose $0 \leq m_0 < m$, $m - m_0 \geq cL$, and both $k + m_0 u, k + mu$ are interior weights. If every floor sequence P with $LD(P) \rightarrow 0$ admits integers j_P such that*

$$j_P \leq J_{m_0}(P) \leq J_m(P) \leq j_P + C_1 LD(P),$$

then $S(qJ_m) \neq 0$ for a bounded positive integer q . Equivalently, some $g\xi^q h^{qm}$ is regular.

- (2) *Suppose the maximal-order bound (4.1) holds on those floor sequences. With the width of Definition 4.35, every $k \in M$ satisfying $\|k\| \leq C_0 L$ admits an invariant eigenfunction $Y = fz^{qk}$ of cost at most $C_2 L$, where q is bounded and positive. Both signs of k admit a common q and common cost. These bounds are uniform at all enlarged scales.*

- (3) Under the preceding hypothesis, suppose $\dim S(nLD) \leq C_* v_i n^p$ eventually for every fixed positive integer n . The space $W_{t,k}$ of regular elements of full character χ^t and fine weight $tu + k$ then satisfies

$$\dim W_{t,k} \leq C_3 v_i n^p$$

uniformly for $0 \leq t \leq C_0 nL$ and $\|k\| \leq C_0 nL$. The constant C_3 is independent of n, t, k .

- (4) For this last assertion no maximal-order condition is needed. Suppose there are a bounded number of opposite invariant pairs $Y_j^\pm$, of weights $\pm k_j$ and costs $m_j \leq AL_j$, where $1 \leq L_j \leq L$. Let u_j be bounded real numbers. Suppose, on every nonzero pure test and every Gauss floor test with $LD \rightarrow 0$, their values are opposite and

$$\sum_j u_j v(Y_j^+) / L_j \leq \delta v(h).$$

For fixed sufficiently large M_0 relative to $1/\delta$, and any bounded rounding errors in $r_j = M_0 u_j L / L_j + O(1)$, there is a regular symbol of full character χ^{qm} and weight

$$q \left(mu - \sum_j r_j k_j \right), \quad 0 < m \leq C_4 M_0 \delta L, \quad 1 \leq q \leq Q.$$

Here C_4 is independent of M_0, δ ; Q may depend on these fixed choices.

Proof. We first obtain sections from the order sandwich and then use them to supply and count the prescribed characters.

Step 1. In this step, we regularize the order sandwich to obtain a section of a bounded multiple. If the neutral field is $\mathbb{C}$, the section dictionary proves the first assertion directly. Otherwise J_m is an ample rational divisor on its ray model. Extract the primes of $a + D$ below a fixed small cutoff. There are finitely many: on a log resolution their positive discrepancies bound the integral monomial weights of every discrepancy below one. The ordinary extraction is obtained by mixing $a + D$ with a small ample generalized klt Calabi–Yau perturbation and replacing its ample nef part by general members. The resulting big ordinary klt boundary keeps the extraction of Fano type. Every extracted prime is a floor prime, since a is nonnegative and integral. On this model the positive coefficients of D occur precisely on floor primes and are at most one.

Put $\rho_P = \min(1, LD(P))$. Choose $c_1 > 0$ with $2c_1 \leq c$, and a correction R_c supported on these primes such that

$$c_1 \rho_P \leq R_c(P) \leq 2c_1 \rho_P, \quad \text{den}(J_m(P) - R_c(P)) \leq C / \rho_P.$$

For example, choose $N_P \leq C / \rho_P$ with $1/N_P \leq c_1 \rho_P$ and a grid point of $N_P^{-1} \mathbb{Z}$ in $[J_m(P) - 2c_1 \rho_P, J_m(P) - c_1 \rho_P]$. Off the floor, J_m is already integral. Minima of sums give $J_m - J_{m_0} \geq (m - m_0)D$. Thus $P = J_m - R_c$ is big and dominates J_{m_0} . Run its MMP to a nef big model P^+ . On a common determination negativity gives

$$J_m \geq (P^+)^* \geq J_{m_0}, \quad R_c^* \leq E := J_m - (P^+)^* \leq (R_c^+)^*, \quad E \geq c_1 D.$$

For the middle comparison, sections of divisible basepoint-free multiples of J_{m_0} remain sections of P^+ ; one avoiding each chosen centre tests the required order inequality.

Choose fixed small $0 < \delta_1 \leq c_1 \delta_2$. For $0 \leq t \leq \delta_2$ take discrepancy

$$a_t = a + (1 - t/\delta_2)\delta_1 D + tE$$

and add nef part $(1 - t/\delta_2)\delta_1 D + tJ_m$ to the original one. The total class is tP^+ . For $t \geq \delta_2$ freeze the discrepancy and add $(t - \delta_2)P^+$ to the nef part. These discrepancies are klt and nondecreasing. On a surviving floor prime, the grid denominator bound and sufficiently small δ_2 give $a_t(P) \leq 1/\text{den}(P^+(P))$. At a nonfloor prime $a = 1, D = E = 0$, and P^+ is integral. Thus the boundary dominates $R(P^+)$.

At growing parameters, $a_t(T) \rightarrow 0$ forces $a(T) = 0, E(T) \rightarrow 0$. Such a primitive prime belongs to the original extracted list, even if subsequently contracted. There $E(T) \geq c_1 \min(1, LD(T))$,

so $LD(T) \rightarrow 0$ and $LD(T) = O(E(T))$. The sandwich now puts $(P^+)^*(T)$ between j_T and $j_T + O(a_t(T))$. Apply Lemma 4.42. Its section of a bounded multiple of P^+ lies in $S(qJ_m)$, proving the first assertion.

Step 2. In this step, we supply invariant eigenfunctions for every bounded-width lattice character. Choose $m_1 = O(L)$, $m_1 > C_0 L$. The two ample ray divisors at $\pm k + m_1 u$ have sections in some divisible degree. Their rational principal orders, denoted $\text{div}(g_\pm)$, satisfy for all angular Gauss tests

$$(P, y)(z^{\pm k} h^{m_1}) + P(g_\pm) \geq 0.$$

These are rational orders only; no root is asserted to belong to K . Put $\ell = -\text{div}(g_+)$ and $G = 2m_1 D + \text{div}(g_+ g_-)$. Adding the inequalities at a height-minimizing vector gives $G \geq 0$. At an actual nonfloor prime the minimizing vector is integral and has zero h -height, so

$$0 \leq \langle y, k \rangle - \ell(P) \leq G(P), \quad \langle y, k \rangle \in \mathbb{Z}.$$

Lemma 4.44 provides a fixed q_0 and integers j_P with $|q_0 \ell(P) - j_P| \leq ALD(P)$ on tiny floors.

Apply the first assertion to $\xi = z^{q_0 k}$. Choose $m_0 = O(L)$ with $m_0 - q_0 m_1 \geq AL$ and put $m = 2m_0$. The positive and negative Gauss inequalities give

$$J_{m_0} \geq q_0 \ell + (m_0 - q_0 m_1)D, \quad J_m \leq q_0 \ell + (m - q_0 m_1)D + q_0 G.$$

Maximality applied to $G/(2m_1)$ makes these the required sandwich. We obtain $Y = gz^{qk}$ of cost $O(L)$. Applying the same argument to $-k$ and taking bounded powers aligns both multipliers and costs. An exceptional sequence would contradict this proof with its own k_i and L_i ; the constants are therefore uniform in the weight. Tiny floors for $L'_i \geq L_i$ are tiny for L_i , and the identical exceptional-sequence argument gives uniformity at enlarged scales.

Step 3. In this step, we bound the dimension of a space with fixed full character and fine weight. For $F \in W_{t,k}$, F/h^t is invariant. At scale nL supply an opposite eigenfunction Y of weight $-qk$ and cost $b \leq CnL$. Then

$$\Phi(F) = F^q Y / h^{qt} \in S((qt + b)D) \subset S(C'nLD).$$

Although Φ is not linear, its derivative at a nonzero F is multiplication by the nonzero function $qF^{q-1}Y/h^{qt}$, hence is injective. It takes values in the same target, by expanding $\Phi(F + zV)$. Thus $\dim W_{t,k} \leq \dim S(C'nLD)$, which proves the third assertion with a constant independent of n, t, k . Characteristic zero is used in this derivative.

Step 4. In this step, we supply the tilted character without assuming the maximal-order bound. Set $\xi = \prod_j (Y_j^+)^{-r_j}$. The opposite values and the regularity of both signs give $|y_j| \leq m_j v(h)$ on every specified test. Consequently

$$\sum_j r_j y_j \leq (M_0 \delta + C_7) L v(h).$$

Choose $M_0 \delta \geq 2C_7 + 2$, an integer m_0 strictly larger than the coefficient on the right and comparable to $M_0 \delta L$, and $m = 2m_0$. The pure inequalities put both weights in the interior. On tiny floor tests $J_{m_0} \geq 0$. At a height-minimizing angular vector the absolute value of the sum is at most $C_8(M_0 + 1)LD(P)$. Hence

$$0 \leq J_{m_0}(P) \leq J_m(P) \leq C_9(M_0, \delta)LD(P).$$

This is the first assertion with integer $j_P = 0$. It supplies $g\xi^q h^{qm}$ with exactly the stated full character and weight. The cost coefficient is independent of M_0, δ , as required. $\square$

4.8. Finite products, rational lines, and the next scale.

Lemma 4.46. *Let V be a finite-dimensional $\mathbb{C}$ -subspace of a field, with $1 \in V$, $\dim V = g$, and $\text{trdeg}_{\mathbb{C}} \mathbb{C}(V) = a \leq n$. If $V(N)$ is the span of products of exactly N members of V , then for every $N \geq 1$*

$$\dim V(N) \geq c_n g N^a, \quad c_n = \frac{1}{(n+1)(2n+2)^n n!}.$$

Proof. For $a = 0$ the field is $\mathbb{C}$. Otherwise choose a smooth point of $\text{Spec } \mathbb{C}[V]$. By Lemma 4.24, its completed local ring is a power-series ring, and members of V vanishing at the point include parameters $z_1, \dots, z_a$. The original ring embeds in this completion. Order Taylor monomials by total degree and then lexicographically. Eliminating leading coefficients gives g distinct leading exponents $E \subset \mathbb{Z}_{\geq 0}^a$. They include $0, e_1, \dots, e_a$. Products add leading exponents, so $\dim V(N) \geq |NE| \geq g$.

Triangulate $\text{conv}(E)$, using every point of E as a vertex. Such a triangulation is obtained by coning compatible face triangulations and inserting the remaining points by stellar subdivision. If it has t top simplices, then $t \geq g - a$: their facet adjacency graph is connected, and along a spanning tree the first simplex contributes $a + 1$ vertices and each later simplex at most one. For each top simplex, positive integral barycentric coefficients summing to N give $\binom{N-1}{a}$ distinct points of NE in its dilated interior. These interiors are disjoint. Therefore

$$|NE| \geq (g - a) \binom{N-1}{a}.$$

Since $g - a \geq g/(a+1)$, this gives $gN^a/((a+1)2^a a!)$ when $N \geq 2a$. For $N < 2a$ use $|NE| \geq g$. The displayed c_n is smaller than both uniform constants. $\square$

Lemma 4.47. *Consider a sequence of original germs and common slices $\mathcal{Q}_i$ as in Definition 4.28, in dimension at most n , and let $L_i \rightarrow \infty$. Suppose the following data are given.*

The baseline is $B(q) = q(h)$, or

$$B(q) = q(h) - \mu q(Y_0^\pm)/L_1$$

for an old opposite pair of cost $O(L_1)$, $L_1 \rightarrow \infty$, $1 \leq L_1 \leq L$, and bounded rational $\mu \geq 0$. On the entire slice $\rho \leq B \leq C$. There is an invariant closed-centred w with finitely generated normal klt graded ring, whose B -normalized volume is within $1 + o(1)$ of the minimum. An old anchor satisfies $|w(Y_0^+)| \geq cL_1$.

There are s selected opposite pairs $Y_j^\pm$, of costs $O(L_j)$, with $L_j = L$ except for the old anchor. Their positive fine weights at w are rationally independent. Their products, and neutral functions G_α including 1, are exact tests. All tests and retained pairs have cost $O(L)$. The accumulated positive variables and neutral functions satisfy the all-degree, onward noncancellation of Lemma 4.33. Some p neutral functions together with the selected positive variables are algebraically independent.

For every fixed N , products of N neutral functions have at least $v_i N^p$ independent initials. Put $R_* = L/L_1$ with an old anchor and $R_* = 1$ otherwise. For every fixed C_0, N , the regular symbols of full character χ^t and w -value at most $C_0 NL$ have dimension at most

$$C(C_0) R_* v_i N^{p+s}, \quad 0 \leq t \leq C_0 NL,$$

where $C(C_0)$ is independent of N, t, i . Finally, assume the tilted supply in Proposition 4.45(4) on every exact rational-height optimizer cone after finitely many bounded normalized both-sign monomial rows have been added to the slice.

Then finitely many further such rows of cost $O(L)$ give a slice containing a closed-centred point on which the selected values are all zero or lie on a rational line. In the latter case an old anchor remains the anchor; if there is none, choose a selected variable as anchor and set $L_1 = L$. The relations have the form

$$d_j y_j = t_j y_0, \quad 1 \leq d_j \leq C', \quad |t_j| = O(L/L_1).$$

The baseline stays unchanged except when a first anchor is chosen, when a rational height of the displayed form may be used. It has fixed positive bounds on the whole new slice. Its exact centred minimizer and an arbitrarily fine refinement both dominate $c'w$ on every regular function; their anchor values are at least $c''L_1$ in absolute value. The refinement satisfies (4.1), and all previous equations and independent initials survive.

Proof. We use the character count to add exact rows and then choose a rational direction with a retained anchor.

Step 1. In this step, we transfer the character count and construct the normalized row spaces. First transfer the stipulated count once. If $v \geq cw$ on every regular function, then $\mathfrak{a}_{A/c}(w) \subset \mathfrak{a}_A(v)$. The induced map of finite-colength quotients is surjective and G -equivariant. Taking any specified eigenspace is exact over $\mathbb{C}$. Thus the same count holds for v , with a constant depending only on the fixed c . In particular the constant for $v \geq w/2$ is fixed before a tilt is chosen.

Write $y_j(q) = q(Y_j^+) = -q(Y_j^-)$ and $x_j = y_j/L_j$. Costs bound these coordinates on the whole slice. Their image K_i is compact and convex: on a simultaneous log resolution the normalized common lc locus is a finite compact polyhedral complex, and Lemma 4.30 interpolates the opposite values exactly. A positive share of a centred valuation gives a centred interpolant dominating that share on all functions.

Add successively integral rows $k \cdot y = 0$ whose normalized coefficients $k_j L_j/L$ are bounded. Each row is imposed by the two monomials using Y_j^+ for positive exponents and Y_j^- for negative exponents, and then exchanging the signs. Their costs are $O(L)$. Keep a centred comparator v_0 with baseline volume within $1 + o(1)$ of the stage minimum. The uniform volume gap in Lemma 4.30 then gives $v_0 = (1 + o(1))w$ on all regular functions. Ratio values change by $o(\text{cost})$, so an old anchor remains nonzero.

Pass to limits of the normalized row spaces R_i . With an old anchor the transverse coefficients are bounded integers and can be fixed. Independence of their transverse rows is equivalent to independence of their normalized limits: a transverse relation leaves only an anchor coefficient, and evaluation at v_0 forces it to vanish. Without an anchor all the rows can be fixed integral rows. Take a Hausdorff limit C_∞ of $C_i = [0, 1]K_i \subset R_i^\perp$.

Step 2. In this step, we exclude a collapse of the limiting radial hull by the character count. Suppose C_∞ does not span $R_\infty^\perp$. Choose a unit u perpendicular to its span inside that space and rational $u_i \rightarrow u$. Uniformly on the current slice, $u_i \cdot x = o(1)$. For a small fixed $\delta > 0$, the rational height

$$b(q) = \delta B(q) - \sum_j u_{ij} y_j(q)/L_j$$

is allowed, after choosing the appropriate sign of each pair, and is $(1 + o(1))\delta B$ everywhere. Its exact optimizer v is centred and has a normal klt cone. Comparing with v_0 , and using the volume gap, gives $v = (1 + o(1))w$ on all regular functions.

On this cone, pure tests are common because every test has weight zero. A Gauss floor test with $LD(P) \rightarrow 0$ is also common: a test of cost $O(L)$ has integral angular order at least $-O(L)D(P)$, hence nonnegative eventually. Sign transfer makes all pair values opposite there. Positivity of the central height, and the uniform upper bound for the central baseline, therefore give

$$\sum_j u_{ij} y'_j/L_j \leq C\delta q'(h)$$

on all those tests. Apply the explicit tilted supply, with $r_j = M_0 u_{ij} L/L_j + O(1)$. It gives an invariant homogeneous ratio Z of weight $-q \sum_j r_j k_j$ and cost at most $C_1 q M_0 \delta L$, $1 \leq q \leq Q$. Its absolute grading value has the same bound: the unrounded value is $-q M_0 L u_i \cdot x(v) = o(q M_0 L)$, and rounding contributes $O(qL)$.

There must be a new bounded normalized integral row exact at v and independent of the previous transverse rows. Suppose otherwise. Every exact tie of any fixed coefficient bound then

lies in the old row span, after a subsequence. A tie in the transverse span differs from such a row by a pure-anchor tie, which vanishes because the anchor is nonzero.

Take unit $\tilde{u}_i \in R_i^\perp$ tending to u . The box $0 \leq e_j \leq \lfloor NL/L_j \rfloor$ contains at least cR_*N^s lattice points. Its normalized projection on $\tilde{u}_i$ has length $O(N)$. For sufficiently large fixed N , some slab of width $qM_0/4$ contains at least

$$cR_*N^{s-1}qM_0$$

points. Multiply their positive monomials by Z^ℓ , $0 \leq \ell \leq \lfloor N/(qM_0\delta) \rfloor$, and by the v_iN^p independent neutral products. For fixed ℓ , these are independent at v : they are separated by fine weights at w , and all-degree noncancellation prevents a new relation after the change of valuation.

Different ℓ have disjoint grading values. A collision would give the exact tie $e - e' - q(\ell - \ell')r$, of bounded normalized size, in the old row span. Its scalar product with $\tilde{u}_i$ is zero. The slab term has magnitude at most $qM_0/4$, while the other term has magnitude at least $qM_0/2$, since $\tilde{u}_i \cdot (r_jL_j/L)_j \geq M_0/2$. This is impossible.

We have at least $c\delta^{-1}R_*v_iN^{p+s}$ independent ratios of cost and absolute grading value at most C_2NL , with C_2 independent of the tilt. A single integral power of h makes their numerators regular of the same full character. The transferred count, whose constant was fixed before δ , is a contradiction for small δ . The order of choices is: the original count constant, then δ , then M_0 , then the bound for q , then fixed N , and only then the index i . No count is used at an index-dependent product degree.

Impose the new row and keep v as comparator. After at most s additions, or $s - 1$ with an old anchor, the limiting radial hull spans the perpendicular row space.

Step 3. In this step, we choose a rational direction and retain an anchor under optimization and refinement. For a small fixed $\lambda > 0$, the set

$$K_i^\lambda = (1 - \lambda)x(v_0) + \lambda K_i$$

has centred realizations dominating $(1 - \lambda)v_0$. Their baseline volumes are within an arbitrarily small fixed relative error of the old minimum as $\lambda \downarrow 0$. The radial hull of K_i^λ contains a relative ball of uniformly positive radius in $R_i^\perp$: choose a nondegenerate limiting simplex and approximate its vertices. With an old anchor choose the ball away from its zero coordinate.

Write the old-anchor row matrix as (b_i, A) , where A is fixed integral of full row rank and $b_i = O(R_*)$. Ratios $z_j = y_j/y_0$ satisfy $Az = -b_i$. A fixed invertible minor gives a rational particular solution with bounded denominator and size $O(R_*)$. The rational kernel has a fixed integral basis up to a fixed denominator. Rounding its coordinates on a sufficiently fine fixed mesh enters the relative ball of feasible ratios. Thus the ratios have bounded denominators and numerators $O(R_*)$. If there is no old anchor, the whole row matrix is fixed. Its perpendicular space is either zero, or contains a rational direction of bounded complexity in that ball with one nonzero coordinate. Choose that coordinate as the first anchor. Imposing both signs of the resulting equations costs $O(d_jL + |t_j|L_1) = O(L)$. The chosen centred mixed point survives.

Only a first anchor needs a new baseline. Let z be the mixed point, with $p_0(z) = z(Y_0^\pm)/L_1 \geq c_1 > 0$, and put $M = \max_{\mathcal{Q}} p_0$. For $0 \leq \mu < 1/M$, $b_\mu = h - \mu p_0$ is allowed and positive on the entire pinned slice. Denote its exact minimizing representative on this new slice by $\hat{v}_\mu$, normalized by $\hat{v}_\mu(h) = 1$. These representatives vary continuously in the all-function topology by Lemmas 4.30 and 4.32. At $\mu_* = (1 - \rho_1)/M$, interpolate a valuation maximizing p_0 with z , giving z weight ρ_1 . The interpolant $\tilde{q}$ satisfies

$$b_{\mu_*}(\tilde{q}) \leq 2\rho_1, \quad \text{vol}(\tilde{q}) \leq \rho_1^{1-n} \text{vol}(z).$$

If V_* is the old ordinary minimum, minimality and $\text{vol}(\hat{v}_{\mu_*}) \geq V_*$ give

$$b_{\mu_*}(\hat{v}_{\mu_*}) \leq 2\rho_1^{1/n} (\text{vol}(z)/V_*)^{1/n} < 1/2$$

for fixed small ρ_1 . Hence $p_0(\widehat{v}_{\mu_*}) \geq M/2$. If $p_0(\widehat{v}_0) \geq c_1/4$, keep $\mu = 0$; minimality gives $\text{vol}(\widehat{v}_0) \leq \text{vol}(z)$. Otherwise choose a rational $0 < \mu < \mu_*$ by continuity with $c_1/4 < p_0(\widehat{v}_\mu) < c_1/3$. Since $p_0(z) \geq c_1$, one has $b_\mu(z) < b_\mu(\widehat{v}_\mu)$, and minimality gives $\text{vol}(\widehat{v}_\mu) \leq \text{vol}(z)$. The optimizer stays close to the old minimum and its anchor is bounded away from zero. The height has fixed positive lower and upper bounds, with bounded rational coefficient.

The volume gap gives domination by a fixed multiple of w , also when the old baseline was retained. Apply balance on the central slice and Lemma 4.32 with errors tending to zero. This preserves the full action, all exact equations, domination, and the anchor. Its regular-symbol estimate becomes (4.1) by the angular dictionary. Noncancellation preserves the independent initials on every subsequent cone. $\square$

Lemma 4.48. *On a sequence of the preceding cones, use the width norm of Definition 4.35. Let E be the field generated by finitely many invariant homogeneous ratios of cost $O(L)$ and absolute grading value $O(L)$. Suppose $\text{trdeg } E = d_0$, their weights span zero or one rational line, and all common valuations are trivial on E^0 , the zero-weight subfield. In the line case let the specified anchor have weight κ , cost $O(L_1)$, and $\|\kappa\| = O(L_1)$, and suppose every generator weight satisfies*

$$d_j k_j = t_j \kappa, \quad 1 \leq d_j \leq B_0, \quad |t_j| = O(L/L_1).$$

Let V_0 be a homogeneous subspace with weights in the subgroup generated by these weights, cost and absolute grading bounded by $O(L)$, and $\dim V_0 \geq cRv_i$, where $R = L/L_1$ in the line case and $R = 1$ otherwise. Suppose the full-character cumulative count is $CRv_i N^{d_0}$ at each fixed scale NL , and the bounded-width supply of Proposition 4.45(2) holds.

Then the whole field generated by $S(LD)$ is algebraic over E^0 and is trivial for every common angular test. Every actual lattice vector k with $\|k\| = O(L)$ lies in the rational span of the weights of E . In the line case, writing $\kappa = \ell k_0$ with k_0 primitive in the actual lattice on that line, the integer ℓ is bounded.

Proof. We use product growth to control the field extension and then count weight classes in the original lattice.

Step 1. In this step, we use product growth to prove algebraicity of the added elements over E . Enlarge V_0 to a space V containing 1, the specified generators of E , a basis of $S(LD)$, and any one supplied symbol of bounded-width weight qk . Every member still has cost and absolute grading value at most $C_1 L$. Products of N members, after multiplication by one power of h , have regular numerators of a single full character and grading at most $C_2 NL$. Thus

$$\dim V(N) \leq C_3 Rv_i N^{d_0}.$$

If $\text{trdeg } \mathbb{C}(V) = a > d_0$, then Lemma 4.46 gives $\dim V(N) \geq c_n c Rv_i N^a$. Choose a constant along a subsequence and then a sufficiently large fixed N . This contradicts the upper bound. Hence all the added elements are algebraic over E . The conclusion is uniform in k , since any sequence of exceptions yields the same contradiction.

Step 2. In this step, we control the zero-weight field and the rational span of the added weights. If a zero-weight element is algebraic over E , clear denominators in an equation and take a nonzero homogeneous component. All its coefficients have the same weight. Dividing by one of them gives an equation over E^0 . A valuation trivial on E^0 is trivial on an algebraic element: a positive value makes the constant term uniquely least, and a negative value makes the leading term uniquely least. This proves the whole-field assertion, at one index for all polynomial degrees. The angular dictionary supplies the common height-minimizing Gauss extension and finite-cover lift of each angular test, so triviality descends to that test.

More generally, a nonzero homogeneous element of weight α , algebraic over E , has a homogeneous polynomial equation with at least two terms. Equality of their weights puts a nonzero integral multiple of α in the weight group of E . Apply this to the supplied weight qk . This proves the rational-span assertion without saturating the actual lattice.

Step 3. In this step, we bound the anchor index in the actual integral lattice. For its index, suppose $\ell \rightarrow \infty$. Supply symbols of weights $q_j j k_0$, $1 \leq j \leq \ell$, with $1 \leq q_j \leq Q$. Their widths, costs, and absolute gradings are $O(L)$. Put $D_0 = \text{lcm}(1, \dots, B_0)$. The generator weight subgroup is $r_0 \mathbb{Z} k_0$, where $r_0 \mid \ell$ and $r_0 \geq \ell/D_0$. This single bound is independent of the number of generators. An integer $q_j j$ has at most Q preimages; each residue class modulo r_0 has at most $QD_0 + 1$ representatives in $[1, Q\ell]$. At least $\ell/[Q(QD_0 + 1)]$ residue classes therefore occur. Multiply V_0 by one supplied symbol in each class. Their weight supports are disjoint, so the sum is direct and has dimension at least

$$\frac{\ell}{Q(QD_0 + 1)} cRv_i.$$

Its common numerators have cost and grading $O(L)$ and one full character. The count at fixed $N = 1$, with a larger fixed cutoff constant, bounds this dimension by $C_4 Rv_i$. This bounds ℓ and proves the assertion in the original integral lattice. $\square$

Theorem 4.49. *Fix an integer $d \geq 1$ and a real number $\epsilon > 0$. There is an integer $P(d, \epsilon) > 0$ with the following property. Let $S \ni o$ be a Gorenstein klt algebraic germ of positive dimension at most d . Let a finite cyclic group G act faithfully on S , fixing o , and suppose that $\pi : (S \ni o) \rightarrow (X \ni x) = (S/G \ni \pi(o))$ is quasi-etale, with $X \ni x$ a $\mathbb{Q}$ -Gorenstein klt germ. Let ω be a divisor-free local canonical generator of full G -character χ , and let h be a nonzero regular function of the same full character such that $(S, \text{div}_S h)$ is lc. Put*

$$D_X = \frac{1}{|G|} \text{div}_X(h^{|G|}).$$

Assume $A_X(E) \geq \epsilon$ for every primitive prime divisor E over X with centre exactly $\{x\}$ and $A_{X, D_X}(E) = 0$. For every closed-centred quasi-monomial valuation T on S with $A_S(T) = T(h) = 1$, there is an integer $1 \leq j \leq P(d, \epsilon)$ and a nonzero regular function s of full character χ^j such that $T(s) < j$. The character χ is not required to be faithful.

The hypothesis tests only primitive divisors centred exactly at x on which the descended boundary has zero log discrepancy.

Proof. Suppose $\dim S = 1$. Normality makes the local rings at o and x discrete valuation rings. The inertia group of ord_o is all of G , since G fixes o and acts trivially on its residue field $\mathbb{C}$. The quotient is etale at this codimension-one point, so its ramification index is one. Lemma 3.8 therefore gives $|G| = 1$. The character is trivial, and $s = 1$ satisfies $T(s) = 0 < 1$. Thus $P(1, \epsilon) = 1$ suffices.

In higher dimension, suppose the conclusion fails. Choose for each i data for which every nonzero section of full character χ_i^j , $1 \leq j \leq i$, has T_i -value at least j . Pass to a subsequence of constant dimension $n \leq d$. By Lemma 4.26, T_i is nonnegative on every invariant ratio of cost at most i . We construct successive common slices on these original germs. Every eventual assertion below persists on further subsequences. There will be at most n stages and a bounded number of row additions within each stage.

Step 1. In this step, we specify the common slices, exact tests, and valuation data retained at each stage. Keep finitely many invariant test ratios of cost $O(L_-)$, initially none and $L_- = 1$. Their common normalized lc slice $\mathcal{Q}$ contains a closed-centred point, and every test is identically zero there. Retain opposite pairs $Y_j^\pm$, their products among the exact tests, and write $y_j = q(Y_j^+) = -q(Y_j^-)$. Their accumulated positive initials are algebraically independent. All their values are zero, or they lie on one rational line with an anchor Y_0 chosen at a scale $L_1 \rightarrow \infty$, $L_1 \leq L_-$. In the line case the both-sign relations are

$$d_j y_j = t_j y_0, \quad d_j > 0 \text{ bounded}, \quad |t_j| = O(L_-/L_1).$$

The anchor costs $O(L_1)$ and the other pairs cost $O(L_-)$.

Before the first anchor set $B(q) = q(h)$. Afterwards use $B(q) = q(h) - \mu q(Y_0^\pm)/L_1$ as in Lemma 4.47. On the entire slice, including its noncentred faces, retain fixed bounds $0 < \rho \leq B \leq C$. Exact rational-height optimization followed by arbitrarily fine refinement gives an actual invariant closed-centred w on $K(S)$, with

$$A_S(w) = w(h) = 1,$$

and a finitely generated normal klt graded ring. Its B -volume is within $1 + o(1)$ of the minimum, and it is multiplicatively $1 + o(1)$ from that minimizer on every regular function. With an anchor, retain $|w(Y_0^+)| \geq cL_1$.

Here refinement has the required central balance: the exact-penalty construction makes the central height positive, the two anchor costs bound it above, and its minimizing value is at least ρ . Lemma 4.30 therefore gives $q(s) \leq C_0 w(s)$ for all common central q and homogeneous regular s . The refinement preserves a fixed balance constant and gives (4.1) on every common angular floor test. By Lemma 4.31, all cones keep the original full cyclic group, quasi-etale quotient, and equality of the canonical character with that of h . The angular dictionary thus applies with the invariant top form ω/h and actual invariant lattice M .

After each stage with $L_- \rightarrow \infty$ we also retain the following two assertions:

- every actual lattice vector of width $O(L_-)$ lies in the anchor line, or is zero;
- every common angular test is trivial on $\mathbb{C}(S(L_-D))$.

In the line case the index of the anchor in the actual lattice on that line is bounded. These assertions will be proved after the stage. They are not hypotheses of the pinning step. All initial requirements hold on the unconstrained slice containing T_i , by optimization and refinement with $B = h$.

Step 2. In this step, we exclude termination on an anchor line by the original quotient discrepancy gap. Use the width norm of Definition 4.35. If there is an anchor of weight κ_0 , its two costs and its value at w give $\|\kappa_0\| \asymp L_1$. If there is no transverse direction, write $\kappa_0 = \ell k_0$ with k_0 primitive in the actual M . The retained index bound makes ℓ bounded. Exactness of invariants in each filtration level, as proved in Lemma 4.31, identifies the entire value group of $w|_{K(X)}$. Its positive generator is

$$\gamma = |w(Y_0^+)|/\ell \rightarrow \infty.$$

The restriction is rank-one Abhyankar and hence divisorial after this normalization. Its primitive order $E_X = w|_{K(X)}/\gamma$ is centred exactly at x . Finite quasi-etale log pullback, or Lemma 4.27, gives

$$A_X(E_X) = E_X(D_X) = 1/\gamma.$$

This contradicts the assumed gap ϵ . There is no rank-zero alternative: a nontrivial valuation on a finite extension cannot restrict trivially to the base, by the unique least term in a minimal polynomial. Without an anchor, there is therefore always a new lattice direction.

Step 3. In this step, we choose the next lattice scale, its neutral field, and opposite character sections. With an anchor, use the quotient norm modulo $\mathbb{R}\kappa_0$ on the projected lattice; without one use the width norm. After a high stage its shortest nonzero length divided by L_- tends to infinity. Otherwise lift a projected vector of width $O(L_-)$ and subtract the nearest multiple of the primitive line vector. The rounding error is $O(L_1)$, by the retained index bound, contradicting the retained width assertion.

Before the first high stage, choose the maximal nonempty block of independent directions of bounded width, if there is one, and take integral $L = O(1)$. This is a bounded stage. Otherwise choose integral $L \rightarrow \infty$ comparable to the first nonzero successive minimum, and take the entire block of successive minima comparable to L . After a subsequence, the next minimum divided by L tends to infinity. Lift projected directions by rounding on the old line. Their widths are $O(L)$,

and $L_- = O(L)$. Let $b \geq 1$ be the new block size. At a high stage put $s = b + 1$ and $R_* = L/L_1$ with an old anchor, and $s = b, R_* = 1$ otherwise. For every fixed N , eventually

$$(4.2) \quad \#\{k \in M : \|k\| \leq CNL\} \leq C'R_*N^s.$$

Indeed the next minimum gap confines these points to the first block modulo the line. Their projected separation is at least cL ; disjoint norm balls bound their number by $O(N^b)$ in bounded dimension. Each line fibre has $O(NL/L_1)$ points. The constants are independent of the norm and of N .

Apply Proposition 4.41 at scale L , with old scale $L_0 = L_-$. The pole maximum of the old neutral initial tests is an integral globally generated Cartier b-order $J \leq CL_-D$, and its zero floor locus is exactly the common angular locus. Maximality follows from the retained refinement. When $L_- \rightarrow \infty$, the extra field hypothesis is precisely the retained whole-field triviality. Let K_*, p, v_i, B_* be the field, transcendence degree, volumes, and constant supplied by that proposition. Lift a basis of $S(B_*LD)$, containing 1 and p algebraically independent members, to actual ratios G_α of integral cost $O(L)$. If the neutral field is $\mathbb{C}$, take just 1, $p = 0, v_i = 1$.

For each new direction, supply both signs of a bounded positive multiple by Proposition 4.45(2). Its maximal-order hypothesis holds: on a floor sequence with $LD \rightarrow 0$, every old test has integral order at least $-CL_-D = o(1)$. It is therefore common. Align the bounded multipliers and lift the resulting opposite symbols to actual pairs $Y_j^\pm$, still of cost $O(L)$ and with the full character at their respective integral costs.

Step 4. In this step, we establish exactness and all-degree noncancellation before adding the pinning equations. Add the G_α and the new pair products as tests, but not yet the individual pair members or proportionality rows. They have value zero at w , so the new slice contains w . On its old cone, nonnegativity on the neutral basis implies nonnegativity on its whole span. The cutoff proposition then forces exact triviality on K_* for common angular tests. Pure tests are already trivial on this field; the Gauss description and finite lifts cover all other common lc tests, with real subcones obtained by linearity and rational density. Each added test has a neutral initial of cost $O(L)$, hence belongs to K_* by the whole-space inclusion. All-cost sign transfer gives its actual value at most zero on the new slice. Its defining inequality gives the reverse bound. The tests are exact and the pairs are opposite. The new minimum lies between the old minimum and the volume at w ; thus w remains within $1 + o(1)$ of the new minimum.

We check all-degree noncancellation before any new optimization. Write $A_j = \text{in}_w Y_j^+$ and $B_0 = \text{in}_w Y_0^-$. With an old anchor, the neutral old relation $A_j^{d_j} A_0^{-t_j}$ is a quotient of neutral functions of cost $O(L_-)$: for $t_j \geq 0$, use

$$\frac{A_j^{d_j} B_0^{t_j}}{(A_0 B_0)^{t_j}};$$

for $t_j < 0$, use $A_j^{d_j} A_0^{-t_j}$ itself. All these numerators and denominators belong to K_* . Without an anchor each old A_j is already neutral and belongs to K_* . The new weights are independent modulo the old line. Thus a weight-kernel monomial has zero exponent in each new variable, and a power of its old part is in K_* . A basis of the full integral weight kernel is therefore algebraic over K_* . The pair products and neutral basis are in K_* , and all common cone tests are trivial there. These are exactly the hypotheses of Lemma 4.33. Its conclusion is simultaneous in all degrees and persists on every subsequent common cone.

The accumulated positive variables remain algebraically independent: group a proposed polynomial by the new independent weights, and use independence of the old variables in each group. Likewise the p chosen neutral variables and the s selected positive variables are independent. In a bounded stage, now add both individual members of each new pair as tests. All costs in all preceding bounded stages lie in a fixed finite interval. The original saturation therefore keeps T_i in this slice. The selected values are forced to zero. Optimize and refine, retaining all independent initials. This completes a bounded stage.

Step 5. In this step, we obtain a fixed character count and pin the selected values at a growing scale. Suppose $L \rightarrow \infty$. The cutoff proposition gives, with the same v_i , at least $v_i N^p$ independent neutral products and $\dim S(NLD) \leq C v_i N^p$ for each fixed degree. Consider regular homogeneous symbols of full character χ^t , $0 \leq t \leq CNL$, and grading value at most CNL . Their ratios by h^t have actual weights $k \in M$. Every pure ray is common because all current tests have fine weight zero. Balance bounds the numerator on each normalized pure ray by $C_1 NL$, and regularity bounds it below by zero. Hence $\|k\| \leq C_2 NL$. For each such weight, the character-supply proposition, part (3), bounds the dimension by $C_3 v_i N^p$. Summing (4.2) gives

$$(4.3) \quad \dim\{\text{these symbols of character } \chi^t\} \leq C_4 R_* v_i N^{p+s}.$$

This is uniform in N, t and counts the full eigenspace at the actual integral degree. For any later $v \geq cw$, exactness of eigenspaces in the quotient of valuation ideals transfers (4.3) with one fixed change of constant.

The tilted-supply hypothesis of Lemma 4.47 follows from part (4) of the character-supply proposition. Indeed every cone of a rational optimizer after a finite row enlargement retains the full group and canonical character by sign transfer, has the angular dictionary, and retains the paired costs $O(L_j)$. Part (4) applies whenever the indicated pure and Gauss inequalities hold; it requires no maximal-order assertion on that tilted cone. Thus all pinning hypotheses have now been checked, including the exact slice, noncancellation, neutral lower count, and (4.3).

Pin the selected values to zero or to a rational line. An old anchor survives, or a first one of scale $L_1 = L$ is chosen. Only in the latter case change the baseline. The exact optimizer and a refinement w' dominate a fixed multiple of w on all regular functions, and retain an anchor value at least $c' L_1$ when present. All independent initials survive. The basic stage data now hold with $L_- = L$.

Step 6. In this step, we verify the next-stage field and lattice assertions and conclude by the bound on algebraically independent variables. On the refined cone let E be generated by the p chosen neutral double initials and the s selected positive double initials. Noncancellation gives $\text{trdeg } E = p + s$, and its weights lie on the pinned line or are zero. Every common test is trivial on E^0 . Indeed the both-sign rows hold there by sign transfer. With an anchor, injectivity of fine weight evaluation and its nonzero value show that equal-weight monomials have the same common value by precisely those rational relations. In the zero case their values are all zero. Onward noncancellation gives the predicted value of each nonzero homogeneous polynomial. Equal-weight ratios therefore have value zero. They generate E^0 : in an identity $P = zQ$ for a zero-weight rational function z , take one nonzero equal-weight component. This proves whole-field triviality.

Let V_0 be the space of double initials of the whole neutral basis. If there was an old anchor, also multiply them by its powers $0, \dots, \lfloor R_* \rfloor$. These functions have cost and absolute grading $O(L)$, and noncancellation keeps them independent. Their weights lie in the subgroup generated by the selected variables. The neutral lower count at $N = 1$ gives

$$(4.4) \quad \dim V_0 \geq c R_* v_i.$$

When the first anchor was just chosen, $R_* = 1$; if the selected values are all zero, V_0 is neutral.

The refined cone has maximality on its common angular locus. A floor prime with $LD \rightarrow 0$ is common because every current test has integral order at least $-O(L)D = o(1)$. The character-supply proposition therefore supplies every actual lattice vector of width $O(L)$. The transferred count (4.3), the space in (4.4), the field E , and its zero-weight triviality verify every hypothesis of Lemma 4.48. That lemma gives triviality on the whole field generated by $S(LD)$, as required at the next stage. It also confines every bounded-width actual lattice vector to the anchor line, or to zero, and bounds the integral anchor index. Its denominator hypothesis is supplied by the pinned relations, with the anchor itself among the generators. This completes the high stage without assuming any next-stage assertion during its construction.

Every nonterminal stage has added a nonempty block of positive variables whose initials remain algebraically independent. The cone field has transcendence degree n , so at most n

such additions are possible. Every terminal alternative has already contradicted the original closed-centred gap. The saturating sequence is therefore impossible. If no uniform integer $P(d, \epsilon)$ existed, choosing counterexamples saturated in all degrees $1, \dots, i$ would give exactly such a sequence. This proves the theorem. $\square$

5. CARTIER CONTACT AND POINTED COMPLEMENTS

We retain the discrepancy and character conventions of Section 2. We first turn Theorem 4.49 into a single complement with a positive discrepancy at every divisor centred at the marked point. We then obtain a Cartier equation on the original germ, first for a relative anticanonical model and then for an arbitrary rational pair. We continue the Cartier-section methods of [Ope26b, Theorem G.1, Proposition H.5 and Lemma H.6] and the complement methods of [Ope26a, Theorem 1.1 and Proposition 2.1], the 25 September 2026 preprints discussed in Section 4. Their main theorems assume neighbourhood discrepancy bounds. The pointed assertions in this section are proved below; Chen's complementary methods are credited at their uses.

Theorem 5.1 (Complements preserving a Kollár component). *Fix an integer $d \geq 1$ and a finite set $I \subset \mathbb{Q} \cap [0, 1]$. There is an integer $N = N(d, I) > 0$ such that every klt germ $(X \ni x, \Delta)$ of dimension d , with $\Delta \in I$, and every Kollár component E over x , admit a local N -complement $\Delta^+ \geq \Delta$ with $A_{X, \Delta^+}(E) = 0$. Thus (X, Δ^+) is lc, $N(K_X + \Delta^+)$ is integral Cartier and locally linearly trivial, and x is an lc centre [Zhu24, Definition 2.12 and Lemma 2.13].*

This is the local complement result. It preserves a specified Kollár component; the construction below will improve positivity at every closed-centred prime simultaneously.

Theorem 5.2 (Extraction of noncanonical places). *Let (X, B) be an lc pair with rational boundary and let $\mathcal{V}$ be a finite collection of divisorial valuations exceptional over X with log discrepancies in $[0, 1]$. There is a projective birational morphism $\pi : Y \rightarrow X$ with purely divisorial exceptional locus, whose exceptional prime divisors are $\mathbb{Q}$ -Cartier and correspond exactly to $\mathcal{V}$, and an anti-effective exceptional divisor which is ample over X . For a singleton $\mathcal{V} = \{\text{ord}_E\}$, the divisor $-E$ is π -ample [Mor20, Theorem 1].*

We use this extraction result for a single exceptional prime captured by an lc boundary. Its ampleness conclusion will give the Fano-type model needed for a second bounded complement.

Lemma 5.3. *Let (U, D) be a rational pair over an algebraically closed field of characteristic zero. Let $\mathfrak{a} \neq 0$ be an ideal generated by regular functions $h_1, \dots, h_s$, and let $0 < c < 1$ be rational. Suppose*

$$A_{U, D}(F) - c \text{ord}_F(\mathfrak{a}) \geq 0$$

for every primitive prime divisor F with centre on U . Then a general scalar linear combination h of the h_i gives an lc pair $(U, D + c \text{div}(h))$ whose lc places are precisely the primitive primes F for which $A_{U, D}(F) = c \text{ord}_F(\mathfrak{a})$.

Thus a finite system can replace an ideal without introducing new lc places when the coefficient is less than one.

Proof. Use Theorem 3.1 to take a log resolution $p : U' \rightarrow U$ of the pair and the ideal, and write

$$K_{U'} + D' = p^*(K_U + D), \quad \mathfrak{a}\mathcal{O}_{U'} = \mathcal{O}_{U'}(-F_0).$$

The subboundary $D' + cF_0$ has simple normal crossings support and coefficients at most one; negative coefficients are allowed. After division by the fixed divisor F_0 , the generators define a basepoint-free system. A general member H is smooth and transverse to the fixed support and contains none of its finitely many strata. Hence $\text{div}(p^*h) = F_0 + H$ and $(U', D' + cF_0 + cH)$ is sub-lc.

For a log-smooth subboundary, zero discrepancy can occur only over strata made entirely of coefficient-one components. Indeed, the discrepancy of the blowup of a smooth centre is its

codimension minus the sum of the coefficients of the components containing it. A transverse direction or a component of coefficient less than one contributes positively. The same calculation after successive blowups, or monomial retraction, describes every zero-discrepancy prime. Since $c < 1$, adding cH preserves precisely these strata and their lc places. This proves both assertions. For a germ the construction is made near the inverse image of its marked point; properness permits shrinking the original representative. $\square$

Lemma 5.4. *Let $n \geq 1$ and $r \geq 0$ be integers, and let $K \subset L$ be finitely generated extensions of an algebraically closed field k , with transcendence degrees n and $n + r$, respectively. Let v be a positive rational multiple of a primitive divisorial valuation of L , trivial on k , and suppose $v|_K \neq 0$. Then $v|_K = a \operatorname{ord}_E$ for a primitive divisorial valuation E of K and a positive rational a .*

An integer-valued restriction has $a \in \mathbb{Z}_{>0}$. This keeps the lower discrepancy bound unchanged after adding variables and taking a finite quotient.

Proof. The nonzero value group of the restriction is discrete. The relative residue-transcendence inequality and the Abhyankar inequality give

$$n + r - 1 = \operatorname{trdeg}_k \kappa(v) \leq \operatorname{trdeg}_k \kappa(v|_K) + r \leq n - 1 + r.$$

The restriction is therefore divisorial up to scaling. Its positive generator is an integer when its value group is contained in $\mathbb{Z}$. $\square$

Lemma 5.5. *Let U be a $\mathbb{Q}$ -Gorenstein klt variety over an algebraically closed field of characteristic zero, and let $r \geq 0$ be an integer. Write $z_1, \dots, z_r$ for the affine coordinates. Let v be a positive rational multiple of a primitive divisorial valuation with centre on $U \times \mathbb{A}^r$, and assume its restriction v_0 to $k(U)$ is nonzero. Let $\bar{v}$ be the joint Gauss extension of v_0 with coordinate weights $v(z_1), \dots, v(z_r)$. Then*

$$A_{U \times \mathbb{A}^r}(v) \geq A_{U \times \mathbb{A}^r}(\bar{v}) = A_U(v_0) + \sum_{i=1}^r v(z_i).$$

The Gauss extension realizes the least discrepancy compatible with the given base restriction and coordinate weights.

Proof. By Lemma 5.4, write $v_0 = a \operatorname{ord}_E$. Take a smooth model of U carrying E . The simple normal crossings inequality for E and the coordinate hyperplanes contributes at least $a + \sum_i v(z_i)$; the relative canonical divisor contributes $a(A_U(E) - 1)$. Their sum is the stated bound. A Gauss valuation is monomial in these parameters, so equality holds. $\square$

Theorem 5.6. *For every integer $n \geq 1$ and real number $\epsilon > 0$, there is an integer $N(n, \epsilon) > 0$ such that every $\mathbb{Q}$ -Gorenstein klt n -dimensional germ $X \ni x$ with $\operatorname{mld}(X \ni x) \geq \epsilon$ admits an effective rational lc $N(n, \epsilon)$ -complement C satisfying*

$$A_{X,C}(E) \geq 1/N(n, \epsilon)$$

simultaneously for every primitive prime E centred exactly at x . Positive-dimensional lc centres through x are allowed.

The same complement controls all the indicated primes. Its construction does not require their enumeration.

Proof. We start with an ordinary bounded lc complement, obtain a bounded character section escaping each of its closed-centred lc places, and put all these sections into one finite system.

Step 1. In this step, we construct an initial bounded complement and its full-character equation. Apply Theorem 4.12 to the identity contraction $X \rightarrow X$, with boundary zero and coefficient set $\mathcal{R} = \{0, 1\}$. This is the relative theorem [Bir19, Theorem 1.8]. The identity is projective, its base has positive dimension, and $-K_X$ is relatively ample. Thus X is of Fano type over itself. We

obtain an effective lc complement B_0 with $b(K_X + B_0) \sim 0$ integral Cartier, where the integer $b \geq 2$ depends only on n . Multiplying a complement index by two ensures the last inequality. For any specified Kollár component, Theorem 5.1 with $I = \{0\}$ also supplies an initial complement while preserving that component as an lc place. The argument below applies to either choice of bounded initial complement.

Use Theorem 3.9 with boundary zero and $m = 1$. Write $\rho : (V \ni o) \rightarrow (X \ni x)$ for its cover, G for its full cyclic group, and θ for its divisor-free canonical generator of faithful character χ . A rational b -canonical tensor η trivializing $b(K_X + B_0)$ satisfies $\text{div}(\eta) + bB_0 = 0$. Therefore

$$h_0 = \frac{\theta^b}{\rho^* \eta}, \quad \text{div}(h_0) = b\rho^* B_0$$

is regular and has full character χ^b . Regularity follows from effectivity and normality. Conversely, a nonzero regular function h of character χ^j defines an effective divisor C_h with $\rho^* C_h = \text{div}(h)/j$. The invariant tensor θ^j/h descends and trivializes $j(K_X + C_h)$ as an integral Cartier divisor. Lemma 3.8 gives lc equivalence upstairs and downstairs, with the actual ramification index at every prime.

Step 2. In this step, we find a bounded character section escaping each closed-centred lc place of the initial complement. Fix any lc place E of (X, B_0) centred at x , and a primitive lift F to V . Put $A_0 = A_V(F)$. This is a positive integer, and $\text{ord}_F(h_0) = bA_0$. Set

$$S = V \times \mathbb{A}^{b-1}, \quad z = (o, 0),$$

and give every new coordinate u_i character χ for the same group G . Its action remains faithful and quasi-etale: every nonidentity fixed locus projects into a fixed locus of codimension at least two on V . The canonical generator $\theta \wedge du_1 \wedge \cdots \wedge du_{b-1}$ has character χ^b , which need not be faithful. The quotient $(Q \ni y) = (S/G \ni \pi(z))$ is $\mathbb{Q}$ -Gorenstein klt: a power of this generator descends, and finite crepant discrepancy comparison applies.

Every primitive prime P over Q centred at y has $A_Q(P) \geq \epsilon$. To check this, extend ord_P to $\mathbb{C}(S)$ and scale its extension v to restrict to ord_P . It is centred at z . Its restriction to $\mathbb{C}(X) \subset \mathbb{C}(Q)$ is nonzero and integer-valued. Lemma 5.4 writes it as $a \text{ord}_D$, where $a \geq 1$ is an integer and D is a primitive prime centred at x . Lemma 5.5 and Lemma 3.8 give

$$A_Q(P) = A_S(v) \geq aA_X(D) + \sum_i v(u_i) \geq \epsilon.$$

This uses the actual invariant field $\mathbb{C}(Q)$, which contains $\mathbb{C}(X)$.

The ideal $\mathfrak{a} = (h_0, u_1^b, \dots, u_{b-1}^b)$ is lc with exponent one. Indeed, for a divisorial v whose base restriction is nonzero,

$$A_S(v) \geq \frac{v(h_0)}{b} + \sum_i v(u_i) \geq \min\{v(h_0), bv(u_1), \dots, bv(u_{b-1})\} = v(\mathfrak{a}).$$

If the restriction is zero, then $v(h_0) = v(\mathfrak{a}) = 0$. Principalize $\mathfrak{a}$ and choose a general member of the divided generating system. The fixed boundary has coefficients at most one, and the moving divisor is smooth and transverse with coefficient one. This gives an lc equation $h' = c_0 h_0 + \sum_i c_i u_i^b$, with $c_0 \neq 0$, of character χ^b .

Let W be the Gauss extension of ord_F with every u_i -weight equal to A_0 . Its value group is $\mathbb{Z}$, and its residue field is $\kappa(F)$ with the $b-1$ independent ratios $u_i/t_F^{A_0}$ adjoined. It is thus a primitive divisorial valuation centred at z , and

$$A_S(W) = bA_0 = W(h').$$

The summands do not cancel in its Gauss initial ring. Apply Theorem 4.49 to $T = W/(bA_0)$ in dimension $d = n + b - 1$. All its hypotheses, including the quotient gap and the equality of full characters, were checked above. It gives a regular section H of character χ^{bj} , where $1 \leq j \leq P(d, \epsilon)$, such that $W(H) < bjA_0$.

Multiply H by an invariant unit clearing its denominator, obtained by taking the product of all translates of a denominator nonvanishing at z . We can then write $H = \sum_{\alpha} f_{\alpha} u^{\alpha}$, where f_{α} is regular on V with character $\chi^{bj-|\alpha|}$. The Gauss formula gives a nonzero coefficient satisfying

$$\text{ord}_F(f_{\alpha}) + A_0|\alpha| < bjA_0.$$

Since $\text{ord}_F(f_{\alpha}) \geq 0$, the integer $k = bj - |\alpha|$ is positive. With $K = bP(n + b - 1, \epsilon)$, we have proved that for each such F there is a regular section s of full character χ^k satisfying

$$(5.1) \quad 1 \leq k \leq K, \quad \text{ord}_F(s) < kA_V(F).$$

Step 3. In this step, we combine the sections into one complement with positive discrepancy at every closed-centred prime. Put $N = 2\text{lcm}(1, \dots, K)$, and set

$$\begin{aligned} R &= \mathcal{O}_{V,o}, \\ R_N &= \{s \in R : \sigma^* s = \chi(\sigma)^N s \text{ for every } \sigma \in G\}, \\ \mathfrak{b} &= R_N R. \end{aligned}$$

The module R_N is finite over $\mathcal{O}_{X,x}$. Choose finite generators, including $h_0^{N/b}$, which exists because $b \leq K$, and represent them on a common neighbourhood. For every divisorial v ,

$$v(\mathfrak{b}) \leq \frac{N}{b} v(h_0) \leq NA_V(v).$$

Apply Lemma 5.3 with coefficient $1/N < 1$. A general linear combination h defines an lc pair $(V, \text{div}(h)/N)$, and every one of its lc places F satisfies

$$(5.2) \quad A_V(F) = \frac{\text{ord}_F(\mathfrak{b})}{N} \leq \frac{\text{ord}_F(h_0)}{b} \leq A_V(F).$$

The section h has character χ^N , so the tensor construction above descends it to an lc N -complement C on X .

Suppose E were an lc place of (X, C) centred at x . For a primitive lift F , equality throughout (5.2) makes F an lc place of the original complement and gives $\text{ord}_F(\mathfrak{b}) = NA_V(F)$. For each $1 \leq k \leq K$ and each section s of character χ^k , the element $s^{N/k}$ belongs to $\mathfrak{b}$. Consequently

$$\frac{N}{k} \text{ord}_F(s) \geq \text{ord}_F(\mathfrak{b}) = NA_V(F),$$

contradicting (5.1). There is no such E . Finally, $N(K_X + C)$ is integral Cartier, so $NA_{X,C}(E)$ is an integer on every smooth model carrying E . Its positivity at every closed-centred prime gives the claimed bound $1/N$. $\square$

Lemma 5.7. *Fix positive integers n and M . There is a positive rational $\gamma(n, M)$ such that the following holds. Let $X \ni x$ be a $\mathbb{Q}$ -Gorenstein klt n -dimensional germ, and let C be an effective rational lc M -complement with $A_{X,C}(E) > 0$ for every primitive prime centred at x . Then there is $0 \neq g \in \mathfrak{m}_{X,x}$ for which $(X, C + \gamma(n, M) \text{div}(g))$ is lc near x . Consequently, for every real $\epsilon > 0$ there is a positive rational $c_0(n, \epsilon)$ such that every $\mathbb{Q}$ -Gorenstein klt n -dimensional germ $X \ni x$ with $\text{mld}(X \ni x) \geq \epsilon$ admits $0 \neq g \in \mathfrak{m}_{X,x}$ for which $(X, c_0(n, \epsilon) \text{div}(g))$ is lc near x .*

Positive-dimensional lc centres of (X, C) impose no condition on the order of the point ideal. We use a threshold prime centred at x to retain this distinction in the complement construction.

Proof. We resolve the point ideal, extract a prime computing its threshold, and take the difference of two locally principal complements. This adapts the construction of [Che24, Proposition 5.1], in arXiv:2403.01154v5; the extraction below allows the given pair to be lc.

For $n = 1$, the germ is smooth. If c_x is the coefficient of C at x , then Mc_x is integral and $c_x < 1$. Thus $1 - c_x \geq 1/M$, and a local parameter works with $\gamma(1, M) = 1/(2M)$. Assume $n \geq 2$. Define

$$\mathcal{R}_M = \{0, 1/M, \dots, 1\}, \quad L = 2n_{\text{lc}}(n, \mathcal{R}_M), \quad \gamma(n, M) = \frac{1}{ML},$$

where $n_{lc}(n, \mathcal{R}_M)$ denotes the complement integer in Theorem 4.12.

Step 1. In this step, we choose a prime computing the threshold of the original point ideal. Resolve (X, C) and $\mathfrak{m} = \mathfrak{m}_{X,x}$, writing

$$K_{X'} + C' = p^*(K_X + C), \quad \mathfrak{m}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_0).$$

Every component P of F_0 maps to x , so its discrepancy is positive. The simple normal crossings criterion gives

$$t = \min_{P \subset \text{Supp } F_0} \frac{A_{X,C}(P)}{\text{coeff}_P(F_0)} > 0$$

as the lc threshold of the ideal. Choose a component E attaining this minimum. If $t \geq 1/2$, the ideal remains lc with coefficient $\gamma \leq 1/2$, and Lemma 5.3 proves the claim. If $t < 1/2$, that lemma gives a general $g_0 \in \mathfrak{m}$ with $\text{ord}_E(g_0) = \text{ord}_E(\mathfrak{m})$ such that $\Delta = C + t \text{div}(g_0)$ is lc and $A_{X,\Delta}(E) = 0$.

Step 2. In this step, we extract the threshold prime and obtain an lc model of Fano type. The prime E is exceptional over X , since it is centred at the closed point and $n \geq 2$. Apply Theorem 5.2 to the rational lc pair (X, Δ) and its single discrepancy-zero prime E . This is [Mor20, Theorem 1]. It gives a projective morphism $\pi : Y \rightarrow X$ whose exceptional locus is exactly the $\mathbb{Q}$ -Cartier prime E , with $-E$ relatively ample. The morphism is an isomorphism off x , and its reduced fibre over x is E : any additional point outside the exceptional locus would be isolated in the connected birational fibre.

Let Δ_Y be the strict transform of Δ , and put $\alpha_0 = A_X(E) > 0$. Then

$$K_Y = \pi^*K_X + (\alpha_0 - 1)E, \quad K_Y + E + \Delta_Y = \pi^*(K_X + \Delta).$$

Thus K_Y and Δ_Y are $\mathbb{Q}$ -Cartier, and (Y, E) is lc. For a prime P centred inside E , effectivity and $\mathbb{Q}$ -Cartierness give $\text{ord}_P(E) > 0$, hence

$$A_Y(P) = A_{Y,E}(P) + \text{ord}_P(E) > 0.$$

Away from E , the variety is the klt open subset of X . This proves that Y is klt. Choose rational $0 < \eta < \min\{1, \alpha_0\}$. The same calculation proves $(Y, (1 - \eta)E)$ klt, and

$$-(K_Y + (1 - \eta)E) \sim_{\mathbb{Q}, X} (\alpha_0 - \eta)(-E)$$

is ample. Thus Y is of Fano type over X .

Let C_Y and D_Y be the strict transforms of C and $\text{div}(g_0)$, and put $\alpha = A_{X,C}(E) > 0$. Comparing coefficients along the only exceptional prime gives

$$K_Y + E + C_Y = \pi^*(K_X + C) + \alpha E, \quad \alpha = t \text{ord}_E(g_0),$$

and

$$K_Y + E + C_Y + tD_Y = \pi^*(K_X + \Delta).$$

The divisor D_Y is $\mathbb{Q}$ -Cartier, since $\pi^* \text{div}(g_0) = D_Y + \text{ord}_E(g_0)E$. Removing tD_Y proves that $(Y, E + C_Y)$ is lc, and $-(K_Y + E + C_Y) \sim_{\mathbb{Q}, X} \alpha(-E)$ is ample.

Step 3. In this step, we obtain a contact equation through x from the principal difference of two complements. The integrality of $M(K_X + C)$ implies that every coefficient of C belongs to $\{j/M : 0 \leq j \leq M\}$. The coefficients of $E + C_Y$ lie in $\Phi(\mathcal{R}_M)$, using denominator one in the definition of that set. Apply Theorem 4.12, or [Bir19, Theorem 1.8], to π and $(Y, E + C_Y)$. The base is positive-dimensional, Y is of Fano type, and the required nefness and coefficient conditions were just checked. We obtain an lc boundary $\Omega_Y \geq E + C_Y$ and a rational function $a \in \mathbb{C}(Y) = \mathbb{C}(X)$ satisfying

$$L(K_Y + \Omega_Y) = \text{div}_Y(a).$$

Its coefficient at E is one. Put $\Omega = \pi_*\Omega_Y$. Pushing down and then pulling back the same principal divisor yields

$$L(K_X + \Omega) = \text{div}_X(a), \quad K_Y + \Omega_Y = \pi^*(K_X + \Omega).$$

In particular (X, Ω) is lc, $\Omega \geq C$, and $A_{X, \Omega}(E) = 0$.

Write $M(K_X + C) = \operatorname{div}_X(b)$ locally. The function

$$g = \frac{a^M}{b^L}, \quad \operatorname{div}_X(g) = ML(\Omega - C)$$

is regular by normality and effectivity. Moreover $\operatorname{ord}_E(g) = ML\alpha > 0$, so $g \in \mathfrak{m}_{X, x}$. The equality $C + \gamma \operatorname{div}(g) = \Omega$ proves the first assertion.

For the last assertion apply Theorem 5.6 and take $c_0(n, \epsilon) = \gamma(n, N(n, \epsilon))$. Removing its effective $\mathbb{Q}$ -Cartier complement boundary preserves log canonicity. $\square$

Proposition 5.8. *For every integer $n \geq 1$ and real number $\epsilon > 0$, there is a rational $u(n, \epsilon) > 0$ with the following property. Let $f : Y \rightarrow X$ be a projective birational contraction of normal quasi-projective n -folds, with Y $\mathbb{Q}$ -Gorenstein klt and $-K_Y$ semiample over X . Let $x \in X$ be closed and suppose $A_Y(P) \geq \epsilon$ for every primitive prime whose centre on X is exactly x . There is a nonzero original function $g \in \mathfrak{m}_{X, x}$ such that $(Y, u(n, \epsilon) \operatorname{div}_Y(f^*g))$ is lc near the whole fibre $f^{-1}(x)$. No $\mathbb{Q}$ -Cartierness of K_X is assumed.*

The auxiliary cone used below has dimension $n + 1$. Adding one coordinate lets us use bare contact in dimension $n + 2$ and recover a degree-zero function on X .

Proof. We first construct a crepant anticanonical cone in the ample case, then use a finite quotient to force a nonzero degree-zero term in its contact equation. Finally we pass from divisorial inequalities over x to the whole fibre and treat semiampleness.

Step 1. In this step, we construct a crepant Gorenstein klt anticanonical cone. Assume first that $-K_Y$ is ample over X . Shrink to $X = \operatorname{Spec} R$, choose a rational top form ω of $K = \mathbb{C}(X) = \mathbb{C}(Y)$, and set $K_Y = \operatorname{div}_Y(\omega)$. Use the all-degree algebra

$$\mathcal{A} = \bigoplus_{j \geq 0} H^0(Y, \mathcal{O}_Y(-jK_Y))t^j \subset K[t], \quad W = \operatorname{Spec} \mathcal{A}.$$

Here at^j belongs to $\mathcal{A}$ exactly when $\operatorname{div}_Y(a) - jK_Y \geq 0$. A Cartier Veronese algebra is finitely generated by relative ampleness; the finitely many residue-degree modules are finite over it. Thus $\mathcal{A}$ is finitely generated. It is normal, since it is the intersection of $K[t]$ with the Gauss valuation rings imposing these inequalities at every prime divisor of Y . Its fraction field is $K(t)$: R has fraction field K , and $H^0(Y, \mathcal{O}_Y(-K_Y)) \neq 0$ on the affine birational base. The positive-degree ideal defines a vertex section $X \rightarrow W$; let o be its point over x .

There is a proper birational map

$$\widetilde{W} = \operatorname{Spec}_Y \bigoplus_{j \geq 0} \mathcal{O}_Y(-jK_Y)t^j \longrightarrow W.$$

For a divisible r , the Veronese relative spectrum is the total space of $\mathcal{O}_Y(rK_Y)$. Its evaluation map to the Veronese affine cone is proper: relative generation realizes it as a closed incidence variety in the product with Y . Both all-degree spectra are finite over their Veronese spectra, and the target is separated. Properness therefore lifts to the displayed map, while equality of fraction fields proves birationality.

The form $\Omega = \omega \wedge dt$ is divisor-free on $\widetilde{W}$. On the smooth locus of Y , write $\operatorname{div}(s) = K_Y$ and use the fibre coordinate $y = st$; then $\Omega = (\omega/s) \wedge dy$. The inverse image of the remaining codimension-two set has codimension at least two, by the finite map to the Veronese line bundle. Locally take the canonical cover of Y from Theorem 3.9. On it the normalized algebra is a polynomial algebra in y , and its deck group acts on ω/s and y by inverse characters. The action on this product is quasi-etale. Its quotient is the local all-degree relative spectrum, so Lemma 3.8 proves that $\widetilde{W}$ is klt. Pushing the divisor of Ω down gives $K_W = 0$. The map is therefore crepant and W is Gorenstein klt.

Step 2. In this step, we compare cone valuations and establish the pointed gap on a finite quotient. Put $S = W \times \mathbb{A}_z^1$. For a divisorial valuation v of $K(t, z)$ with nonzero restriction v_0 to K , that restriction is divisorial up to scaling by Lemma 5.4. Define

$$q_v = v(t) + v_0(K_Y), \quad b_v = v(z).$$

If v is centred on S , then $q_v, b_v \geq 0$. Indeed, properness supplies a centre of v_0 on Y , and a generated Cartier anticanonical section avoiding its generic centre has value $dq_v \geq 0$, for a divisible d . The simple normal crossings calculation used in Lemma 5.5, now with rational form $\omega \wedge dt \wedge dz$, gives

$$(5.3) \quad A_S(v) \geq A_Y(v_0) + q_v + b_v.$$

To see the correction term, take a uniformizer of the primitive restriction on a smooth model of Y . The discrepancy measured by ω there is $A_Y(v_0) + v_0(K_Y)$. Multiplying t, z by powers of this uniformizer makes their values nonnegative; the powers cancel in the differential-form formula. The remaining contributions are $v(t) + v(z)$, which gives (5.3).

For each original primitive prime P centred at x and each rational $q > 0$, its joint Gauss extension v_q defined by

$$v_q|_K = \text{ord}_P, \quad v_q(t) = q - \text{ord}_P(K_Y), \quad v_q(z) = q$$

is divisorial up to scaling and centred exactly at $(o, 0)$. In fact, for every homogeneous section,

$$(5.4) \quad \begin{aligned} v_q(at^i z^j) &= (i + j)q + \text{ord}_P(\text{div}_Y(a) - iK_Y), \\ A_S(v_q) &= A_Y(P) + 2q. \end{aligned}$$

The second summand in the value formula is nonnegative; positive degree therefore has positive value, while degree zero has the centre x .

Let $c = c_0(n + 2, \epsilon)$ be supplied by Lemma 5.7. Choose an integer $\ell \geq \max\{2, 4/c\}$. Let μ_ℓ act on W by its grading and on z with weight one. The action on W is faithful because $t \in \text{Frac } \mathcal{A}$ has weight one. A nonidentity fixed locus on W is proper; its product with the fixed locus $\{0\}$ in the extra coordinate has codimension at least two. Thus $S \rightarrow Z = S/\mu_\ell$ is quasi-etale. The form $\Omega \wedge dz$ has character of weight two, so a power descends and Z is $\mathbb{Q}$ -Gorenstein klt. Let z_0 be the image of $(o, 0)$.

For a primitive prime Q over Z centred at z_0 , normalize an extension v to $\mathbb{C}(S)$ to restrict to ord_Q . Its restriction to $K \subset \mathbb{C}(Z)$ is nonzero and integer-valued, hence is $a \text{ord}_P$ for an integer $a \geq 1$ and an original primitive P over x . Finite comparison and (5.3) give

$$A_Z(Q) = A_S(v) \geq aA_Y(P) + q_v + b_v \geq \epsilon.$$

The bare-contact lemma applies to this $(n + 2)$ -dimensional germ.

Step 3. In this step, we recover a nonzero degree-zero contact equation and prove log canonicity near the whole fibre. Choose $0 \neq h \in \mathfrak{m}_{Z, z_0}$ with $(Z, c \text{div}(h))$ lc. Clearing a denominator that is a unit at z_0 , take $h \in \mathcal{A}[z]^{\mu_\ell}$. It has a finite total-degree expansion

$$h = g + \sum_{j>0} h_j, \quad g \in R \cap \mathfrak{m}_{Z, z_0} = \mathfrak{m}_{X, x}.$$

Every nonzero positive degree is divisible by ℓ . Pullback is lc, so (5.4) implies

$$c v_q(h) \leq A_Y(P) + 2q, \quad v_q(h) \geq \min\{\text{ord}_P(g), \ell q\}$$

when $g \neq 0$. If $g = 0$, the lower bound is ℓq for all rational $q > 0$, contradicting the first inequality for large q , since $c\ell > 2$. Thus $g \neq 0$. Take $q = \text{ord}_P(g)/\ell > 0$. We obtain, simultaneously for every original primitive P over x ,

$$\frac{c}{2} \text{ord}_P(g) \leq \left(c - \frac{2}{\ell}\right) \text{ord}_P(g) \leq A_Y(P).$$

Set $u(n, \epsilon) = c/2$.

These inequalities imply lc near the whole fibre. On a log resolution of $(Y, u \operatorname{div}(f^*g))$, suppose a negative-discrepancy component has image containing x . Choose a closed point on that component over x , and take monomial valuations with arbitrarily large integral weight on its equation and weight one on the other local parameters. Their discrepancies are eventually negative, whereas their centres on X are exactly x . This contradicts the inequalities just proved. The finitely many negative-discrepancy components therefore have images avoiding x ; projectivity permits removing these closed images. The remaining log-smooth boundary has coefficients at most one.

Step 4. In this step, we reduce the semiample case to the ample anticanonical model. Let $h : Y \rightarrow Y_a$ be the relative ample model of $-K_Y$. It is birational over X . For a divisible r , write $-rK_Y = h^*L$ with L Cartier and ample over X . Pushforward identifies $L = -rK_{Y_a}$, using compatible canonical divisors. Hence $K_Y = h^*K_{Y_a}$, and Y_a is $\mathbb{Q}$ -Gorenstein klt with the same discrepancies at all original primes. Apply the ample case to $Y_a \rightarrow X$ and pull back its lc pair crepantly to Y . $\square$

Theorem 5.9. *For every integer $n \geq 1$ and real number $\epsilon > 0$, there is a positive rational number $c_{\text{ct}}(n, \epsilon)$ with the following property. Let $(X \ni x, B)$ be an n -dimensional klt germ over an algebraically closed field of characteristic zero. Assume that B is rational and $\operatorname{mld}(X \ni x, B) \geq \epsilon$. Then there is a nonzero $g \in \mathfrak{m}_{X,x}$ such that $(X, B + c_{\text{ct}}(n, \epsilon) \operatorname{div}(g))$ is lc near x . For every primitive prime divisor E centred at x ,*

$$\operatorname{ord}_E(\mathfrak{m}_{X,x}) \leq \operatorname{ord}_E(g) \leq \frac{A_{X,B}(E)}{c_{\text{ct}}(n, \epsilon)}.$$

The constant is independent of the coefficients and the number of components of B .

Only the total divisor $K_X + B$ is required to be $\mathbb{Q}$ -Cartier. The relative proposition allows us to preserve this hypothesis throughout the reduction.

Proof. We first work over $\mathbb{C}$. We lower the closed-centred discrepancy to a fixed positive rational number on a resolution, extract a prime attaining it, and use relative contact after an anticanonical MMP. We then transfer the ideal inequality to the original field. The relative reduction follows the method of [Che24, Proposition 6.1 and Section 2.6], in arXiv:2403.01154v5, with the pointed inequalities verified below.

For $n = 1$, choose any rational $0 < c_{\text{ct}}(1, \epsilon) \leq \min\{\epsilon, 1/2\}$. The normal curve germ is smooth, and its boundary coefficient at x is at most $1 - \epsilon$. A local parameter gives the assertion over any field in the statement. Assume $n \geq 2$, choose a fixed rational

$$0 < \delta \leq \min\{\epsilon/2, 1/4\}, \quad u = u(n, \delta), \quad c_{\text{ct}}(n, \epsilon) = \min\{1/2, \delta u\}.$$

Step 1. In this step, we construct a klt perturbation with a prescribed positive discrepancy at the marked point. Take an affine klt representative and a projective log resolution $p : Z \rightarrow X$ of its pair and point ideal, with

$$K_Z + B_Z = p^*(K_X + B), \quad \mathfrak{m}_{X,x} \mathcal{O}_Z = \mathcal{O}_Z(-F_0).$$

The support of F_0 is the reduced inverse image of x . Let P_i be its components and put $m_i = \operatorname{ord}_{P_i}(\mathfrak{m}_{X,x}) > 0$ and $a_i = A_{X,B}(P_i) \geq \epsilon$. Define

$$t = \min_i \frac{a_i - \delta}{m_i} > 0,$$

and let E be a component attaining the minimum. On Z every component of F_0 has discrepancy at least δ for the ideal pair with exponent t ; all other components of the fixed simple normal crossings support still have positive discrepancy. For any primitive prime Q centred at x , its centre on Z lies in $\operatorname{Supp} F_0$. At least one local equation of a component of F_0 has positive integral Q -value. The simple normal crossings discrepancy inequality therefore gives

$$A_{X,B}(Q) - t \operatorname{ord}_Q(\mathfrak{m}_{X,x}) \geq \delta.$$

Equality holds for the chosen E .

Choose a general linear combination h_0 of generators of $\mathfrak{m}_{X,x}$ so that $\operatorname{div}(p^*h_0) = F_0 + H_0$, where H_0 is smooth and transverse to the fixed support and contains none of its strata, as in Lemma 5.3. If $t \geq 1/2$, the subboundary $B_Z + (F_0 + H_0)/2$ has all coefficients less than one. This proves contact with the stated constant in that case.

Suppose $t < 1/2$, and put $D = B + t \operatorname{div}(h_0)$. The pair (X, D) is klt. The new component H_0 has discrepancy $1 - t > 1/2 \geq \delta$, so the same simple normal crossings calculation proves

$$(5.5) \quad \begin{aligned} A_{X,D}(Q) &\geq \delta \quad (\operatorname{centre}_X(Q) = \{x\}), \\ A_{X,D}(E) &= \delta, \\ t \operatorname{ord}_E(\mathfrak{m}_{X,x}) &= A_{X,B}(E) - \delta \geq \delta. \end{aligned}$$

Step 2. In this step, we extract the chosen prime and run an anticanonical MMP while retaining the pointed discrepancy bound. The prime E is exceptional since $n \geq 2$. Apply Theorem 4.8, namely [BCHM10, Corollary 1.4.3], to the rational klt pair (X, D) and $\{E\}$. We obtain a projective birational morphism $f : Y \rightarrow X$ with Y $\mathbb{Q}$ -factorial and E its only exceptional prime; small exceptional loci are allowed. Let D_Y be the strict transform of D , and write

$$\Gamma_Y = D_Y + (1 - \delta)E, \quad K_Y + \Gamma_Y = f^*(K_X + D).$$

This is an effective klt pair, so Y is klt and $\mathbb{Q}$ -Gorenstein. In particular $A_Y(E) = 1$.

The variety Y is of Fano type over X . To verify this, use that every divisor is relatively big for a birational contraction. Write $\Gamma_Y \sim_{\mathbb{Q},X} A + G_0$, with A relatively ample and $G_0 \geq 0$. For a sufficiently small rational $\eta > 0$, the effective boundary $\Theta = (1 - \eta)\Gamma_Y + \eta G_0$ is klt and

$$-(K_Y + \Theta) \sim_{\mathbb{Q},X} \eta A.$$

Theorem 4.7, from [BCHM10, Corollary 1.3.2 and following discussion], applies to this $\mathbb{Q}$ -factorial log Fano pair over the affine base. Run a $-K_Y$ -MMP over X . Since $-K_Y$ is relatively big, it ends in a model $f' : Y' \rightarrow X$ on which $-K_{Y'}$ is nef and semiample.

Let Γ' be the pushforward of Γ_Y . Pushing down at each divisorial contraction, and comparing in codimension one at each flip, preserves the equality

$$K_{Y'} + \Gamma' = f'^*(K_X + D).$$

Thus this pair is klt and Γ' is effective and $\mathbb{Q}$ -Cartier. For every original primitive prime P over x ,

$$(5.6) \quad A_{Y'}(P) \geq A_{Y',\Gamma'}(P) = A_{X,D}(P) \geq \delta.$$

Bare log discrepancies weakly decrease along this $-K_Y$ -MMP. On a common resolution $r : W' \rightarrow Y$, $s : W' \rightarrow Y'$, negativity at the $-K_Y$ -negative steps gives

$$r^*(-K_Y) = s^*(-K_{Y'}) + N_0, \quad N_0 \geq 0.$$

Here one uses Lemma 4.9. Consequently

$$(5.7) \quad A_Y(P) \geq A_{Y'}(P)$$

for every original prime P .

Step 3. In this step, we use relative contact to bound the perturbation coefficient from below. Apply Proposition 5.8 to $f' : Y' \rightarrow X$ and (5.6). It supplies $0 \neq g_1 \in \mathfrak{m}_{X,x}$ with $(Y', u \operatorname{div}(f'^*g_1))$ lc near the fibre. Apply its inequality to the same primitive valuation E , whether or not E survives as a divisor on Y' . By (5.7),

$$u \operatorname{ord}_E(\mathfrak{m}_{X,x}) \leq u \operatorname{ord}_E(g_1) \leq A_{Y'}(E) \leq A_Y(E) = 1.$$

Combining this with (5.5) gives $t \geq \delta u$. The original equation h_0 therefore satisfies $(X, B + c_{\text{ct}}(n, \epsilon) \operatorname{div}(h_0))$ lc. Only n and ϵ entered the chosen constants.

Step 4. In this step, we transfer the ideal inequalities and choose a contact equation over the original field. For general k , use Lemma 3.11 to realize the pair, point, a Cartier multiple of $K_X + B$, and the fixed resolution principalizing its maximal ideal over a common algebraically closed field embedded in both k and $\mathbb{C}$. The coefficients on its log resolution preserve the klt condition, and the cited lemma preserves the closed-point minimum. The complex assertion implies log canonicity of the ideal pair with exponent c_{ct} , because the order of the ideal is at most the order of its contact equation. On the fixed resolution this is exactly the finite set of coefficient inequalities

$$\text{coeff}_P(B_Z + c_{\text{ct}}F_0) \leq 1.$$

They are unchanged under either field extension. The ideal pair is therefore lc over k . Since $c_{\text{ct}} \leq 1/2$, Lemma 5.3 gives a general k -linear combination $0 \neq g \in \mathfrak{m}_{X,x}$ with the required lc property. This transfers the ideal inequalities and chooses the equation over the original field.

Finally, membership in $\mathfrak{m}_{X,x}$ and log canonicity give, for every primitive prime E centred at x ,

$$\text{ord}_E(\mathfrak{m}_{X,x}) \leq \text{ord}_E(g), \quad c_{\text{ct}} \text{ord}_E(g) \leq A_{X,B}(E).$$

These are the two asserted contact bounds. $\square$

6. ORIGINAL COMPUTING ORBITS

We retain the original cover data of Definition 3.10. We first bound the orbits of divisors of discrepancy less than one on a rational surface. The threefold argument uses this bound for the traces of the prescribed divisors. At discrepancy one, a Cartier perturbation gives a finite set of discrepancies strictly below one while keeping the cover and the prescribed divisors fixed.

Theorem 6.1 (Minimal rational group surfaces). *Let a finite group G act on a smooth projective rational surface S . Suppose S admits no nontrivial G -equivariant birational morphism to a smooth surface. Then either S is del Pezzo with $\text{rk Pic}(S)^G = 1$, or it has a G -equivariant conic bundle $S \rightarrow \mathbb{P}^1$ with $\text{rk Pic}(S)^G = 2$ [DI09, Theorem 3.8].*

These two models allow us to count the possible centres of a valuation without bounding the number of exceptional curves on the original surface.

Theorem 6.2 (Moduli degree on an lc-trivial fibration). *Let $f : (S, B) \rightarrow C$ be a projective contraction from a normal variety to a smooth projective curve, and let D be a rational divisor on C . Let $A^*(S, B)$ be the b -divisor whose trace on a model with discrepancy divisor $\sum a_i E_i$ is $\sum_{a_i > -1} a_i E_i$. Assume B is rational, (S, B) is sub-lc over the generic point of C , $K_S + B \sim_{\mathbb{Q}} f^*D$, and*

$$\text{rank } f_* \mathcal{O}_S([A^*(S, B)]) = 1.$$

For each closed $p \in C$, set $b_p = 1 - \text{lt}_{\eta_p}(S, B; f^*p)$, $B_C = \sum_p b_p p$, and $M_C = D - K_C - B_C$. Then M_C is nef [FG14, Definition 3.2, Section 3.4 and Theorem 3.6].

For an effective lc pair over $\mathbb{P}^1$ with trivial log canonical class, this bounds the degree of the discriminant by two. We check the rank condition when applying the theorem.

Lemma 6.3 (Equivariant plt extraction). *Let a finite group G preserve a klt germ $(X \ni x, B)$ and fix x . There is a projective G -equivariant birational morphism $\pi : Y \rightarrow X$ extracting a single G -invariant prime divisor E over x , such that $-E$ is π -ample and $(Y, E + B_Y)$ is plt, with B_Y the strict transform of B [Mor21, Lemma 2.18, with nef part zero].*

We use the reduced exceptional prime to read the action on the traces of the original divisors.

Theorem 6.4 (Plt adjunction and the different). *Let X be a normal variety, S a reduced divisor, and $B = \sum_j b_j B_j \geq 0$ rational with $\lfloor B \rfloor = 0$ and no component in common with S . Assume $K_X + S + B$ is $\mathbb{Q}$ -Cartier. Then $(X, S + B)$ is plt near S if and only if S is normal*

and $(S, \text{Diff}_S B)$ is klt. In the plt case, for a prime $P \subset S$, let m_P be the local index of S at its generic point. There are nonnegative integers n_{Pj} such that the different coefficient at P is

$$\frac{m_P - 1 + \sum_j b_j n_{Pj}}{m_P}$$

[Pro01a, Theorem 2.2.6 and Corollary 2.2.8].

The contributions of all boundary components will bound how many original divisors can contain the same trace.

Theorem 6.5 (Plt surface germs). *A normal surface germ with a reduced curve $(S \ni p, C)$ for which (S, C) is plt is analytically isomorphic to $(\mathbb{C}^2 \ni 0, \{x = 0\})/\mu_r(1, a)$ for some integers $r \geq 1$ and a with $\gcd(a, r) = 1$ [Pro01a, Theorem 2.1.2].*

On this smooth covering germ the boundary contributions are ordinary intersection multiplicities.

Theorem 6.6 (A rationally connected subvariety in a birational fibre). *Let (X, Δ) be an effective klt pair over a field of characteristic zero and let $X' \rightarrow X$ be proper birational. For every point $p \in X$, its fibre contains a geometrically rationally connected subvariety defined over $\kappa(p)$ [HX09, Theorem 1.3].*

We apply this result to a twisted pair over $\mathbb{C}(t)$. The field of definition of the subvariety is essential.

Theorem 6.7 (Rational points over a complex function field). *Let C be a smooth curve and V a smooth proper geometrically rationally connected variety over $\mathbb{C}(C)$. Then $V(\mathbb{C}(C)) \neq \emptyset$ [GHS03, Theorems 1.1–1.2].*

After resolving the subvariety in the preceding theorem, a rational point and specialization will give a fixed point in the original fibre.

Lemma 6.8. *Let $U \ni P$ be a smooth surface germ, and let a finite group K fix P and preserve an effective rational divisor D . Suppose (U, D) is lc and every positive coefficient of D is at least $1/M$, where $M \geq 1$ is an integer. Let r be the number of irreducible analytic branches of $\text{Supp}(D)$ at P , and let K_0 be the kernel of the permutation action on these branches. Then*

(i) $r \leq 2M$ and $[K : K_0] \leq (2M)!$.

(ii) K_0 fixes every primitive prime divisor F centred at P with $A_{U,D}(F) < 1$.

In particular, each such divisor has K -orbit of size at most $(2M)!$.

The same subgroup fixes all the indicated valuations, including those appearing after arbitrarily many point blowups.

Proof. We bound the number of branches and then follow the blowups that can produce a positive crepant boundary coefficient.

Step 1. In this step, we bound the number of branches and the index of their permutation kernel. The ordinary blowup of P has discrepancy $2 - \text{mult}_P(D)$. Log canonicity therefore gives

$$\frac{r}{M} \leq \text{mult}_P(D) \leq 2, \quad [K : K_0] \leq r! \leq (2M)!.$$

Step 2. In this step, we follow the equivariant resolution and node blowups to fix every prime of discrepancy less than one. Resolve the reduced curve germ by point blowups at the points where its total transform is not simple normal crossing. Every centre is fixed by K_0 . Indeed, an invariant irreducible analytic branch has one point above P on its normalization, and consequently one successive infinitely near point at each stage. The other possible centres are intersections of exceptional curves. The exceptional graph over a smooth point is a tree, and two of its curves meet at most once. Once these curves are individually invariant, their intersection point is

fixed. Induction shows that every exceptional curve and every strict branch in this resolution is individually invariant.

Let D_R be the crepant boundary on the resulting smooth model. Its coefficients are at most one and may be negative. A further point blowup at a point on zero, one, or two boundary components has exceptional coefficient respectively

$$-1, \quad d_1 - 1, \quad d_1 + d_2 - 1.$$

If the centre is outside the boundary nodes, the new coefficient is nonpositive. All subsequent exceptional coefficients above that centre remain nonpositive: at most one strict transform of an original local branch can occur, its coefficient is at most one, and all the other components have nonpositive coefficients. Thus a divisor with positive crepant coefficient cannot arise after a sequence of centres leaves the nodes. A divisor with $A_{U,D}(F) < 1$ has positive crepant coefficient, so it is either already present or is obtained by successive node blowups. Each such node is fixed, since the resolution tree and its strict-branch legs are individually invariant. Subdividing an edge preserves this property. Every resulting prime, and hence its primitive valuation, is fixed by K_0 . $\square$

Lemma 6.9. *Let S be a smooth projective surface, let $D \geq 0$ be rational, and let $M \geq 1$ be an integer. Suppose (S, D) is lc and $M(K_S + D)$ is integral Cartier and linearly equivalent to zero. Let a finite group K act on S , preserving D . Assume one of the following conditions.*

- (i) *$-K_S$ is ample and $K_S^2 \leq 9$.*
- (ii) *There is a K -equivariant projective contraction $f : S \rightarrow \mathbb{P}^1$ with general fibre $\mathbb{P}^1$. Every fibre is either smooth and reduced or consists of two smooth reduced curves meeting transversely at one point. For $t \in \mathbb{P}^1$, let τ_t be the supremum of the real numbers $u \geq 0$ for which $(S, D + uf^*t)$ is lc near $f^{-1}(t)$. Assume that $\sum_t (1 - \tau_t)t$ is an effective divisor of degree at most two.*

Then every primitive prime divisor F over S with $A_{S,D}(F) < 1$ has K -orbit of size at most $18M^2(2M)!$.

This estimate counts the possible centres of a valuation and then uses Lemma 6.8 at each centre. In the second case only fibres with a positive discriminant coefficient enter the count.

Proof. We use the Cartier multiple to obtain a discrepancy grid, then count the possible centres on each model. The integral Cartier multiple gives

$$A_{S,D}(F) \in M^{-1}\mathbb{Z}_{\geq 0}$$

for every primitive prime F : pull back $M(K_S + D)$ to a smooth model $\tilde{S}$ carrying F and compare it with the integral divisor $MK_{\tilde{S}}$. In particular, $A_{S,D}(F) < 1$ implies $A_{S,D}(F) \leq 1 - 1/M$. Also MD is integral, so every positive coefficient of D is at least $1/M$.

Step 1. In this step, we bound the valuation orbits when the anticanonical divisor is ample. Put $d = K_S^2$ and write $R = \text{Supp}(D) = \sum_{i=1}^{\ell} C_i$ as a reduced divisor. Since $R \leq MD$, ampleness gives

$$\ell \leq (-K_S) \cdot R \leq Md \leq 9M.$$

A prime on S with discrepancy less than one is a component of R , so this already bounds its orbit.

An exceptional prime with discrepancy less than one is centred at a singular point of R . At any other point, the nonpositive coefficient argument in Lemma 6.8 applies. Normalization of each integral curve gives $p_a(C_i) = g(\tilde{C}_i) + \sum_P \delta_P(C_i)$, and each singular point contributes at least one. Distinct curves contribute at least one to their intersection number at every intersection point. Adjunction and the Hodge index theorem give

$$\# \text{Sing}(R) \leq \sum_i p_a(C_i) + \sum_{i < j} C_i \cdot C_j = \ell + \frac{R^2 + K_S \cdot R}{2} \leq \ell + \frac{R^2}{2},$$

$$R^2 \leq \frac{((-K_S) \cdot R)^2}{d} \leq M^2 d.$$

Consequently

$$\# \text{Sing}(R) \leq 9M + \frac{9M^2}{2} \leq 18M^2.$$

This is a K -invariant finite set. The orbit of a centre has size at most $18M^2$, and its stabilizer has valuation orbits of size at most $(2M)!$ by Lemma 6.8. Multiplying the two bounds proves the assertion in this case.

Step 2. In this step, we bound the valuation orbits using the discriminant of the conic bundle. Write $D = D_{\text{hor}} + D_{\text{ver}}$ according to whether a component dominates $\mathbb{P}^1$. For a general fibre C , adjunction and $K_S + D \sim_{\mathbb{Q}} 0$ give

$$D_{\text{hor}} \cdot C = D \cdot C = 2.$$

There are at most $2M$ horizontal boundary components. A prime valuation dominating the base is the order of its curve centre on S ; if its discrepancy is less than one, this curve is one of those components.

For a vertical prime F of discrepancy less than one, let t be its image on the base. The positive integer $\text{ord}_F(f^*t)$ and the definition of τ_t give

$$\tau_t \leq \frac{A_{S,D}(F)}{\text{ord}_F(f^*t)} \leq A_{S,D}(F) \leq 1 - \frac{1}{M}.$$

Thus $1 - \tau_t \geq 1/M$. The discriminant degree assumption bounds the number of such base points by $2M$. Their set is K -invariant. If F is on S , it is one of at most two components of this fibre, and its orbit has size at most $4M$.

Otherwise its centre P is either on $\text{Supp}(D_{\text{hor}}) \cap f^{-1}(t)$ or is the node of a reducible fibre. At all other points the boundary is empty or has only one smooth local branch, which cannot support an exceptional prime of discrepancy less than one. Intersection with the total fibre gives

$$\#(\text{Supp}(D_{\text{hor}}) \cap f^{-1}(t)) \leq M D_{\text{hor}} \cdot f^*t = 2M.$$

Including the possible node gives a set of at most $2M + 1$ points invariant under the stabilizer of t . Applying Lemma 6.8 to the stabilizer of P bounds the orbit of F by

$$(2M)(2M + 1)(2M)! \leq 18M^2(2M)!.$$

The horizontal curves, fibre components, and exceptional primes exhaust the possible centres, proving the assertion. $\square$

Theorem 6.10. *Fix a positive integer M . Let Z be a normal projective rational klt surface, and let $B \geq 0$ be a rational divisor such that (Z, B) is lc and $M(K_Z + B)$ is integral Cartier and linearly equivalent to zero. Let a finite group H act regularly on Z , preserving B . Then every primitive prime divisor E over Z with $A_{Z,B}(E) < 1$ has H -orbit of size at most $18M^2(2M)!$. The action of H need not be faithful.*

The assertion includes valuations whose centres become points on an auxiliary birational model.

Proof. We pass crepantly to a smooth equivariantly minimal surface and verify the hypotheses of Lemma 6.9. The kernel of the action fixes the function field and all its valuations, so the geometric constructions may use the faithful image of H .

Step 1. In this step, we obtain an effective boundary on the equivariant minimal resolution. Let $\pi : Y \rightarrow Z$ be the minimal resolution. Its uniqueness lifts the group action. Write

$$K_Y + B_Y = \pi^*(K_Z + B), \quad B_Y = B^{\text{str}} + \sum_i c_i C_i,$$

where C_i are the exceptional curves. Minimality and adjunction give $K_Y \cdot C_i \geq 0$. Indeed, if $p_a(C_i) = 0$, then C_i is a smooth rational curve and $C_i^2 \leq -2$; if $p_a(C_i) \geq 1$, the same inequality follows from $C_i^2 < 0$. Since $B^{\text{str}} \cdot C_i \geq 0$, we have

$$\sum_i c_i (C_i \cdot C_j) = -(K_Y + B^{\text{str}}) \cdot C_j \leq 0.$$

The exceptional intersection matrix is negative definite with nonnegative off-diagonal entries, and its inverse has nonpositive entries. Thus $c_i \geq 0$. Log canonicity gives $c_i \leq 1$, so B_Y is effective. The same integer M makes $M(K_Y + B_Y)$ integral Cartier and linearly trivial.

Step 2. In this step, we pass crepantly to a smooth equivariantly minimal rational surface. Contract disjoint group orbits of exceptional curves of the first kind as long as a smooth equivariant contraction is possible. The Picard number decreases, so this ends at a smooth rational surface S minimal for equivariant morphisms to smooth surfaces. Let D be the pushforward of B_Y . Pushing down the principal divisor equation for $M(K_Y + B_Y)$ shows that $M(K_S + D)$ is the divisor of the same rational function. Pulling it back proves that the contraction is crepant. Therefore $D \geq 0$, (S, D) is lc, and every original valuation retains its discrepancy and its group orbit.

By Theorem 6.1, either S is del Pezzo or it has an equivariant conic bundle $f : S \rightarrow \mathbb{P}^1$. A smooth rational del Pezzo surface has $1 \leq K_S^2 \leq 9$, giving the first alternative of Lemma 6.9.

Step 3. In this step, we verify the discriminant bound needed for the conic-bundle alternative. In the second case the fibres are reduced conics. A singular fibre has two smooth rational components meeting transversely at one point. The morphism has connected fibres, the pair is effective lc, and $K_S + D \sim_{\mathbb{Q}} 0$. Over the generic point of the base its smooth fibre is a genus-zero curve. The coefficients of the effective boundary there lie in $[0, 1]$. Coefficients strictly less than one give rounded-up discrepancy zero, and coefficient-one components are omitted from A^* . Hence $\mathcal{O}([A^*])$ on the generic fibre is its structure sheaf, whose space of sections has dimension one. This verifies the rank hypothesis of Theorem 6.2.

Let τ_t be the fibre thresholds and put $D_{\mathbb{P}^1} = \sum_t (1 - \tau_t)t$. That theorem gives a nef moduli divisor $M_{\mathbb{P}^1}$ with

$$K_{\mathbb{P}^1} + D_{\mathbb{P}^1} + M_{\mathbb{P}^1} \sim_{\mathbb{Q}} 0.$$

Log canonicity gives $\tau_t \geq 0$. Each fibre has a reduced component whose discrepancy is at most one, so $\tau_t \leq 1$. Thus $D_{\mathbb{P}^1} \geq 0$ and $\deg D_{\mathbb{P}^1} \leq 2$. The second alternative of Lemma 6.9 applies. That lemma bounds every valuation on the smooth model; crepancy and the unchanged action on the function field give the required bound for every original E . $\square$

Lemma 6.11. *For every rational $b \in (0, 1)$, there is an integer $N(b) > 0$ with the following property. Let S be a normal projective surface of Fano type, let a finite cyclic group K act on S , and let R be an invariant reduced divisor. Assume (S, bR) is lc and that there is an effective rational divisor $D_0 \geq bR$ such that (S, D_0) is lc and $K_S + D_0 \sim_{\mathbb{Q}} 0$. Then a single subgroup of K of index at most $N(b)$ preserves every prime component of R . The action on S need not be faithful.*

No bound on the number of components of R is assumed. The complement is constructed on the quotient to keep its index independent of the group order.

Proof. We construct an invariant complement of bounded index on the original surface, then apply Theorem 6.10. Let $\overline{K}$ be the faithful image of K . Write $\pi : S \rightarrow Q = S/\overline{K}$ for the quotient, and define Δ_Q by

$$K_S + bR = \pi^*(K_Q + \Delta_Q).$$

Choose an integer ℓ clearing bR and making $\ell(K_S + bR)$ Cartier. On an invariant affine neighbourhood of an orbit, choose a section of its natural log-canonical line nonzero at an orbit point. Since $\overline{K}$ is cyclic, one of its character projections is nonzero there, and equivariance

makes that eigen-section nonzero along the whole orbit. Shrinking gives the local eigen-generator required by Lemma 3.8. Its finite crepant comparison gives coefficients

$$1 - \frac{1 - \delta}{u}, \quad \delta \in \{0, b\}, \quad u \geq 1.$$

Thus (Q, Δ_Q) is effective lc, with coefficients in $\Phi(\{1, 1 - b\})$, in the notation of Theorem 4.12. All quotient comparisons here are local comparisons of Cartier multiples of the total log divisor.

Step 1. In this step, we construct a nef model of the negative adjoint on the quotient. Average a klt log Fano boundary on S over the finite group. The average is still klt, and its negative log canonical class is ample. Its finite crepant quotient is effective klt by Lemma 3.8; finite descent of ampleness makes it a log Fano boundary on Q . In particular Q is of Fano type. Average D_0 as well and descend the result. This gives an effective lc divisor $\Gamma_Q \geq \Delta_Q$ with $K_Q + \Gamma_Q \sim_{\mathbb{Q}} 0$. Consequently $-(K_Q + \Delta_Q)$ is pseudo-effective.

A klt surface is $\mathbb{Q}$ -factorial. Apply Theorem 4.7 to a log Fano boundary on Q and run the divisor MMP for $-(K_Q + \Delta_Q)$. It terminates at a nef model Q' . In dimension two its birational steps are divisorial contractions, and the resulting surfaces remain of Fano type by Lemma 4.14. At each contraction $f_i : Q_i \rightarrow Q_{i+1}$ with exceptional curve C_i , put $L_i = K_{Q_i} + \Delta_i$. Since $L_i \cdot C_i > 0$,

$$L_i - f_i^* L_{i+1} = d_i C_i, \quad d_i = \frac{L_i \cdot C_i}{C_i^2} < 0.$$

The pushforwards of Γ_Q remain effective lc complements: their principal log canonical multiples pull back to the same principal divisors. Hence each (Q_i, Δ_i) is lc, and its nonzero boundary coefficients still belong to the same hyperstandard set.

Step 2. In this step, we obtain an invariant complement whose index depends only on b . Theorem 4.12 supplies an lc complement $\Omega' \geq \Delta_{Q'}$ with index M_b depending only on b . Pull it back crepantly through the contractions. At each step the identity

$$\Omega_i - \Delta_i = f_i^*(\Omega_{i+1} - \Delta_{i+1}) + f_i^* L_{i+1} - L_i \geq 0$$

shows that the pulled-back boundary dominates Δ_i . Both terms are effective: the first is the pullback of an effective $\mathbb{Q}$ -Cartier divisor, and the second is $-d_i C_i$. Crepancy preserves log canonicity and the integral principal multiple $M_b(K_{Q_i} + \Omega_i)$. Finite crepant pullback from Q to S now gives a K -invariant effective boundary $D_* \geq bR$ with (S, D_*) lc and

$$M_b(K_S + D_*) \sim 0$$

integral Cartier. In particular no group order enters M_b .

Step 3. In this step, we find one subgroup of bounded index fixing every original component. The surface S is klt and rational. For rationality, apply Theorem 4.13 to a log Fano boundary. A smooth resolution is then rationally connected. A smooth rationally connected surface has Kodaira dimension minus infinity and irregularity zero. Its minimal model is $\mathbb{P}^2$ or a ruled surface with genus-zero base, so it is rational. Theorem 6.10 applies to (S, D_*) . Each prime C of R has $A_{S, D_*}(C) \leq 1 - b < 1$, and therefore has orbit size at most

$$L_b = 18M_b^2(2M_b)!$$

Set $N(b) = \text{lcm}\{1, \dots, L_b\}$. In the original cyclic group K , a subgroup of index $d \leq L_b$ contains every $N(b)$ -th power, since $d \mid N(b)$. Thus the subgroup generated by these powers fixes every such C simultaneously. Its index is $\gcd(|K|, N(b)) \leq N(b)$. This also handles a nonfaithful action, whose kernel fixes every component. $\square$

Lemma 6.12. *Let a finite cyclic group Q preserve an effective rational klt pair (U, Δ) on a normal quasi-projective variety, and let $u \in U$ be a fixed closed point. Let $h : Y \rightarrow U$ be a Q -equivariant projective birational morphism with Y normal. Then $h^{-1}(u)$ contains a Q -fixed closed point. The action need not be faithful.*

The assertion concerns the full cyclic group. We obtain its fixed point from a rational point on a twisted fibre.

Proof. We descend along a cyclic extension, find a rational point on the twisted fibre, and specialize to a fixed point in the original fibre.

Step 1. In this step, we descend the equivariant pair and morphism along a cyclic field extension. If Q is trivial, any closed point of the fibre suffices. Otherwise choose a generator σ , put $q = |Q|$, and let

$$L = \mathbb{C}(v), \quad L_0 = \mathbb{C}(t), \quad t = v^q.$$

Identify $\text{Gal}(L/L_0)$ with Q by $v \mapsto \zeta v \leftrightarrow \sigma$, where ζ is a primitive q -th root of unity. Give the base-changed diagram its diagonal semilinear action. Descent produces

$$h^{\text{tw}} : Y^{\text{tw}} \longrightarrow U^{\text{tw}}, \quad (U^{\text{tw}}, \Delta^{\text{tw}}), \quad u^{\text{tw}} \in U^{\text{tw}}(L_0).$$

This descent is effective: a tensor product of translates of an ample bundle gives an invariant linearized ample bundle on a quasi-projective model. The same construction with a relatively ample bundle preserves projectivity of the morphism. The descended pair is effective klt, as can be checked after the separable extension L/L_0 on a log resolution.

Step 2. In this step, we obtain a rational point on the twisted fibre. By Theorem 6.6, the projective fibre over u^{tw} contains a geometrically rationally connected subvariety defined over L_0 . Take its closure and a smooth projective resolution. Theorem 6.7 gives a rational point on this resolution, and hence on the twisted fibre. After extension to L , it is an L -point of the original fibre $h^{-1}(u)$, satisfying the diagonal descent condition.

Step 3. In this step, we specialize the rational point to a point fixed by the full cyclic group. Regard this point as a rational map from the v -line to $h^{-1}(u)$. Properness extends it over the discrete valuation ring $\mathbb{C}[v]_{(v)}$. The descent condition identifies substitution $v \mapsto \zeta v$ with the action of σ or its inverse, according to the chosen descent convention. Separatedness extends this equality across $v = 0$. The substitution acts trivially on the residue field there, so the specialized point is fixed by σ , and hence by all of Q . $\square$

Lemma 6.13. *Fix a rational $b \in (0, 1)$, and let $N(b)$ be the integer in Lemma 6.11. Let a finite cyclic group Q fix a closed point P on a normal threefold Z . Let $D = \sum_i D_i$ be an invariant reduced $\mathbb{Q}$ -Cartier divisor whose every prime component contains P . Let $\Delta \geq 0$ be rational and invariant, with no component in common with D , and suppose $(Z, \Delta + bD)$ is klt near P . Then each D_i has Q -orbit of size at most*

$$N(b)(\lceil 1/b \rceil - 1).$$

The bound allows a kernel acting trivially on the auxiliary surface: it also controls how many original divisors can have the same surface trace.

Proof. We use a plt extraction to produce curve traces on a log Fano surface. Adjunction bounds the number of divisors sharing a trace, and Lemma 6.11 bounds the permutation of all traces.

Step 1. In this step, we construct the exceptional log Fano surface by an equivariant plt extraction. Apply Lemma 6.3 to $(Z \ni P, \Delta + bD)$. It gives an equivariant extraction $\mu : Z' \rightarrow Z$ with one invariant $\mathbb{Q}$ -Cartier prime S , with $-S$ relatively ample, and with $(Z', S + \Delta' + bD')$ plt. Primes denote strict transforms. Here S is reduced. In the finite-quotient construction of the cited lemma, a coefficient-one prime with ramification index j has upstairs crepant coefficient

$$j - (j - 1) = 1$$

by the finite comparison of Lemma 3.8. Thus ramification multiplicity is not assigned as a boundary coefficient to the exceptional prime.

The reduced fibre of μ over P is S . Indeed, a contracted curve outside S has nonnegative intersection with the effective $\mathbb{Q}$ -Cartier divisor S , contradicting relative ampleness of $-S$. Any

additional isolated component is excluded by connectedness of proper birational fibres over a normal base. Set $\alpha = A_{Z, \Delta + bD}(S) > 0$. Then

$$K_{Z'} + S + \Delta' + bD' = \mu^*(K_Z + \Delta + bD) + \alpha S.$$

Theorem 6.4 gives a normal projective surface S and an effective klt different Θ satisfying

$$-(K_S + \Theta) \sim_{\mathbb{Q}} \alpha(-S|_S).$$

The right-hand side is ample, so S is of Fano type.

Step 2. In this step, we retain the complete curve traces and bound their multiplicities in adjunction. Each D'_i meets S , since its proper image is D_i and $P \in D_i$. At an intersection point an equation for a positive Cartier multiple of S is a nonzero nonunit on the integral surface D'_i . Its zero locus has a curve component. Consequently D'_i has a complete prime curve trace on S . Let R be the reduced union of all these traces. It is $\mathbb{Q}$ -invariant.

For a prime C of R , the different formula in Theorem 6.4 reads

$$\text{coeff}_C \Theta = 1 - \frac{1}{d_C} + \frac{bn_C + v_C}{d_C}, \quad d_C \geq 1, \quad v_C \geq 0.$$

Here n_C is a positive integer at least the number of original strict transforms D'_i containing C . To see the positivity of each contribution, take a transverse surface at a general point of C and use Theorem 6.5. On its smooth cyclic cover the pullback of each such D'_i meets the reduced floor with positive integral intersection multiplicity. This does not require individual $\mathbb{Q}$ -Cartierness of the D'_i on Z' . The term v_C retains all contributions from Δ' .

Since (S, Θ) is klt, $bn_C + v_C < 1$, whence

$$\#\{i : C \subset D'_i\} \leq n_C \leq \lceil 1/b \rceil - 1.$$

Moreover

$$\text{coeff}_C \Theta - bn_C = \left(1 - \frac{1}{d_C}\right)(1 - bn_C) + \frac{v_C}{d_C} \geq 0.$$

It follows that $\Theta \geq bR$. The surface is $\mathbb{Q}$ -factorial, so decreasing the boundary gives a klt pair (S, bR) . Choose a large divisible j and a general member A of the base-point-free system $|-j(K_S + \Theta)|$. Bertini on a log resolution shows that $\Theta + A/j$ is klt for sufficiently large j ; it is an effective rational complement dominating bR .

Step 3. In this step, we bound the orbit of each original component through its surface trace. Lemma 6.11 gives a subgroup Q_0 of index at most $N(b)$ fixing all prime traces. Choose a trace C of D'_i . Every member of its Q_0 -orbit contains C , so that orbit has size at most $\lceil 1/b \rceil - 1$. Multiplication by $[Q : Q_0]$ gives the stated bound. Strict transformation is an equivariant bijection on the original prime divisors, so it has not changed their orbit sizes. $\square$

Proposition 6.14. *Let $a \in \mathbb{Q} \cap (0, 1)$, set $b = \min\{1/2, 1 - a\}$, and use the integer $N(b)$ of Lemma 6.11 to define*

$$C_{<1}(a) = N(b)(\lceil 1/b \rceil - 1).$$

Let $(X \ni x, B)$ be a klt threefold germ over an algebraically closed field of characteristic zero, with rational boundary. Let $\rho : (V \ni o) \rightarrow (X \ni x)$ be a connected normal finite cyclic Galois cover, quasi-etale and with a single underlying point over x , and write G for its deck group. Let E be any primitive prime divisor centred at x with $A_{X,B}(E) = a$. Then every primitive extension F to $k(V)$ satisfies

$$[G : \text{Stab}_G(F)] \leq C_{<1}(a).$$

The prescribed divisor need not compute a minimum. This permits the same cover and prime to be used after perturbing the boundary.

Proof. We extract the prescribed prime, normalize in the given covering field, and divide by its inertia subgroup. The resulting effective exceptional boundary allows us to apply Lemma 6.13 at a common fixed point.

Step 1. In this step, we transfer the marked diagram to the complex field and extract the prescribed prime. We restrict to an affine neighbourhood of x . Apply Lemma 3.11 to the pair, the finite cover, a model carrying E , and the normalization of that model with all primitive lifts labelled. This finite diagram has a common realization over an algebraically closed subfield of finite transcendence degree and over $\mathbb{C}$. The discrepancy of the marked prime and the permutation of all its primitive lifts are preserved. Thus it suffices to prove the assertion over $\mathbb{C}$.

Theorem 5.2 gives a projective birational morphism $\pi : Y \rightarrow X$ extracting exactly E , with E $\mathbb{Q}$ -Cartier and $-E$ relatively ample. Its reduced fibre over x is E , by the relative-ameness and connectedness argument in Lemma 6.13. With B_Y the strict transform, we have

$$K_Y + B_Y + (1 - a)E = \pi^*(K_X + B).$$

Step 2. In this step, we normalize in the original covering field and form the quotient by inertia. Normalize the main component of $Y \times_X V$, obtaining $\nu : W \rightarrow Y$ finite and $g : W \rightarrow V$ projective birational. Its reduced exceptional fibre is $T_W = \sum_{i=1}^t F_i$, where the F_i are exactly the primitive extensions of E to $k(V)$. They form one G -orbit. Put $H = \text{Stab}_G(F)$, $I = I_G(F)$, and $e = |I|$. Cyclicity makes I the inertia subgroup of every F_i . The finite valuation comparison of Lemma 3.8 gives

$$t = [G : H], \quad \nu^*E = eT_W.$$

In particular T_W is $\mathbb{Q}$ -Cartier and $-T_W$ is ample over V .

Form the actual quotients

$$U = V/I, \quad Z = W/I, \quad \lambda : W \rightarrow Z, \quad h : Z \rightarrow U.$$

The equivariant projective morphism g descends to the projective birational morphism h . The subgroup I acts trivially on the function field of each F_i , hence on that reduced integral divisor. The images T_i of the F_i are therefore distinct. They are the components of the reduced exceptional fibre $T = \sum_i T_i$, and

$$\lambda^*T = eT_W.$$

Taking norms of local Cartier equations shows that T is $\mathbb{Q}$ -Cartier; finite descent of relative ameness makes $-T$ ample over U . The residual group $Q = G/I$ permutes the T_i in the same orbit of size t .

Step 3. In this step, we identify the effective crepant boundary after the inertia quotient. Let B_U be the finite Weil pullback of B , and let B_Z be its strict transform on Z . The map $U \rightarrow X$ is quasi-etale, so $K_U + B_U$ is the pullback of $K_X + B$, and (U, B_U) is klt by Lemma 3.8. The ramification indices at F_i for $k(V)/k(X)$ and $k(V)/k(U)$ are both e ; in the valuation tower, the primitive valuation of T_i consequently restricts to ord_E with index one. The same lemma gives $A_{U, B_U}(T_i) = a$. Therefore

$$K_Z + B_Z + (1 - a)T = h^*(K_U + B_U).$$

This is an effective klt pair. Its total log divisor is $\mathbb{Q}$ -Cartier by the displayed equality; no separate pullback of a canonical divisor or of the boundary was used.

Step 4. In this step, we obtain a common fixed point and bound the orbit of the original lift. The full group Q fixes the point of U over x . Lemma 6.12, applied to (U, B_U) and h , gives a Q -fixed point $P \in T$. It belongs to one component, and transitivity forces it to belong to all T_i . Since $b \leq 1 - a$ and T is effective $\mathbb{Q}$ -Cartier, $(Z, B_Z + bT)$ is klt. Apply Lemma 6.13 at P with $\Delta = B_Z$ and $D = T$. Its bound is

$$t \leq N(b)(\lceil 1/b \rceil - 1) = C_{<1}(a).$$

Every component counted is the image of an original lift, so $t = [G : H]$ is the required original orbit size. $\square$

Lemma 6.15. *Let $(X \ni x, B)$ be a klt threefold germ over an algebraically closed field of characteristic zero, with rational boundary and $\text{mld}(X \ni x, B) \geq 1$. Set*

$$c = \min\{1/2, c_{\text{ct}}(3, 1)\}, \quad \mathcal{A} = \left\{1 - \frac{cj}{2} : 1 \leq j \leq \lfloor 1/c \rfloor, j \in \mathbb{Z}\right\}.$$

There is a nonzero $g \in \mathfrak{m}_{X,x}$ such that the pair

$$(X, B^\sharp), \quad B^\sharp = B + \frac{c}{2} \text{div}(g),$$

is klt near x , and every primitive prime E centred at x with $A_{X,B}(E) = 1$ satisfies

$$A_{X,B^\sharp}(E) = 1 - \frac{c}{2} \text{ord}_E(g) \in \mathcal{A} \subset [1/2, 1 - c/2] \subset (0, 1).$$

The same g works for all these primes, and $\mathcal{A}$ depends only on the contact constant.

The divisor E need not compute the minimum of the perturbed pair. Its primitive lifts and their stabilizers on any fixed cover are unchanged by this perturbation.

Proof. Theorem 5.9 gives a nonzero $g \in \mathfrak{m}_{X,x}$ such that $(X, B + c \text{div}(g))$ is lc near x . Here decreasing the contact coefficient to c preserves log canonicity. Since $\text{div}(g)$ is Cartier, $K_X + B^\sharp$ is $\mathbb{Q}$ -Cartier. For every primitive prime P over this neighbourhood,

$$A_{X,B^\sharp}(P) = \frac{1}{2} A_{X,B}(P) + \frac{1}{2} A_{X,B+c \text{div}(g)}(P) > 0.$$

Thus the perturbed pair is klt. For a prime E as in the statement, its primitive order and closed centre imply that $j = \text{ord}_E(g)$ is a positive integer. Log canonicity gives $1 - cj \geq 0$, so $1 \leq j \leq \lfloor 1/c \rfloor$. The claimed discrepancy formula and interval follow. $\square$

Theorem 6.16. *For every rational $a \in (0, 1]$, there is an integer $C_{\text{orb}}(a) > 0$ such that for any original cover data of exact minimum a ,*

$$[G : H] \leq C_{\text{orb}}(a).$$

The bound depends only on a , and includes every prescribed original computing divisor over every algebraically closed field of characteristic zero.

The bound concerns the orbit of each prescribed lift. It is independent of the boundary coefficients and of the integer used to construct the original cover.

Proof. Let E be the computing divisor and F its prescribed primitive lift in the original cover data. If $0 < a < 1$, Proposition 6.14 applies to the given cover and gives the result with $C_{\text{orb}}(a) = C_{<1}(a)$.

Suppose $a = 1$. Use the function g , constant c , finite set $\mathcal{A}$, and pair $(X, B^\sharp)$ of Lemma 6.15. Set

$$C_{\text{orb}}(1) = \max_{\alpha \in \mathcal{A}} C_{<1}(\alpha).$$

This is a positive integer depending only on $c_{\text{ct}}(3, 1)$. The same original E has discrepancy in $\mathcal{A}$ for the klt pair $(X, B^\sharp)$. The given cover remains connected, cyclic, quasi-etale, and has one point over x . Thus Proposition 6.14 applies to $(X, B^\sharp)$, this cover, and the same prime E . It does not require E to compute the perturbed minimum. Neither the covering field nor the group action on F has changed, so its bound is exactly $[G : H] \leq C_{\text{orb}}(1)$. The proposition holds over every algebraically closed field of characteristic zero and for every prescribed lift, which proves the full assertion. $\square$

7. TERMINAL PAIR MODELS AND MARKED EXITS

We retain the integral-log cover and the natural linearizations of Section 3. We study effective terminal models and the contraction of a prescribed computing prime. Throughout the section, m is a positive integer.

Theorem 7.1 (Gorenstein terminal threefold points). *A singular threefold point is terminal of canonical index one if and only if it is an isolated compound Du Val point [Rei83, Theorem (1.1), pp. 138–139].*

In particular, the singular Gorenstein terminal case has an isolated hypersurface presentation.

Theorem 7.2 (Discrepancy of a non-Gorenstein terminal point). *Let $X \ni x$ be a terminal threefold germ with canonical index $r > 1$. Then $\text{mld}(X \ni x) = 1 + 1/r$ [Kaw93, Appendix, pp. 193–195].*

Lemma 7.3. *Let V be a normal variety with a faithful action of a finite group H , and suppose that its finite quotient $\rho : V \rightarrow W = V/H$ exists. Let $f : Q \rightarrow W$ be a projective birational morphism with Q normal, and let $\pi : Y \rightarrow Q$ be the normalization in $k(V)$. Then the following hold:*

- (i) *The original field action induces an action of H on Y , with quotient $Y/H = Q$, and an H -equivariant projective birational morphism $g : Y \rightarrow V$.*
- (ii) *For every rational Weil divisor D_0 on Q , the divisor D_0 is $\mathbb{Q}$ -Cartier if and only if π^*D_0 is $\mathbb{Q}$ -Cartier. The variety Y is H - $\mathbb{Q}$ -factorial if and only if Q is $\mathbb{Q}$ -factorial.*
- (iii) *For any two such models Q_1, Q_2 , with normalizations Y_1, Y_2 , a small birational map $Q_1 \dashrightarrow Q_2$ over W induces a small H -equivariant birational map $Y_1 \dashrightarrow Y_2$ over V .*

This construction keeps the original function field while allowing the birational operations to take place on a $\mathbb{Q}$ -factorial quotient.

Proof. We construct the normalized quotient diagram and then compare its divisors and small maps.

Step 1. In this step, we construct the equivariant normalization and identify its quotient. Normalization is finite for varieties. The variety Y is also the normalization of the component of $Q \times_W V$ dominating V . This gives the projective birational morphism g . The action on $k(V)$ preserves every integral closure, so all these maps carry the original H -action.

To identify the quotient, take an affine open of Q with normal coordinate ring A , and let A' be its integral closure in $k(V)$. An element of $(A')^H$ belongs to $k(V)^H = \text{Frac}(A)$ and is integral over A . It therefore belongs to A , proving $(A')^H = A$. These affine identifications give $Y/H = Q$.

Step 2. In this step, we prove $\mathbb{Q}$ -Cartier descent, compare equivariant $\mathbb{Q}$ -factoriality, and lift small maps. The pullback of a $\mathbb{Q}$ -Cartier divisor is $\mathbb{Q}$ -Cartier. For the converse, suppose that π^*D_0 is $\mathbb{Q}$ -Cartier. Over the semilocal fibre of any point of Q , some positive integral multiple $n\pi^*D_0$ has a rational equation u : an invertible module over a semilocal ring is free of rank one. Taking the field norm gives a local equation with divisor

$$\text{div}(\text{Norm}_{k(V)/k(W)} u) = \pi_* \text{div}(u) = |H|nD_0.$$

Thus D_0 is $\mathbb{Q}$ -Cartier at each point of Q .

Now suppose that Q is $\mathbb{Q}$ -factorial, and let D be an H -invariant integral Weil divisor on Y . The primes over each prime of Q form one H -orbit and have a common ramification index. Divide the common coefficient of D on that orbit by this index. The resulting rational Weil divisor D_1 on Q satisfies $D = \pi^*D_1$. Since D_1 is $\mathbb{Q}$ -Cartier, so is D .

Conversely, if Y is H - $\mathbb{Q}$ -factorial, the finite Weil pullback of every integral Weil divisor on Q is invariant and $\mathbb{Q}$ -Cartier. The descent just proved makes the original divisor $\mathbb{Q}$ -Cartier, establishing (ii).

Finally, a small map between Q_1 and Q_2 is an isomorphism between open subsets with complements of codimension at least two. Finite morphisms preserve the dimensions of closed subsets, so their inverse images have complements of codimension at least two in Y_1, Y_2 . Normalization in the same field identifies these inverse images H -equivariantly. This proves (iii). $\square$

Lemma 7.4. *Let (V, B_V) be a klt threefold pair, let $m \geq 1$ be an integer such that $m(K_V + B_V)$ is integral Cartier, and let a finite cyclic group H preserve the pair. Let $g : (Y, B_Y) \rightarrow (V, B_V)$ be an H -equivariant projective birational morphism. Assume that Y is H - $\mathbb{Q}$ -factorial, $B_Y \geq 0$, (Y, B_Y) is terminal, and*

$$K_Y + B_Y = g^*(K_V + B_V).$$

Then the following hold:

- (i) *The underlying variety Y is terminal.*
- (ii) *There is a natural H -equivariant isomorphism*

$$\mathcal{O}_Y(m(K_Y + B_Y)) \simeq g^*\mathcal{O}_V(m(K_V + B_V)).$$

- (iii) *Every log discrepancy of (Y, B_Y) belongs to $m^{-1}\mathbb{Z}$, and, for every closed point $P \in Y$,*

$$\text{mld}(Y \ni P) \geq \text{mld}(Y \ni P, B_Y) \geq 1 + \frac{1}{m}.$$

- (iv) *For every closed point $P \in Y$, one has $\text{ind}_P(K_Y) \leq m$.*

The discrepancy grid gives a uniform gap above one and bounds the canonical index at every closed point of a terminal pair model.

Proof. We first compare the bare and pair discrepancies, then identify the natural line and its discrepancy grid. The terminal-point formula converts this grid into the index bound.

Step 1. In this step, we prove terminality of the underlying variety. The invariant rational divisor B_Y is $\mathbb{Q}$ -Cartier by H - $\mathbb{Q}$ -factoriality. Thus K_Y is also $\mathbb{Q}$ -Cartier. For every prime D exceptional over Y , effectivity gives

$$A_Y(D) = A_{Y, B_Y}(D) + \text{ord}_D(B_Y) > 1.$$

Indeed, a positive Cartier multiple of B_Y has regular local equations, so its order along D is nonnegative. This proves (i).

Step 2. In this step, we identify the natural log line and its discrepancy grid. Multiplying the crepant equality by m identifies the two invertible sheaves in (ii). On the common birational open set the identification is pullback of rational differential tensors. Their orders agree along every prime divisor on Y , so this identification extends over Y with its natural H -action.

On a normal model $f : T \rightarrow Y$ carrying a prime D , we have

$$mA_{Y, B_Y}(D) = m + m \text{coeff}_D(K_T) - \text{coeff}_D(f^*[m(K_Y + B_Y)]) \in \mathbb{Z}.$$

Every prime centred at a closed point of the threefold Y is exceptional over Y . Terminality and this integrality give $A_{Y, B_Y}(D) \geq 1 + 1/m$. The comparison in (i), applied to all these primes and then to their infima, proves (iii).

Step 3. In this step, we use the discrepancy grid to bound each local canonical index. For (iv), fix a closed point P and put $r = \text{ind}_P(K_Y)$, which is finite by (i). If $r = 1$, the assertion follows from $m \geq 1$. For $r > 1$, apply Lemma 3.11 to the existing terminal-model diagram and the marked bare pair $(Y \ni P, 0)$. It gives a common realization $Y_0 \ni P_0$ over an algebraically closed subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$, together with an embedding $k_0 \hookrightarrow \mathbb{C}$. Write $Y_{\mathbb{C}} \ni P_{\mathbb{C}}$ for the base change of this realization through that embedding; the base change of $Y_0 \ni P_0$ to k is $Y \ni P$. Terminality is retained, and the lemma applied to the bare pair preserves

both the exact local canonical index and the infimum over all primes centred at the marked point. Hence

$$\mathrm{ind}_{P_{\mathbb{C}}} K_{Y_{\mathbb{C}}} = \mathrm{ind}_P K_Y = r, \quad \mathrm{mld}(Y_{\mathbb{C}} \ni P_{\mathbb{C}}) = \mathrm{mld}(Y \ni P).$$

Theorem 7.2 applies to this complex terminal threefold germ, with that same index. Together with (iii), it gives

$$1 + \frac{1}{r} = \mathrm{mld}(Y \ni P) \geq 1 + \frac{1}{m}, \quad r \leq m.$$

The marked point was arbitrary, so (iv) holds throughout the model over the chosen neighbourhood. $\square$

Proposition 7.5. *Let $(V \ni o, B_V)$ be a klt threefold germ with $m(K_V + B_V)$ integral Cartier, and let a finite cyclic group H preserve it. There is an H -equivariant projective birational morphism $g : (Y, B_Y) \rightarrow (V, B_V)$ such that Y is H - $\mathbb{Q}$ -factorial, B_Y is effective, (Y, B_Y) is terminal and*

$$K_Y + B_Y = g^*(K_V + B_V).$$

It extracts exactly the exceptional primitive divisors with log discrepancy at most one, including those of discrepancy one. The natural m -log line is the equivariant pullback of the original line. The underlying Y is terminal and $\mathrm{ind}_P(K_Y) \leq m$ for every closed point P over a sufficiently small neighbourhood of o . Here H - $\mathbb{Q}$ -factorial means that every H -invariant integral Weil divisor is $\mathbb{Q}$ -Cartier.

The model retains primes of discrepancy one even though their crepant boundary coefficients are zero.

Proof. We first locate all primes to be extracted, construct the model on the finite quotient, and normalize in the original field. The crepant-model lemma then gives the natural line and the local canonical-index bound.

Step 1. In this step, we prove that only finitely many exceptional primes have discrepancy at most one. Only finitely many exceptional primitive primes over V have log discrepancy at most one. Choose a projective log resolution $f : T \rightarrow V$ by Theorem 3.1, and write

$$K_T + D_T = f^*(K_V + B_V).$$

All coefficients of the simple normal crossing divisor D_T are less than one, though some may be negative.

Blow up intersections of pairs of positive-coefficient components until those components are disjoint. To see that this process is finite, consider a coordinate cone of the initial simple normal crossing divisor, with coefficients $b_1, \dots, b_s < 1$. Every exceptional prime produced by a stratum blowup has a primitive nonnegative integral weight vector w , and its log discrepancy is

$$\sum_{i=1}^s (1 - b_i) w_i.$$

There are only finitely many such vectors with this sum less than one, since each $1 - b_i$ is positive. There are finitely many initial strata and their components. Hence only finitely many positive-coefficient primes can appear. Each blowup separates the chosen pair of strict transforms permanently, and it blows up their whole intersection. There are only finitely many pairs of the possible positive components, so the process terminates.

On the resulting smooth model $f' : T' \rightarrow V$, write $K_{T'} + D_{T'} = (f')^*(K_V + B_V)$. The positive part $(D_{T'})_+$ is a disjoint union of smooth divisors with coefficients less than one. Let F_0 be a prime exceptional over T' , and let $z_1, \dots, z_d$ be regular parameters at its generic centre, where $d \geq 2$. Put $\alpha_i = \mathrm{ord}_{F_0}(z_i) \geq 1$. A logarithmic top differential has at most a simple pole along F_0 ;

multiplying by $z_1 \cdots z_d$ gives $A_{T'}(F_0) \geq \sum_i \alpha_i$. If one positive component, of coefficient $b < 1$, contains the centre, take its equation to be z_1 . Then

$$A_{T', (D_{T'})_+}(F_0) \geq (1 - b)\alpha_1 + \sum_{i=2}^d \alpha_i > 1.$$

If none contains the centre, the same inequality has no subtracted boundary term and is at least $d \geq 2$. Since $(D_{T'})_+ - D_{T'}$ is effective,

$$A_{V, B_V}(F_0) = A_{T', D_{T'}}(F_0) \geq A_{T', (D_{T'})_+}(F_0) > 1.$$

Thus every exceptional prime over V with discrepancy at most one already appears on this fixed model. Their finite set $\mathcal{E}$ is H -invariant.

Step 2. In this step, we extract the primitive restrictions of the selected primes on the quotient. Replace V by an invariant affine neighbourhood of o , and write $\rho : V \rightarrow W = V/H$ for the quotient. An eigenvector lift of the fibre of the natural m -log line at o generates that line after invariant shrinking. Apply Lemma 3.8 to the faithful image of H . Give W its finite crepant boundary Θ , so

$$K_V + B_V = \rho^*(K_W + \Theta).$$

The total divisor downstairs is $\mathbb{Q}$ -Cartier by the norm descent in Lemma 7.3. The coefficient and discrepancy formulas of Lemma 3.8 show that $\Theta \geq 0$ and (W, Θ) is klt.

Let $\mathcal{E}_0$ be the set of primitive restrictions of $\mathcal{E}$ to $k(W)$. Its members are exceptional over W , since finite maps preserve centre dimensions, and have log discrepancy at most one by the same finite comparison. Put this finite set on a common projective log resolution of (W, Θ) . The normal variety W is an excellent Noetherian scheme of equal characteristic zero and has a dualizing complex. The pair is effective and klt, and the selected primes have classical discrepancy at most zero. Thus Theorem 3.7 gives a projective $\mathbb{Q}$ -factorial model $Q \rightarrow W$ extracting exactly $\mathcal{E}_0$.

Step 3. In this step, we normalize in the original field and obtain the effective terminal pair model. For the application of Lemma 7.3, use the faithful image of H ; the original action factors through this image. Normalization of Q in $k(V)$ supplies the projective birational H -equivariant morphism $g : Y \rightarrow V$ and H - $\mathbb{Q}$ -factoriality of Y .

The primes over a prime of Q form one orbit. Since the original pair is invariant, all these extensions have the same discrepancy. Finite morphisms preserve centre dimensions. Consequently the g -exceptional primes are exactly $\mathcal{E}$. Define B_Y by the crepant equality. A strict transform of a component of B_V keeps its coefficient, whereas the coefficient of an exceptional prime $D \in \mathcal{E}$ is $1 - A_{V, B_V}(D) \geq 0$. Hence B_Y is effective. A prime exceptional over Y is exceptional over V ; if its discrepancy were at most one, it would already be a prime on Y . This proves terminality of (Y, B_Y) , including the extraction of discrepancy-one primes with zero coefficient.

Lemma 7.4 now supplies bare terminality, the natural log-line identity, and $\text{ind}_P(K_Y) \leq m$ at every closed point over the chosen neighbourhood of o . $\square$

Theorem 7.6 (Terminal singularities in codimension two). *A terminal variety is smooth in codimension two. In particular a terminal surface is smooth [CKM88, Proposition 6.10 and Corollary 6.13].*

Theorem 7.7 (The ordinary point blowup). *Let $X \ni x$ be a smooth germ of dimension $n \geq 2$, and let $\mathfrak{a}$ be an effective $\mathbb{R}$ -ideal with $\text{ord}_x \mathfrak{a} \leq 1$. The exceptional divisor of the ordinary blowup of x computes $\text{mld}(X \ni x, \mathfrak{a}) = n - \text{ord}_x \mathfrak{a}$. If $\text{ord}_x \mathfrak{a} < 1$, it is the unique computing prime [Kaw17, Proposition 6].*

Theorem 7.8 (Algebraic relative vanishing). *Let $\pi : X \rightarrow V$ be proper between varieties, (X, B) an effective lc pair, and L a $\mathbb{Q}$ -Cartier integral Weil divisor. If $L - (K_X + B)$ is π -nef and π -big and is big on every lc centre over its image, then $R^q \pi_* \mathcal{O}_X(L) = 0$ for $q > 0$. For klt pairs the condition on lc centres is vacuous [Fuj09b, Theorem 2.48 and Corollary 2.49].*

Lemma 7.9. *A terminal variety over an algebraically closed field of characteristic zero is smooth in codimension two.*

Proof. Choose a log resolution of the bare variety, and include it and a Cartier multiple of the canonical divisor in a common realization as in Lemma 3.11. On the resolution the crepant boundary is supported on the exceptional divisor and all its coefficients are negative. These finitely many strict inequalities persist in the complex realization. They imply terminality there: a further exceptional prime has discrepancy at least its discrepancy over the smooth resolution, hence at least two. Apply Theorem 7.6 over $\mathbb{C}$. The nonsmooth locus and its dimension are preserved by algebraically closed field extension. The codimension bound therefore holds over the original field as well. $\square$

Lemma 7.10. *Let $f : M \rightarrow T$ be a projective birational morphism of normal varieties over an algebraically closed field of characteristic zero. Suppose that (M, D) is an effective rational klt pair and $K_M + D = f^*L$ for a $\mathbb{Q}$ -Cartier divisor L on T . If $t \in T$ is a closed point and $Z_t = f^{-1}(t)_{\text{red}}$ has dimension at most one, then*

$$R^i f_* \mathcal{O}_M = 0 \quad (i > 0), \quad H^1(Z_t, \mathcal{O}_{Z_t}) = 0.$$

Proof. We first establish relative vanishing over the given field and then restrict to the reduced fibre.

Step 1. In this step, we establish relative vanishing over the stated ground field. Over $\mathbb{C}$, apply Theorem 7.8 with integral Weil divisor zero. The difference $-(K_M + D) = -f^*L$ is f -nef. It is also f -big, since the generic fibre of the birational morphism has dimension zero. The pair is klt, so there is no lc-centre condition to check. This proves the first assertion over $\mathbb{C}$.

For the original field, realize the morphism, boundary, crepant identity, a log resolution and its discrepancy coefficients over an algebraically closed subfield of finite transcendence degree over $\mathbb{Q}$, as in Lemma 3.11. The resolution retains the effective klt pair after passage to $\mathbb{C}$. Projectivity, birationality and the Cartier identity are retained as parts of the finite diagram. On an affine open of the base, compute cohomology on its inverse image by a finite affine cover. The scheme is separated, so the intersections are affine. Tensoring its Čech complex with a field extension is exact and computes the extended cohomology. Faithful flatness therefore descends the complex vanishing to the field of realization and then transfers it to the original field. No resolution of the target and no separate $\mathbb{Q}$ -Cartierness of K_T is used.

Step 2. In this step, we use a two-affine-open cover to obtain vanishing on the reduced fibre. Restrict to an affine neighbourhood T' of t and choose a projective embedding $f^{-1}(T') \subset \mathbb{P}_{T'}^n$. Choose a hyperplane containing no component of Z_t , and a second hyperplane avoiding the finite intersection of the first with Z_t . For a zero-dimensional fibre choose the first hyperplane disjoint from it. The two hyperplanes have no common point on the fibre. Their common intersection with $f^{-1}(T')$ has closed image by properness; shrink T' to an affine neighbourhood avoiding that image. The two hyperplane complements give an affine cover of $f^{-1}(T')$, with affine intersection. Consequently every coherent sheaf on $f^{-1}(T')$ has vanishing second cohomology.

Let $\mathcal{I}$ be the ideal of the reduced fibre on this inverse image. The cohomology sequence of

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_{Z_t} \longrightarrow 0$$

places $H^1(Z_t, \mathcal{O}_{Z_t})$ between $H^1(f^{-1}(T'), \mathcal{O}_M) = 0$ and $H^2(f^{-1}(T'), \mathcal{I}) = 0$. This proves the second assertion directly for the reduced fibre. $\square$

Lemma 7.11. *Let Z be a nonempty reduced connected projective scheme over k , with $\dim Z \leq 1$ and $H^1(Z, \mathcal{O}_Z) = 0$. Let a finite cyclic group K act on Z . Then $H^0(Z, \mathcal{O}_Z) = k$, and Z has a K -fixed closed point. If $\dim Z = 1$, every irreducible component of Z is a smooth projective $\mathbb{P}^1$, and the bipartite graph of components and points on at least two components is a tree, with one edge for each incidence.*

The junction vertices allow several components to meet at one point; no nodal hypothesis is imposed on Z .

Proof. We first determine the components and then use normalization to determine the incidence graph. Properness and reducedness make $H^0(Z, \mathcal{O}_Z)$ a finite reduced k -algebra. Connectedness therefore gives $H^0(Z, \mathcal{O}_Z) = k$. If $\dim Z = 0$, the scheme Z is a single reduced point, which is fixed by K .

Step 1. In this step, we identify each component as a smooth rational curve. Suppose $\dim Z = 1$. Connectedness excludes an isolated zero-dimensional component, so Z is pure of dimension one. For an irreducible component C_1 , the ideal sequence

$$0 \longrightarrow \mathcal{I}_{C_1} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{C_1} \longrightarrow 0$$

gives $H^1(C_1, \mathcal{O}_{C_1}) = 0$, since a coherent sheaf on Z has no second cohomology. For the normalization $\nu_1 : \tilde{C}_1 \rightarrow C_1$, write

$$0 \longrightarrow \mathcal{O}_{C_1} \longrightarrow (\nu_1)_* \mathcal{O}_{\tilde{C}_1} \longrightarrow \mathcal{Q}_1 \longrightarrow 0.$$

Both curves have only constant global functions. Consequently $H^0(C_1, \mathcal{Q}_1)$ injects into $H^1(C_1, \mathcal{O}_{C_1}) = 0$. The sheaf $\mathcal{Q}_1$ has finite support, so it is zero. Thus C_1 is normal and has genus zero. Over k , it is a smooth projective $\mathbb{P}^1$.

Step 2. In this step, we prove that the component-junction incidence graph is a tree. Write the irreducible components as $Z_1, \dots, Z_t$. The normalization sequence is

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^t \mathcal{O}_{Z_i} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

The sheaf $\mathcal{Q}$ is supported at junction points. If r_z components meet at such a point z , branch residues induce a surjection

$$\mathcal{Q}_z \longrightarrow k^{r_z} / \text{diag}(k).$$

Hence $\text{length}(\mathcal{Q}_z) \geq r_z - 1$. Cohomology of the normalization sequence gives

$$\sum_z \text{length}(\mathcal{Q}_z) = t - 1.$$

The incidence graph is connected. Its number of edges minus its number of vertices plus one is

$$\sum_z (r_z - 1) - t + 1 \geq 0.$$

The preceding length inequalities force equality. Thus the graph is a tree. They also give $\text{length}(\mathcal{Q}_z) = r_z - 1$ at every junction.

Step 3. In this step, we obtain a fixed point for the cyclic action. Repeated deletion of the leaves of a finite tree leaves a vertex or an edge preserved by every automorphism. The action here preserves the component and junction types, so it cannot invert an edge. There is therefore a fixed vertex. A fixed junction vertex is already a fixed point of Z . A fixed component vertex gives a K -stable $\mathbb{P}^1$. The image of K in $\text{PGL}_2(k)$ is cyclic. If it is nontrivial, a generator has two eigenlines, each fixed by the whole image; if it is trivial, every point is fixed. This supplies the required closed point in either case. $\square$

Lemma 7.12. *Let $(X \ni x, B)$ be a klt threefold germ with rational effective boundary, mB integral and $a = \text{mld}(X \ni x, B) = p/q > 1$. Use its original integral-log cover $(V \ni o) \rightarrow (X \ni x)$ and group G . For any subgroup $K \leq G$, let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a K -equivariant effective terminal pair model from Proposition 7.5. Its reduced fibre $Z = g^{-1}(o)_{\text{red}}$ is connected, has dimension at most one, and satisfies*

$$H^0(Z, \mathcal{O}_Z) = k, \quad H^1(Z, \mathcal{O}_Z) = 0.$$

If it is one-dimensional, each component is a smooth projective $\mathbb{P}^1$, and the bipartite graph of components and junction points is a tree with its actual K -action. The fibre has a K -fixed closed point; such a point need not be a computing centre.

For a prescribed original computing lift F , put $H = \text{Stab}_G(F)$, $I = I_G(F)$, and take the model for $K = H$. If its centre is a curve C , then $C \simeq \mathbb{P}^1$ and the kernel of the H -action on C is exactly I . If $q \mid m$, this action is faithful and F is the ordinary generic blowup valuation of C .

The fixed point of the whole fibre and the centre of the prescribed computing divisor serve different purposes.

Proof. We prove vanishing on the whole fibre, then identify the kernel on the prescribed computing curve by its actual generic transverse quotient.

Step 1. In this step, we prove the rational-tree and fixed-point assertions for the whole fibre. If the fibre contained a surface, it would be a prime D exceptional for g , with centre o and $A_{V,B_V}(D) \leq 1$. Its primitive restriction D_X has centre x . Lemma 3.8 gives

$$A_{X,B}(D_X) = \frac{A_{V,B_V}(D)}{|I_G(D)|} \leq 1,$$

contrary to $a > 1$. Thus $\dim Z \leq 1$. The proper birational morphism g has normal target, so $g_*\mathcal{O}_Y = \mathcal{O}_V$; its fibres are connected. The crepant identity and effectivity allow us to apply Lemma 7.10, giving $H^1(Z, \mathcal{O}_Z) = 0$. Lemma 7.11 now gives the component, incidence-tree and fixed-point assertions. A curve centre C of F is then an irreducible component of Z , and hence is isomorphic to $\mathbb{P}^1$.

Step 2. In this step, we identify the exact transverse minimum of the quotient by the curve-action kernel. Use the model with $K = H$, and put $J = \ker(H \rightarrow \text{Aut}_k k(C))$. The valuation gives an H -equivariant inclusion $k(C) \hookrightarrow \kappa(F)$, whence $I \leq J$. By Lemma 7.9, Y is smooth near the generic point of C . On an invariant neighbourhood form the quotient $\pi : Y \rightarrow Q = Y/J$, with finite crepant boundary Θ , and let $C_0 = \pi(C)$. The residue field at the generic point of C_0 is $k(C)$. The norm argument of Lemma 7.3 and Lemma 3.8 show that (Q, Θ) is an effective klt pair.

Let D_0 be any primitive prime with centre C_0 , and extend it to a primitive prime D_1 of $k(Y)$. The inverse image of C_0 has just the component C , since its components form one J -orbit and C is J -stable. Thus the centre of D_1 is C . Its primitive restriction $(D_1)_X$ has centre x , and the original field actions give

$$A_{Q,\Theta}(D_0) = \frac{A_{Y,B_Y}(D_1)}{|I_J(D_1)|} = \frac{|I_G(D_1)|}{|I_J(D_1)|} A_{X,B}((D_1)_X) \geq a.$$

For the prescribed F , one has $I_J(F) = I_G(F) = I$. Its primitive restriction F_0 to $k(Q)$ therefore has discrepancy exactly a .

Choose a nonconstant $\bar{t} \in k(C)$, lift it to the local ring of Y at the generic point of C , and average the lift over J . This gives a J -invariant rational function t_0 with residue $\bar{t}$. After shrinking, it is regular and defines a smooth morphism to the affine line near that generic point: $d\bar{t} \neq 0$ in characteristic zero. Write $K_0 = k(t_0)$. The generic fibre is a smooth surface M near a closed point c , with

$$\mathcal{O}_{M,c} = \mathcal{O}_{Y,\eta_C}, \quad \kappa(c) = k(C),$$

and its quotient $M_0 = M/J$ has a closed point c_0 with the same residue field. Invariants commute with localization and separable scalar extension, since averaging is an idempotent projection. The quotient is therefore the generic fibre of Q . Restrict the boundary to obtain Θ_0 on M_0 .

Every prime over M_0 centred at c_0 spreads to a prime over Q with centre C_0 ; its function field and primitive valuation are unchanged. Cancelling the nonvanishing base differential dt_0 from a

canonical three-form shows that its surface discrepancy equals its threefold discrepancy. The preceding all-prime comparison consequently gives

$$\mathrm{mld}(M_0 \ni c_0, \Theta_0) = a,$$

computed by F_0 .

This equality also holds at each geometric point over c_0 . Indeed, choose an algebraic closure Ω of K_0 . A divisor on a geometric model descends to a finite separable extension of K_0 . Such a constant extension is étale on every model over K_0 , so it has ramification index one at each divisorial local ring and preserves its discrepancy. Restriction therefore gives an original primitive prime of the same discrepancy. Conversely, F_0 has an extension above every embedding of its centre's residue field, with the same discrepancy. These two directions preserve the exact minimum, not just the value of one chosen prime. The group J fixes each geometric point over c , since it is trivial on $k(C)$, and acts faithfully on its smooth surface germ.

Step 3. In this step, we identify the geometric transverse quotient as a reflection quotient. Work at one such geometric point, writing $M_\Omega \ni c'$ for the smooth cover and $Q' \ni c'_0$ for its quotient. The bare canonical divisor of Q' is $\mathbb{Q}$ -Cartier: its finite Weil pullback is the canonical divisor of the smooth cover minus the ramification divisor, and the norm descends a Cartier multiple. Hence the effective quotient boundary is $\mathbb{Q}$ -Cartier. For every prime centred at c'_0 ,

$$A_{Q'}(D_0) \geq A_{Q', \Theta_\Omega}(D_0) \geq a > 1.$$

A normal surface is smooth away from finitely many points on a neighbourhood of c'_0 . After removing the other singular points these inequalities make Q' terminal. By Lemma 7.9, it is smooth.

Put $h_0 = |J|$. Diagonalize its cotangent action and average lifts of the two eigenvectors to obtain regular parameters z_1, z_2 on the cover. Let u, v be the orders of their characters. In the invariant completed local ring the monomials z_1^u, z_2^v are indecomposable. Every other indecomposable invariant monomial would give a third independent cotangent vector. Since the quotient is smooth of dimension two, all invariant monomials are products of these two. Thus the invariant exponent lattice is generated by $(u, 0), (0, v)$. The action is faithful, so the index of this lattice is $h_0 = uv$. The quotient parameters are $Z_1 = z_1^u, Z_2 = z_2^v$, with ramification orders u, v . The finite boundary formula and the discrepancy of the ordinary point blowup give

$$a \leq 2 - \mathrm{ord}_{c'_0} \Theta_\Omega \leq \frac{1}{u} + \frac{1}{v}.$$

Since $a > 1$, one of u, v equals one. Interchanging parameters, the quotient has the form

$$(z_1, z_2) \mapsto (Z_1, Z_2) = (z_1^{h_0}, z_2).$$

Step 4. In this step, we identify the curve-action kernel with the inertia of the prescribed computing prime. Set $b_0 = \mathrm{ord}_{c'_0} \Theta_\Omega$. The preceding blowup has discrepancy $2 - b_0 \geq a > 1$, so $b_0 < 1$. Theorem 7.7, applied to the local equations of this effective boundary, identifies the ordinary blowup as its unique computing prime. Its complex scope is sufficient here. Realize the smooth pair and its ordinary blowup over a common algebraically closed subfield as in Lemma 3.11; orders and the minimum are preserved. If another prime computed over Ω , include its model in that finite diagram. Its complex realization would be a distinct computing prime, contradicting the cited uniqueness. This checks both the existence and uniqueness after field comparison.

In particular every geometric extension of the original F_0 is this ordinary blowup valuation. The finite cover field is generated by z_1 : its faithful character has an orbit of size h_0 . The equation $z_1^{h_0} = Z_1$ is Eisenstein for the ordinary blowup valuation. It has a unique extension, with ramification index h_0 and unchanged residue field. In its normalized blowup chart put $w_0 = Z_2/Z_1$. Then

$$Z_1 = z_1^{h_0}, \quad z_2 = z_1^{h_0} w_0.$$

The whole J fixes the residue parameter w_0 and acts trivially on the residue field. Uniqueness identifies this extension with the geometric extension of the prescribed F . The original residue field embeds equivariantly in that field, so J already acts trivially on $\kappa(F)$. Together with $I \leq J$, this proves $J = I$.

Finally, if $q \mid m$, Theorem 3.9 gives $I = 1$. The ordinary transverse point blowup is then the blowup of the maximal ideal of $\mathcal{O}_{Y, \eta_C}$. Its exceptional prime is geometrically integral over $k(C)$, so the identification descends to the original primitive valuation. Thus the same F is the ordinary generic blowup of C , and the H -action on C is faithful. $\square$

Lemma 7.13. *Let Y be a normal threefold smooth outside a finite set, let E be an integral $\mathbb{Q}$ -Cartier divisor, and let $R \geq 0$ be a rational divisor not containing E . Suppose a finite group H preserves Y, E, R . Let $m \geq 1$, $0 < a \leq 1$, $ma \in \mathbb{Z}$, and mR integral. Suppose that the natural reflexive sheaf*

$$\mathcal{L} = [\omega_Y^{[m]} \otimes \mathcal{O}_Y(m(1-a)E + mR)]^{**}$$

is invertible and has an H -eigen-generator of character χ_0 . For the normalization $\nu : S \rightarrow E$, there are a $\mathbb{Q}$ -Cartier rank-one reflexive conormal sheaf $\mathcal{N}$ and an effective rational divisor Δ , with $m\Delta$ integral, such that

$$[\omega_S^{[m]} \otimes \mathcal{O}_S(m\Delta) \otimes \mathcal{N}^{[ma]}]^{**} \simeq \mathcal{O}_S \otimes k_{\chi_0}$$

naturally H -equivariantly. In codimension one, Δ is the full conductor divisor plus the pullback to S of $R|_E$. If rE is Cartier, then

$$\mathcal{N}^{[r]} \simeq \nu^*(\mathcal{O}_Y(-rE)|_E).$$

Proof. We calculate the full conductor at each codimension-one point, apply hypersurface adjunction, and extend the resulting identity reflexively. Put $U = Y_{\text{reg}}$ and $S^\circ = \nu^{-1}(E \cap U)$. Its complement in S is finite. On U , the divisor E is a Cartier hypersurface and R is rational Cartier.

Step 1. In this step, we compute normalized canonical modules with the full conductor. At a codimension-one point of $E \cap U$, let A be its one-dimensional local ring and A' its finite normalization. Write $A = T_0/(f_0)$, where T_0 is the two-dimensional regular local ring of Y at that point. The ring A is Gorenstein. Dualizing the two-term free resolution of A over T_0 gives $\omega_A = \text{Ext}_{T_0}^1(A, \omega_{T_0})$. The finite algebra A' is Cohen–Macaulay of dimension one. Its canonical module is computed by a free T_0 -resolution in the same way. Since $f_0 A' = 0$, the change-of-rings calculation from that two-term resolution has its only nonzero row in degree one and gives

$$\omega_{A'} = \text{Ext}_{T_0}^1(A', \omega_{T_0}) = \text{Hom}_A(A', \omega_A).$$

This is the finite-duality calculation needed here, at a hypersurface local ring.

Trivialize ω_A for this calculation. Every A -linear map $A' \rightarrow A$ becomes multiplication by an element b after passing to the common fraction field. The condition is exactly $bA' \subset A$. Taking $1 \in A'$ shows $b \in A$, so these elements form the conductor ideal $\mathfrak{c} = \{b \in A' : bA' \subset A\}$. At each branch, A' is a DVR, and this nonzero ideal has a nonnegative integral order. Write it as $\mathcal{O}_{S^\circ}(-C_{\text{cond}})$ in codimension one. We obtain

$$\nu^* \omega_{E \cap U} = \omega_{S^\circ}(C_{\text{cond}}) \quad \text{in codimension one.}$$

This computation places no upper bound on a conductor coefficient and makes no seminormality assumption.

Step 2. In this step, we extend the boundary and conormal sheaf across the finite omitted set. Define, in codimension one of S° ,

$$\Delta = C_{\text{cond}} + \nu^*(R|_{E \cap U}), \quad \mathcal{N}|_{S^\circ} = \nu^* \mathcal{O}_{E \cap U}(-E).$$

A local equation of a component of R restricts to a nonzero regular function on the integral hypersurface E , and stays regular on its normalization. Thus the second divisor in Δ is effective.

Its multiplicities are integral multiples of the coefficients of R , proving $m\Delta$ integral. Close these Weil divisors on S and extend $\mathcal{N}$ reflexively; the omitted finite set contributes no Weil prime. For rE Cartier, the displayed formula for $\mathcal{N}^{[r]}$ holds on S° . Both sides are reflexive on the normal surface, so it holds everywhere. Its right side is a line bundle; this proves that $\mathcal{N}$ is $\mathbb{Q}$ -Cartier.

Step 3. In this step, we obtain the natural adjunction identity with the original character. The identity

$$K_Y + (1 - a)E + R = K_Y + E + R - aE$$

and hypersurface adjunction on U give, in codimension one of S ,

$$\nu^*(\mathcal{L}|_E) = [\omega_S^{[m]} \otimes \mathcal{O}_S(m\Delta) \otimes \mathcal{N}^{[ma]}]^{**}.$$

The sign of the conductor here is positive, by the preceding canonical-module calculation; the conormal exponent is ma . All maps are natural: adjunction is the residue map, the canonical-module map is induced by the free resolution, and the conductor is identified by multiplication in the original function field. They therefore commute with H , without changing a linearization by a character. Reflexivity extends the equality uniquely to S . Restriction of the given eigen-generator of $\mathcal{L}$ now proves the claimed equivariant identity. The calculation also applies when $a = 1$; then the conormal exponent is exactly m . $\square$

Lemma 7.14. *Let $f : M \rightarrow T$ be a projective birational morphism of surfaces over an algebraically closed field of characteristic zero, with M smooth and T normal. Suppose that (M, D) is an effective rational klt pair and $K_M + D = f^*L$ for a $\mathbb{Q}$ -Cartier divisor L . If the reduced fibre over a closed point has dimension one, it is a simple normal crossing tree of smooth rational curves.*

Proof. The fibre is connected. Apply Lemmas 7.10 and 7.11, with the trivial group, to obtain smooth rational components and the incidence tree. It remains to determine the intersections on the smooth surface.

At a junction z with r_z branches, let δ_z be the colength of the local ring in its normalization. The proof of Lemma 7.11 gives $\delta_z = r_z - 1$. In the regular surface local ring, each smooth branch is principal. For two unions of branches with no common component, the exact sequence

$$0 \longrightarrow \mathcal{O}_{A \cup B} \longrightarrow \mathcal{O}_A \oplus \mathcal{O}_B \longrightarrow \mathcal{O}_{A \cap B} \longrightarrow 0$$

adds their intersection multiplicity to the normalization colength. Induction on the number of branches therefore gives

$$\delta_z = \sum_{i < j} (Z_i \cdot Z_j)_z \geq \binom{r_z}{2}.$$

Hence $r_z \leq 2$. At a junction there are exactly two branches, with intersection multiplicity one. They meet transversely, proving the assertion. $\square$

Lemma 7.15. *Let $f : M \rightarrow T$ be a projective contraction of normal varieties over k , and let (M, D) be an effective rational klt pair such that $-(K_M + D)$ is f -ample. If a $\mathbb{Q}$ -Cartier divisor L is numerically trivial on every curve contracted by f , then there are a positive integer q_0 and a line bundle $\mathcal{L}_T$ on T such that*

$$\mathcal{O}_M(q_0 L) \simeq f^* \mathcal{L}_T.$$

In particular, a negative extremal contraction of an effective rational klt pair is good in the sense of Theorem 3.3.

Proof. Choose $r_0 > 0$ with $r_0 L$ integral Cartier. Both $r_0 L$ and $-r_0 L$ are f -nef. For either sign,

$$(\pm r_0 L) - (K_M + D) \equiv_f -(K_M + D)$$

is f -ample, hence f -big and f -nef. The normal varieties M, T are integral, excellent and Noetherian of characteristic zero, and T has a dualizing complex. Thus Theorem 3.2 applies to both Cartier divisors, with its integer parameter equal to one. For some common positive multiple q_0 of r_0 , both $\mathcal{O}_M(q_0 L)$ and its inverse are generated relative to f .

Fix a point of T and a point in its fibre. Relative generation supplies local sections s and t of these two inverse lines, each nonzero at the latter point. Their product is a regular function on the inverse image of a base neighbourhood. Since $f_*\mathcal{O}_M = \mathcal{O}_T$, it comes from a function on that neighbourhood. The product is nonzero at the chosen fibre point, so after shrinking the base it is invertible. Hence s trivializes $\mathcal{O}_M(q_0L)$ over the whole inverse image. The transition units between such trivializations likewise descend through $f_*\mathcal{O}_M = \mathcal{O}_T$. They define the line bundle $\mathcal{L}_T$ and the required isomorphism.

For a negative extremal contraction, the anti-adjoint is relatively ample and every contracted curve has class in its extremal ray. A rational line bundle numerically trivial on that ray therefore satisfies the proved assertion, which is exactly goodness. $\square$

Theorem 7.16. *Let original cover data be given with $0 < a \leq 1$. After shrinking around o , there are an integer $N \geq 0$, H -equivariant projective birational models $g_i : Y_i \rightarrow V$ with effective terminal pairs (Y_i, B_i) , a sequence of small modifications over V , and a divisorial contraction $h : Y_N = Y \rightarrow Z$ over V . Every Y_i is bare terminal and H - $\mathbb{Q}$ -factorial, and*

$$K_{Y_i} + B_i = g_i^*(K_V + B_V), \quad B_i = (1 - a)F_i + R_i,$$

where F_i is the same original prime, $R_i \geq 0$ avoids it, and all coefficients belong to $\{0, 1/m, \dots, (m-1)/m\}$. The natural m -log line is the equivariant pullback of the original line. For every closed point $P \in Y_i$, $\text{ind}_P(K_{Y_i}) \leq m$. The sole exceptional prime of h is the same complete integral surface $E = F_N$. The source pair has the form

$$(Y, (1 - a)E + R), \quad R \geq 0, \quad E \not\subset \text{Supp } R,$$

and is terminal. The divisor $-E$ is h -ample. The centre $h(E)$ is a complete point or integral curve over o . The maps are crepant for the pairs and retain the original natural m -log character. The pair on Z is klt, and E is its unique exceptional divisor of log discrepancy at most one. The quotient Z/H is $\mathbb{Q}$ -factorial.

For the normalization $\nu : S \rightarrow E$, with its induced faithful H -action, there are a $\mathbb{Q}$ -Cartier rank-one reflexive conormal sheaf $\mathcal{N}$ and an effective rational divisor Δ , with $m\Delta$ integral, with a natural H -equivariant reflexive identity

$$[\omega_S^{[m]} \otimes \mathcal{O}_S(m\Delta) \otimes \mathcal{N}^{[ma]}]^{**} \simeq \mathcal{O}_S \otimes k_{\chi|H}.$$

The divisor Δ is the full conductor plus the normalized restriction of R , defined in codimension one and closed on S . It need not be a boundary or make (S, Δ) lc. If $h(E)$ is a point, $\mathcal{N}$ is ample. If it is a curve, the Stein factorization $S \rightarrow C \rightarrow h(E)$ has C smooth projective, degree at most two in its second map, and geometric generic fibre $\mathbb{P}^1$.

For $c = c_{\text{ct}}(3, a)$, there is an original $g \in \mathfrak{m}_{X,x} \setminus \{0\}$ whose pullback satisfies

$$\text{div}_Y(g) = \ell E + D, \quad 1 \leq \ell \leq a/c, \quad D \geq 0, \quad E \not\subset \text{Supp } D,$$

with $(Y, (1 - a)E + R + c \text{div}_Y(g))$ lc near E . Its leading section belongs to $H^0(S, \mathcal{N}^{[\ell]})^H$. Every primitive divisor P with centre contained in E satisfies

$$\frac{A_{Y, (1-a)E+R}(P)}{|I_H(P)|} \geq a.$$

The prime E retains its original valuation and residue field. In particular, the construction at $a = 1$ must retain the marked prime even though its boundary coefficient is zero.

Proof. We construct a good scaling divisor on the quotient and stop its MMP at the first divisorial step. Normalization gives the marked contraction in the original field. Normalized adjunction supplies the full character, the transverse fibre calculation gives the Stein degree, and the original contact function supplies the invariant section.

Step 1. In this step, we construct a small klt perturbation by the marked prime on the quotient. Apply Proposition 7.5 with the original H . Its model Y_0 contains the prescribed prime F_0 , including when $a = 1$ and its coefficient is zero. Put $W = V/H$ and $Q_0 = Y_0/H$. The quotient

is projective birational over W : a power of the tensor product of the translates of a relatively ample line descends to an ample line on the finite quotient. It is $\mathbb{Q}$ -factorial by Lemma 7.3. Define the quotient boundaries Θ_W, Θ_0 by the finite crepant identities. The coefficient formula in Lemma 3.8 and norm descent make these effective klt pairs, and $Q_0 \rightarrow W$ is crepant.

Let $\bar{F}_0$ be the image of F_0 . Since H stabilizes the original prime and its inertia is one, the finite pullback of $\bar{F}_0$ is F_0 , with multiplicity one. Its coefficient in Θ_0 is $1 - a$. On a log resolution, the klt condition for $(Q_0, \Theta_0 + \epsilon \bar{F}_0)$ is a finite collection of strict inequalities in ϵ . Choose a positive rational $\epsilon < a$ satisfying them, and shrink the base around the image of o . Then

$$K_{Q_0} + \Theta_0 + \epsilon \bar{F}_0 \equiv_W \epsilon \bar{F}_0.$$

The perturbation is present also at $a = 1$; its construction does not use a positive original coefficient of F_0 .

Step 2. In this step, we construct a good scaling divisor and verify the relative MMP hypotheses. Write $f_0 : Q_0 \rightarrow W$ and $\Theta_\epsilon = \Theta_0 + \epsilon \bar{F}_0$. Choose an f_0 -ample Cartier divisor L_0 , and set

$$A_0 = nL_0 - (K_{Q_0} + \Theta_\epsilon)$$

for a sufficiently large positive integer n . This is a $\mathbb{Q}$ -Cartier divisor. To check ampleness, choose $d_1 > 0$ for which $d_1(K_{Q_0} + \Theta_\epsilon)$ is Cartier. The line

$$\mathcal{O}_{Q_0}(d_1 A_0) = \mathcal{O}_{Q_0}(d_1 L_0)^{\otimes n} \otimes \mathcal{O}_{Q_0}(-d_1(K_{Q_0} + \Theta_\epsilon))$$

is relatively ample for n sufficiently large, as an ample twist of a fixed line bundle. Thus A_0 is f_0 -ample, and

$$K_{Q_0} + \Theta_\epsilon + A_0 = nL_0$$

is f_0 -nef.

Both Q_0 and W are integral excellent Noetherian varieties of characteristic zero, Q_0 is normal, and W has a dualizing complex. The morphism f_0 is projective and surjective. The boundary Θ_ϵ is effective and rational, its pair is klt, and its adjoint is $\mathbb{Q}$ -Cartier. Lemma 3.5 therefore applies: A_0 is a good scaling divisor, including the required etale-local effective representatives preserving klt.

The source is $\mathbb{Q}$ -factorial. Moreover, $K_{Q_0} + \Theta_\epsilon \equiv_W \epsilon \bar{F}_0$ is relatively pseudoeffective. The termination hypothesis of Theorem 3.6 holds with its rational parameter equal to zero: Θ_ϵ is f_0 -big because the generic fibre of the birational morphism has dimension zero. All hypotheses of that theorem now hold over the original field, so the MMP with scaling of A_0 exists and terminates with a minimal model.

At every step the perturbed pair is klt. Its negative extremal contraction has relatively ample anti-adjoint, so Lemma 7.15 makes that contraction good. Starting from Q_0 , Theorem 3.4 preserves $\mathbb{Q}$ -factoriality at each flip, and Theorem 3.3 gives it for the target of a divisorial step. Their base and source hypotheses hold at each stage, since all models are normal varieties over the same field and projective over W .

Step 3. In this step, we prove that the first divisorial contraction contracts the marked prime. Write Q_i for these $\mathbb{Q}$ -factorial models. Until the first divisorial step, the maps are small. The divisor $K_{Q_i} + \Theta_i$ is the pullback of $K_W + \Theta_W$: equality holds on the common open set in codimension one and determines the Weil divisor on the small model. Thus

$$K_{Q_i} + \Theta_i + \epsilon \bar{F}_i \equiv_W \epsilon \bar{F}_i.$$

Every negative contracted curve lies in $\bar{F}_i$, since an effective $\mathbb{Q}$ -Cartier divisor has nonnegative intersection with a curve outside its support. A divisorial step can therefore contract no other prime: a general contracted curve in another exceptional prime would meet its complement of $\bar{F}_i$. A fibre-type step is impossible for a model birational to the three-dimensional base W .

Nor can the process terminate with $\bar{F}_i$ still present and nef over W . Apply Lemma 4.9 with pulled-back divisor zero, nef divisor $\bar{F}_i$, effective divisor zero, and exceptional divisor $-\bar{F}_i$. Its conclusion would be $-\bar{F}_i \geq 0$, a contradiction. For its complex scope, realize the projective

diagram and divisor over a common field as in Lemma 3.11. Relative nefness is preserved by algebraically closed extension: a negative curve would spread and specialize to a negative effective one-cycle over the smaller field. Thus the complex negativity statement gives the same contradiction for the original diagram. Consequently a first divisorial step occurs. Stop there, writing $\bar{h} : Q_N \rightarrow Q_{\text{fin}}$. Its exceptional locus is exactly $\bar{F}_N$, and $-\bar{F}_N$ is $\bar{h}$ -ample because this is the negative extremal contraction with relative Picard number one.

Step 4. In this step, we lift the sequence by normalization while retaining the original prime, field, and natural log line. Normalize every Q_i and Q_{fin} in the same original field $k(V)$, obtaining Y_i and Z . Lemma 7.3 gives

$$Y_i/H = Q_i, \quad Z/H = Q_{\text{fin}}.$$

In particular, Z/H is $\mathbb{Q}$ -factorial, every Y_i is H - $\mathbb{Q}$ -factorial, and the small maps lift in the original field. The last step lifts to a projective birational morphism $h : Y_N \rightarrow Z$. There is a single prime over $\bar{F}_N$, namely the same F_N , with multiplicity one, because its original decomposition group is H and its inertia is trivial. Outside it the downstairs morphism is an isomorphism, so the normalization is too. Thus the exceptional locus of h is exactly $E = F_N$. Relative ampleness of $-E$ follows from finite pullback of $-\bar{F}_N$ on the normalization of the dominating component of the base change.

Define B_i by $K_{Y_i} + B_i = g_i^*(K_V + B_V)$. On small models it is the strict transform of B_0 , hence effective, with coefficients in $m^{-1}\mathbb{Z} \cap [0, 1)$. It has the form $(1 - a)F_i + R_i$, with $R_i \geq 0$ avoiding F_i . The small maps preserve the set of prime divisors on the models. Consequently a prime exceptional over Y_i is exceptional over Y_0 , and its crepant pair discrepancy is greater than one. Every (Y_i, B_i) is terminal. Lemma 7.4 supplies bare terminality and the naturally linearized pullback of the original m -log line. Its index bound also gives $\text{ind}_P(K_{Y_i}) \leq m$ for every closed point P on every small model.

Set $B_Z = h_*B_N$, and let $g_Z : Z \rightarrow V$ be the induced map. With canonical divisors chosen using the original rational log tensor, equality of the divisorial orders after pushforward gives

$$K_Z + B_Z = g_Z^*(K_V + B_V), \quad K_{Y_N} + B_N = h^*(K_Z + B_Z).$$

In particular $m(K_Z + B_Z)$ is integral Cartier. These are divisor equalities: the original tensor gives its generator, so no descent of a merely numerically trivial line is being assumed. The same generator has character $\chi|H$ throughout. The pair on Z is klt by these equalities. Among primes exceptional over Z , the only one already appearing on Y_N is E , of discrepancy a . Every other one is exceptional over the terminal pair on Y_N , and has discrepancy greater than one. This proves uniqueness of the exceptional prime of discrepancy at most one.

The prime E lies in the projective fibre over o , so it is a complete integral surface. Its image $T = h(E)$ is complete and integral, and has dimension zero or one since h is birational. The normalization S is projective. Its function field is the original $\kappa(F)$, on which H acts faithfully because $I_G(F) = 1$.

Step 5. In this step, we obtain the natural adjunction line on the normalization with the full conductor. By Lemma 7.9, $Y = Y_N$ is smooth away from finitely many points. The divisor E is $\mathbb{Q}$ -Cartier by H - $\mathbb{Q}$ -factoriality, since it is invariant. The original cover data give $ma \in \mathbb{Z}$, and the coefficient grid gives mR integral. The naturally linearized m -log line on Y has the original eigen-generator of character $\chi|H$. All hypotheses of Lemma 7.13 therefore hold. It supplies $\mathcal{N}$, the full conductor plus normalized restriction Δ , its effectivity and integrality, and the required reflexive identity. No step requires Δ to have coefficients at most one or the normalized pair to be lc. If T is a point, the formula

$$\mathcal{N}^{[r]} = \nu^*(\mathcal{O}_Y(-rE)|_E)$$

for a sufficiently divisible positive r proves ampleness by restriction and finite pullback of the h -ample line.

Step 6. In this step, we bound the Stein degree over a curve image by two. Suppose $\dim T = 1$. Remove the finitely many images in T of the singular points of Y . Choose a nonconstant element of $k(T)$, lift it to $\mathcal{O}_{Z, \eta_T}$, and use its generic fibre over the corresponding rational function field. This is the generic transverse construction in the proof of Lemma 7.12, now applied to h . The point corresponding to η_T has residue field $k(T)$. The source is a smooth surface near its fibre: its local rings are localizations of the regular local rings of Y , and characteristic zero makes regularity geometric. The restricted boundary is effective, and the restricted morphism is projective, birational and pair-crepant. Cancelling the base differential preserves all discrepancies. After algebraic closure of the constant field, every new prime descends through a finite separable constant extension, whose divisorial ramification index is one. Thus the source pair remains klt at every geometric point under consideration.

Lemma 7.14 applies to this actual geometric surface morphism. Its reduced exceptional fibre is a simple normal crossing tree of smooth rational curves. Every component comes from the single integral divisor E . The generic fibre of $E \rightarrow T$ is integral over $k(T)$, so its geometric components form one transitive Galois orbit. If the tree has v_0 vertices, all have the same degree d_0 . Hence

$$v_0 d_0 = 2(v_0 - 1),$$

which permits only one vertex, or two joined vertices.

Normalization commutes with localization and with these separable constant extensions. It follows that the geometric generic fibre of $S \rightarrow T$ consists of one or two disjoint copies of $\mathbb{P}^1$. In its Stein factorization $S \rightarrow C \rightarrow T$, the curve C is normal, hence smooth over k , and projective. In characteristic zero the finite map $C \rightarrow T$ is generically separable; its degree equals the number of connected components of the geometric generic fibre of $S \rightarrow T$. The degree is therefore at most two, and the geometric generic fibre of $S \rightarrow C$ is $\mathbb{P}^1$. Normalization and Stein factorization are canonical, so these maps retain the actual H -action.

Step 7. In this step, we construct the invariant conormal section from the original contact function and retain the all-prime comparison. We apply Theorem 5.9 to $(X \ni x, B)$, with $\epsilon = a$, and retain its original function g . Lemma 3.8 and pair crepancy show that

$$(Y, (1 - a)E + R + c \operatorname{div}_Y(g))$$

is lc near the fibre over o . Let E_X be the original computing prime whose prescribed lift is F . By Definition 3.10, this lift has inertia one. Consequently

$$\ell = \operatorname{ord}_E(g) = \operatorname{ord}_{E_X}(g), \quad 1 \leq \ell \leq \frac{A_{X,B}(E_X)}{c} = \frac{a}{c}.$$

Since g is regular, the divisor $D = \operatorname{div}_Y(g) - \ell E$ is effective and avoids E .

We construct the leading section on the regular locus of Y . There E has a local equation t . The rational function g/t^ℓ has no pole along any prime divisor, so it is regular by normality. Its restriction to E is generically nonzero. Under a change of equation for E , these restrictions transform as a section of $\mathcal{O}_E(-\ell E)$. Pulling back to S gives a nonzero section of $\mathcal{N}^{[\ell]}$ away from finitely many points. Reflexivity extends it uniquely over those points. The original function g is H -invariant, and taking its initial term along E commutes with the natural action. Thus the extended section lies in $H^0(S, \mathcal{N}^{[\ell]})^H$.

Finally let P be any primitive prime whose centre on Y is contained in E , and let P_X be its primitive restriction to $k(X)$. Its centre on X is x . The original field actions give $I_H(P) = H \cap I_G(P)$. Applying Lemma 3.8 to the original cover yields

$$\frac{A_{Y, (1-a)E+R}(P)}{|I_H(P)|} = \frac{|I_G(P)|}{|I_H(P)|} A_{X,B}(P_X) \geq A_{X,B}(P_X) \geq a.$$

This comparison tests all primes with centre contained in E , including those not fixed by H . $\square$

8. LATTICE APPROXIMATION AND NATURAL CHARACTERS

We retain the discrepancy and character conventions of Section 2. We first prove that the local canonical index of a terminal threefold kills every integral $\mathbb{Q}$ -Cartier divisor class. We then construct equivariant local charts and compare their natural log-canonical characters, retaining the full inertia group. The final comparison applies to any finite abelian action on a smooth ambient germ preserving a normal klt hypersurface.

Theorem 8.1 (Torsion in a punctured complete intersection). *Let $(A, \mathfrak{m}, \kappa)$ be a three-dimensional local complete intersection in a Noetherian ring R , with $\text{char } \kappa = 0$. Assume every ring obtained from R by finitely many localizations at prime ideals and completions at the resulting maximal ideals is factorial and Cohen–Macaulay. Then*

$$\text{Pic}(\text{Spec } A \setminus \{\mathfrak{m}\})$$

is torsion-free [Rob76, Section 1, Theorem (ii)].

On an isolated hypersurface, a $\mathbb{Q}$ -Cartier divisor gives a torsion line bundle on the punctured spectrum. The theorem therefore applies without assuming that every Weil divisor is $\mathbb{Q}$ -Cartier.

Lemma 8.2. *Let $Y \ni P$ be a terminal threefold germ over an algebraically closed field of characteristic zero. Set $r = \text{ind}_P(K_Y)$, and let $\pi : (W \ni z) \rightarrow (Y \ni P)$ be the degree- r canonical cover of Theorem 3.9, applied to $(Y, 0)$ with multiplier one. Then W is Gorenstein terminal and is either smooth or an isolated hypersurface singularity of embedding dimension four. Every integral $\mathbb{Q}$ -Cartier Weil divisor on W is Cartier near z .*

This lemma supplies both the local divisor calculation and the hypersurface structure needed for the equivariant charts below.

Proof. We check terminality of the existing cover, apply the two local structure theorems over $\mathbb{C}$, and transfer their consequences to the stated ground field.

Step 1. In this step, we prove terminality of the original canonical cover. Theorem 3.9 gives a connected quasi-étale cover with a single point above P , and $K_W = \pi^*K_Y$ is Cartier. An exceptional primitive prime $E^\sharp$ over W restricts, after primitive normalization, to an exceptional prime E over Y : their centres have the same dimension under the finite morphism. For the positive ramification integer t , Lemma 3.8 gives

$$A_W(E^\sharp) = tA_Y(E) > 1.$$

Thus W is Gorenstein terminal.

Step 2. In this step, we show that integral $\mathbb{Q}$ -Cartier divisor classes on the complex cover are Cartier. Suppose first that the ground field is $\mathbb{C}$. If W is singular, Theorem 7.1, namely [Rei83, Theorem (1.1), pp. 138–139], makes it an isolated cDV point. In particular its completed local ring is

$$\hat{\mathcal{O}}_{W,z} \simeq \mathbb{C}[[x_1, x_2, x_3, x_4]]/(f), \quad f \in (x_1, x_2, x_3, x_4)^2,$$

and the singularity is isolated. The hypersurface property descends from the completion to the excellent algebraic local ring; its cotangent space has dimension four. The power-series ambient ring is regular. Every successive localization and local completion of it is again regular local, hence factorial and Cohen–Macaulay. It therefore satisfies the hypotheses of Theorem 8.1.

Let D_W be an integral $\mathbb{Q}$ -Cartier Weil divisor on W , and let $\mathcal{L} = \mathcal{O}_W(D_W)_z$ be its rank-one reflexive module. Some reflexive power $\mathcal{L}^{[n]}$ is free. Its completed module is rank-one reflexive over $\hat{\mathcal{O}}_{W,z}$: flatness of completion and base change for duals of finite modules make completion commute with reflexive powers. On the punctured spectrum U of this completed normal local ring it is a line bundle, since U is regular, and its n -th power is trivial. Theorem 8.1 gives $\hat{\mathcal{L}}|_U \simeq \mathcal{O}_U$. Reflexive extension across the closed point then gives $\hat{\mathcal{L}} \simeq \hat{\mathcal{O}}_{W,z}$. Faithfully flat descent along completion makes $\mathcal{L}$ free of rank one. Thus D_W is Cartier. When W is smooth, its regular local ring is factorial, which gives the same conclusion directly.

Step 3. In this step, we transfer the local structure and divisor assertions to the stated ground field. For arbitrary k , use Lemma 3.11 on the canonical cover diagram, its marked point, a terminal log resolution, and the canonical Cartier trivialization. For the divisor assertion, include the specified D_W , its divisorial sheaf, and one Cartier multiple. These finite data have a common realization over an algebraically closed field k_0 embedded in k and in $\mathbb{C}$, retaining terminality and canonical index one. Apply the complex argument to that realization. Smoothness or the isolated complete-intersection property and the dimension of the cotangent space descend to k_0 and ascend to k . Likewise, invertibility of the specified divisorial sheaf descends faithfully from $\mathbb{C}$ and ascends to k . This proves both assertions over k , without asserting an embedding of the entire field k into $\mathbb{C}$. $\square$

Lemma 8.3. *Let $Y \ni P$ be a terminal threefold germ over an algebraically closed field of characteristic zero, and put $r = \text{ind}_P(K_Y)$. Every integral $\mathbb{Q}$ -Cartier Weil divisor D on Y satisfies that rD is Cartier near P .*

The multiple is the exact local canonical index; no separate bound for the divisor class is required.

Proof. Take the degree- r cover $\pi : (W \ni z) \rightarrow (Y \ni P)$ of Lemma 8.2. Its Weil pullback $D_W = \pi^*D$ is integral, since π is quasi-étale, and is $\mathbb{Q}$ -Cartier, since a Cartier multiple of D pulls back to the same multiple of D_W . The lemma makes D_W Cartier. The cover is local at its unique point over P , so write $D_W = \text{div}(h)$ for $h \in k(W)^*$. The field norm then satisfies

$$\text{div}_Y(\text{Norm}_{k(W)/k(Y)} h) = \pi_* \text{div}_W(h) = \pi_* \pi^* D = rD.$$

Indeed, at each prime of Y the norm order is the sum of the upstairs orders weighted by residue degrees, and the sum of ramification indices times residue degrees is r . Thus rD is principal on the local scheme, and Cartier on a neighbourhood of P . $\square$

Theorem 8.4 (Semi-invariant inversion of adjunction). *Let $G \subset \text{GL}_N(k)$ be finite of order d , without pseudoreflections, over an algebraically closed field of characteristic zero. Set $X = \mathbb{A}^N/G$ and let $Z \subset X$ be the branch locus. Let $\bar{Y} \subset \mathbb{A}^N$ be a codimension- c complete intersection defined by semi-invariants $f_1, \dots, f_c$. Assume $Y = \bar{Y}/G$ is klt and $\text{codim}_Y(Y \cap Z) \geq 2$. For an $\mathbb{R}$ -ideal $\mathfrak{a}$ on X with nonzero restriction to Y , and a closed $y \in Y$, one has*

$$\text{mld}(Y \ni y, \mathfrak{a}\mathcal{O}_Y) = \text{mld}(X \ni y, (f_1^d \cdots f_c^d)^{1/d} \mathfrak{a})$$

[NS26a, Theorem 7.1].

The branch condition concerns the underlying complete intersection. The additional ideal may supply a boundary of coefficient one.

Theorem 8.5 (Completed smooth-quotient formula with ideals). *Let $G \subset \text{GL}_N(k)$ be a finite group without pseudoreflections, where k is algebraically closed of characteristic zero. Put $d = |G|$, choose a primitive d -th root ζ , and set $X = \text{Spec } k[[x_1, \dots, x_N]]^G$, with closed point o . Let $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\delta_j}$, where $\mathfrak{a}_j \neq 0$ and $\delta_j > 0$. For each $\gamma \in G$, choose eigen-coordinates with eigenvalues ζ^{e_i} , $1 \leq e_i \leq d$, and put $\text{age}'(\gamma) = \sum_i e_i/d$. Substitution $x_i \mapsto t^{e_i/d} u_i$ defines*

$$\lambda_\gamma : k[[x_1, \dots, x_N]]^G \longrightarrow k[[u_1, \dots, u_N]][[t]].$$

For $w = (w_1, \dots, w_r) \in \mathbb{Z}_{\geq 0}^r$, let $C_{\gamma, w} \subset (\mathbb{A}^N)_\infty$ consist of arcs $u(t)$ satisfying

$$\text{ord}_t \lambda_\gamma(\mathfrak{a}_j)(u(t)) = w_j \quad (1 \leq j \leq r).$$

With cylinder codimension computed in this ordinary affine arc space, and empty cylinders assigned infinite codimension,

$$\text{mld}(X \ni o, \mathfrak{a}) = \inf_{\gamma, w} \left\{ \text{codim } C_{\gamma, w} + \text{age}'(\gamma) - \sum_j \delta_j w_j \right\}.$$

The substitutions automatically send the maximal ideal of o to positive t -order [NS25a, Section 7.1, Theorem 7.9 and Remark 7.10, $c = 0$].

We use this formula for invariant powers of a semi-invariant equation, with its full completed series retained.

Lemma 8.6. *Let $S \ni o$ be a normal Gorenstein algebraic germ of positive dimension, let a finite abelian group J_0 act faithfully on it, fixing o , and let C be an invariant effective rational Cartier divisor. Give $Q = S/J_0$ its crepant quotient boundary C_Q , and assume that $K_Q + C_Q$ is $\mathbb{Q}$ -Cartier and $\text{mld}(Q \ni o/J_0, C_Q) = a_0 > 0$. Let $\beta_1, \dots, \beta_j$ be characters of J_0 , where $j \geq 1$, and let J_0 act on*

$$S^* = S \times \mathbb{A}_{s_1, \dots, s_j}^j, \quad C^* = \text{pr}_S^* C + \sum_{i=1}^j \text{div}(s_i)$$

by its given action on S and the characters β_i on the coordinates. Let (Q^*, C_{Q^*}) be the crepant quotient pair. Then

$$\text{mld}(Q^* \ni (o, 0)/J_0, C_{Q^*}) = a_0.$$

Moreover, if a primitive prime T centred at o satisfies $A_{S,C}(T)/|I_{J_0}(T)| = a_0$, and $e = |I_{J_0}(T)|$, there are integers $1 \leq N_i \leq e$ and a primitive algebraic prime T^* centred at $(o, 0)$ such that

$$\begin{aligned} \text{ord}_{T^*} |_{k(S)} &= \text{ord}_T, & I_{J_0}(T^*) &= I_{J_0}(T), \\ A_{S^*}(T^*) &= A_S(T) + \sum_i N_i, & A_{S^*, C^*}(T^*) &= A_{S,C}(T). \end{aligned}$$

The coordinate divisors have coefficient one. Their contribution cancels the extra canonical discrepancy, even when the original action has divisorial ramification.

Proof. We prove the lower bound for one coordinate and then construct the computing extension. Both arguments can be iterated.

Step 1. In this step, we identify the primitive divisorial restriction to the original field. Let w be a primitive divisorial valuation of $k(S)(s)$ centred at $(o, 0)$. Its restriction to $k(S)$ is nonzero; write it as $\rho\nu$, where $\rho \geq 1$ is an integer and ν has value group $\mathbb{Z}$. If $d = \dim S$, the residue field of w has transcendence degree d . Adjoining one transcendental element increases residue transcendence degree by at most one, whereas a nontrivial discrete valuation on $k(S)$ has residue transcendence degree at most $d - 1$. Thus ν has residue transcendence degree $d - 1$, so it is an algebraic divisorial valuation. Its centre is o .

Step 2. In this step, we prove the lower bound for logarithmic product discrepancies. We claim that

$$(8.1) \quad A_{S \times \mathbb{A}^1, \text{pr}_S^* C + \text{div}(s)}(w) \geq \rho A_{S,C}(\nu).$$

At the generic point of a model carrying ν , choose a uniformizer t and separating residue parameters. Put $b_1 = w(s) > 0$, $g_1 = \gcd(\rho, b_1)$, $u_1 = \rho/g_1$, and $v_1 = b_1/g_1$. On the two-dimensional toric chart with primitive ray (u_1, v_1) , a uniformizer v has $w(v) = g_1$, and $R = s^{u_1}/t^{v_1}$ is an invertible coordinate. A local canonical generator on S , wedged with ds/s , has order $u_1 A_S(\nu) - 1$ along $v = 0$. The discrepancy of w on this smooth chart is at least $w(v) = g_1$: in a uniformizer for w , differentiating v lowers its order by at most one, and the other regular differentials have nonnegative order. Consequently its logarithmic discrepancy is at least

$$g_1(u_1 A_S(\nu) - 1) + g_1 = \rho A_S(\nu).$$

The pullback of C has order $\rho\nu(C)$, proving (8.1).

The inclusion of residue fields gives $I_{J_0}(w) \subseteq I_{J_0}(\nu)$. Write their orders as e_w and e_ν . By Lemma 3.8 and the original quotient minimum,

$$\frac{A_{S \times \mathbb{A}^1, \text{pr}_S^* C + \text{div}(s)}(w)}{e_w} \geq \rho \frac{A_{S,C}(\nu)}{e_\nu} \geq \rho a_0 \geq a_0.$$

Every prime over the product quotient has a primitive finite extension, so this proves the lower bound for all such primes.

Step 3. In this step, we construct a computing extension preserving the full inertia. For the opposite inequality, the positive minimum on the fixed rational quotient pair is attained: its discrepancies belong to $r_0^{-1}\mathbb{Z}$ for an integral Cartier multiple $r_0(K_Q + C_Q)$. If T is not already specified, lift a computing prime to T over S . Set $I = I_{J_0}(T)$, let $e = |I|$, and let λ_T be its faithful conormal character. Choose $1 \leq N \leq e$ with $\beta_1|_I = \lambda_T^N$. The Gauss extension with $T^*(s) = N$ has value group $\mathbb{Z}$, restricts to T , and has residue field $\kappa(T)(\overline{s/t^N})$. It is an algebraic divisorial valuation, realized by the monomial ray $(1, N)$ over the generic T -DVR. Its centre is $(o, 0)$, and

$$A_{S \times \mathbb{A}^1}(T^*) = A_S(T) + N, \quad A_{S \times \mathbb{A}^1, \text{pr}_S^* C + \text{div}(s)}(T^*) = A_{S, C}(T).$$

Every element of I fixes the new residue $\overline{s/t^N}$. Conversely, inertia for T^* fixes the embedded field $\kappa(T)$, and hence belongs to I . Thus the full inertia is still I , and the quotient discrepancy is a_0 . The same construction starting with any specified computing T proves the additional assertion. Iteration gives the stated formulas for all j coordinates. $\square$

Lemma 8.7. *Let $W \ni z$ be a normal algebraic threefold germ with a faithful action of a finite abelian group A fixing z . Assume that W is smooth, or that its local ring is a hypersurface of embedding dimension four. Let C be an invariant effective rational divisor, let $m \geq 1$ satisfy mC integral, and suppose the natural m -log-canonical line has an eigen-generator τ . Then W has a faithful equivariant smooth algebraic ambient germ M of dimension three or four, respectively. In dimension four its ideal has a semi-invariant generator f . There is a regular semi-invariant h on M with $\text{div}_W(h|_W) = mC$, and the character of τ is the character of*

$$\Omega_M^{\otimes m}/h \quad \text{or} \quad \Omega_M^{\otimes m}/(f^m h),$$

respectively.

The ambient equations are algebraic equations of the given germ. The construction does not replace its function field by a completed field.

Proof. In the smooth case take $M = W$. In the singular case, embed an invariant affine representative of W equivariantly in a finite-dimensional representation $\mathbb{A}^n$, sending z to the origin. To obtain this embedding, take the span of the finite group translates of algebra generators vanishing at z . The kernel of the ambient cotangent map has dimension $n - 4$. Choose an eigenbasis of that kernel and lift it to the defining ideal by the corresponding character projections. The resulting $n - 4$ equations have independent differentials. Their common zero germ is therefore smooth of dimension four and contains W . Denote it by M .

The remaining hypersurface ideal in $\mathcal{O}_{M, z}$ is principal. Project a generator onto the character of its one-dimensional fibre; Nakayama's lemma shows that the projection is still a generator. Call it f . The action on M is faithful because its restriction to W is faithful.

Put $D = \dim M$. Choose regular eigenparameters $x_1, \dots, x_D$ on M by splitting its cotangent map with character projections, and put $\Omega_M = dx_1 \wedge \dots \wedge dx_D$. An eigen-generator of the dualizing line of W is $\theta = \Omega_M$ in the smooth case and $\theta = \text{Res}(\Omega_M/f)$ otherwise. The rational function $h_W = \theta^{\otimes m}/\tau$ is semi-invariant and has divisor mC . Effectivity and normality make it regular. Lift h_W to M , and project the lift onto its character. This gives a regular semi-invariant h whose restriction is h_W . The identity $\tau = \theta^{\otimes m}/h_W$ gives the asserted natural character, including the contribution of the full equation f . $\square$

Proposition 8.8. *Let $(Y \ni P_0, B_Y)$ be a klt germ whose underlying threefold is terminal. Suppose a faithful cyclic group J fixes P_0 and preserves the pair, and that mB_Y is integral and $m(K_Y + B_Y)$ has a natural generator with faithful character χ . Let $b > 0$ be the exact closed-point minimum of the quotient pair. For $r = \text{ind}_{P_0}(K_Y)$, the connected local canonical cover $W \rightarrow Y$ carries an*

abelian group extension

$$1 \longrightarrow \mu_r \longrightarrow \tilde{J} \longrightarrow J \longrightarrow 1.$$

The cover W is Gorenstein terminal and has one point over P_0 . It has an actual equivariant smooth ambient germ M_0 , of dimension three if W is smooth and of dimension four with semi-invariant equation f otherwise. Let h_B be the regular equation of mB_W , lifted equivariantly to the ambient ring. The inflated natural character is that of

$$\frac{\Omega^m}{h_B} \quad \text{or} \quad \frac{\Omega^m}{f^m h_B},$$

respectively, and its kernel is exactly μ_r . Every primitive algebraic ambient prime F centred at the closed point satisfies

$$\frac{A_{M_0}(F) - F(f) - F(h_B)/m}{|I_{\tilde{J}}(F)|} \geq b,$$

where the f -term is omitted in the smooth case. The ambient infimum is b and is attained.

The character kernel records the degree of the canonical cover. It must be retained when the ambient action is used to estimate $|J|$.

Proof. We lift the cyclic action to the canonical cover and construct the natural ambient log line. In the hypersurface case, logarithmic auxiliary coordinates make inversion of adjunction applicable; restriction of an algebraic computing prime then gives attainment on the original ambient germ.

Step 1. In this step, we lift the original cyclic action to the abelian canonical-cover extension. Use the canonical cover $\pi : (W \ni z) \rightarrow (Y \ni P_0)$ of Lemma 8.2. In its cyclic algebra construction, choose an eigen-generator α of $\omega_Y^{[r]}$. Such a choice is possible by projecting any generator to the character of its fibre at P_0 . Write

$$\alpha = \eta^{\otimes r} / c_Y$$

for a nonzero rational three-form η . Thus the function field of the resulting cover is $k(Y)(v)$, with $v^r = c_Y$. Its degree, connectedness, unique point over P_0 , and quasi-étaleness are those of that same canonical cover.

Let σ generate J , let $h = |J|$, and write $\sigma^* \alpha = \epsilon \alpha$ and $\sigma^* \eta = u \eta$. Then $\sigma^* c_Y = \epsilon^{-1} u^r c_Y$. Choose a constant ξ with $\xi^r = \epsilon^{-1}$. The rule

$$\hat{\sigma}|_{k(Y)} = \sigma^*, \quad \hat{\sigma}(v) = \xi u v$$

preserves $v^r = c_Y$, and hence preserves the normalization defining W . Since $\sigma^{h^*} \eta = \eta$, the product of the successive transforms of u is one, and $\hat{\sigma}^h(v) = \xi^h v$. This is a deck transformation. Moreover $\hat{\sigma}$ commutes with every deck transformation, since roots of unity are constants. The group generated by $\hat{\sigma}$ and μ_r is therefore abelian and fits into the required exact sequence. These algebraic automorphisms extend over an invariant neighbourhood of the marked point.

Step 2. In this step, we identify the natural log character and construct the actual algebraic ambient chart. The cover is Gorenstein terminal by Lemma 8.2. Lemma 3.8 gives the klt property of (W, B_W) , where B_W is the effective Weil pullback.

Let τ be the given natural log generator on Y . Quasi-étale pullback of rational differential tensors gives the natural equivariant equality

$$\mathcal{O}_W(m(K_W + B_W)) = \pi^* \mathcal{O}_Y(m(K_Y + B_Y)).$$

Thus $\pi^* \tau$ has the inflated character χ , whose kernel is exactly μ_r . If θ is an eigen-generator of ω_W , then

$$h_W = \theta^{\otimes m} / \pi^* \tau, \quad \text{div}_W(h_W) = mB_W.$$

This is a regular semi-invariant by effectivity and normality. In particular the boundary equation is obtained without assuming $r \mid m$. By Lemma 8.2, the singular cover is an isolated hypersurface of

embedding dimension four. Lemma 8.7 therefore supplies the stated algebraic ambient equations and their natural character in both cases.

Step 3. In this step, we establish the comparison and attainment when the canonical cover is smooth. The invariant field and local ring of $W/\tilde{J}$ are those of Y/J . Composing the finite crepant equalities shows that the two quotient boundaries agree, including their ramification terms. If W is smooth, $M_0 = W$, and Lemma 3.8 identifies each displayed ambient quotient discrepancy with the discrepancy of its primitive restriction to this same quotient. The lower bound is therefore b . The positive minimum on the fixed rational quotient pair is attained by discreteness of its discrepancies, and a primitive finite extension of a computing prime gives ambient attainment.

Step 4. In this step, we use logarithmic auxiliary coordinates and inversion of adjunction on the hypersurface chart. Suppose now that W is singular, and use its hypersurface chart in M_0 . Choose a faithful representation of $\tilde{J}$ by j characters, and form

$$W^* = W \times \mathbb{A}^j, \quad M_0^* = M_0 \times \mathbb{A}^j.$$

On W^* use the boundary $B^* = \text{pr}_W^* B_W + \sum_i \text{div}(s_i)$. On M_0^* use

$$C^* = \text{div}(f) + \frac{1}{m} \text{div}(h_B) + \sum_i \text{div}(s_i).$$

Denote their crepant quotient boundaries by B_q^* and C_q^* . Each nonidentity element acts nontrivially on both W and some auxiliary coordinate. Thus both product quotient maps are étale in codimension one. The underlying W^* and M_0^* , and their quotients, are klt, by finite discrepancy comparison with zero boundary. Set $H = h_B \prod_i s_i^m$; its restriction to W^* is nonzero. We claim that

$$(8.2) \quad \text{mld}(M_0^*/\tilde{J} \ni (z, 0)/\tilde{J}, C_q^*) = \text{mld}(W^*/\tilde{J} \ni (z, 0)/\tilde{J}, B_q^*),$$

To apply Theorem 8.4 to this algebraic germ, embed M_0^* equivariantly in a linear representation V , with the marked point at the origin. As in Lemma 8.7, choose semi-invariant local equations $\ell_1, \dots, \ell_{c_0}$ with independent differentials, and lift f, H to semi-invariants $\tilde{f}, \tilde{H}$ on V . The two complete intersections near the origin are defined by $(\ell_1, \dots, \ell_{c_0})$ and $(\ell_1, \dots, \ell_{c_0}, \tilde{f})$.

The representation on $T_{(z,0)} M_0^*$ is a quotient of V . Each nonidentity element acts nontrivially on $T_z M_0$, since regular eigenparameters detect a finite-order automorphism, and on an auxiliary coordinate. Hence V has no pseudoreflections. Its branch locus meets each complete intersection in codimension at least two, by the product fixed-locus calculation. Their quotients are klt as just checked. All these conditions hold on an invariant neighbourhood of the origin, which suffices for the closed-point formula.

Put $d_0 = |\tilde{J}|$. Apply Theorem 8.4, that is [NS26a, Theorem 7.1], first to M_0^* with the invariant ideal $((\tilde{f}^m \tilde{H})^{d_0})^{1/(md_0)}$, and then to W^* with $(\tilde{H}^{d_0})^{1/(md_0)}$. Both restrictions are nonzero. Both applications give the same ideal expression on $V/\tilde{J}$:

$$(\ell_1^{d_0} \dots \ell_{c_0}^{d_0})^{1/d_0} (\tilde{f}^{d_0})^{1/d_0} (\tilde{H}^{d_0})^{1/(md_0)}.$$

Because the product quotient maps have no divisorial ramification, the restricted ideals give exactly C_q^* and B_q^* . This proves (8.2). By Lemma 8.6, its right-hand side is b .

Step 5. In this step, we recover the minimum and its attainment in the original algebraic ambient field. Extend any primitive ambient prime centred at z on M_0 by the Gauss construction in the proof of Lemma 8.6. That construction preserves its pair discrepancy and full inertia without any assumption that the prime computes a minimum. The product lower bound then proves the required original ambient lower bound.

The same equality also supplies attainment in the original algebraic field. The product quotient is a fixed rational pair with positive minimum b , so choose a computing prime and lift it primitively to w on M_0^* . Its restriction to $k(M_0)$ is $\rho\nu$, with $\rho \geq 1$ and ν primitive algebraic

divisorial, by the residue argument in the product lemma. Writing e_w and e_ν for the full inertia orders, repeated use of (8.1) gives

$$b = \frac{A_{M_0^*, C^*}(w)}{e_w} \geq \rho \frac{A_{M_0}(\nu) - \nu(f) - \nu(h_B)/m}{e_w} \geq \rho \frac{A_{M_0}(\nu) - \nu(f) - \nu(h_B)/m}{e_\nu} \geq \rho b \geq b.$$

Here $I_{\tilde{J}}(w) \subseteq I_{\tilde{J}}(\nu)$, and the numerator is positive by the original lower bound just obtained. Equality forces $\rho = 1$, $e_w = e_\nu$, and equality in the original ambient comparison. Thus no formal or analytic prime is substituted for an algebraic computing prime. $\square$

For an action with cyclic image, write $\beta^P(g) = \beta(g)^P$. This is again a homomorphism, since all its values lie in the same cyclic subgroup.

Lemma 8.9. *Let $p : A \twoheadrightarrow H_0$ be a surjection of finite abelian groups, let $B \leq H_0$, and let $\alpha_1, \dots, \alpha_N$ be characters of A . If $Q \geq 1$ is an integer and*

$$|H_0| > |B|Q^N,$$

then there is $\eta \in A$ with $p(\eta) \notin B$ such that every $\alpha_i(\eta)$ has a signed phase in $(-1/Q, 1/Q)$. For a cyclic action $\beta : H_0 \rightarrow \text{Aut}(\mathbb{P}^1)$ and an integer $P \geq 1$,

$$|\ker(\beta^P)| = |\ker \beta| \gcd(|\text{im } \beta|, P).$$

The recorded characters need not be faithful, and the kernel of p does not enter the estimate.

Proof. Choose representatives in $[0, 1)$ for the values of all N characters. Partition $[0, 1)^N$ into the Q^N half-open cubes of side $1/Q$. Put $K = \ker p$. Since

$$|A| = |K||H_0| > |K||B|Q^N,$$

one cube contains more than $|K||B| = |p^{-1}(B)|$ elements of A , counted as elements even if their character vectors coincide. These elements cannot all belong to a single coset of $p^{-1}(B)$. Choose a_1, a_2 in that cube and in distinct cosets, and put $\eta = a_1 a_2^{-1}$. Then $p(\eta) \notin B$. The coordinate differences of their representatives are signed phases of the $\alpha_i(\eta)$, and each lies strictly between $-1/Q$ and $1/Q$. This also covers $Q = 1$, and the argument with the single zero-dimensional cube covers an empty list of characters.

Let $d = |\text{im } \beta|$. The P -th power map on a cyclic group of order d has kernel of order $\gcd(d, P)$. Its inverse image under β is $\ker(\beta^P)$, and each fibre of β has $|\ker \beta|$ elements. This gives the second formula. $\square$

Definition 8.10. Let $d, m \geq 1$ be integers, let a finite abelian group J_0 act diagonally on $k[[x_1, \dots, x_d]]$, and let F_0 be a nonzero semi-invariant series. For $g \in J_0$, let $\theta_i(g) \in (0, 1]$ be the positive phase of its action on x_i . The *coefficient arcs in the sector of g* are the substitutions

$$x_i = t^{\theta_i(g)} X_i(t), \quad X_i(t) \in k[[t]].$$

A cylinder is the inverse image of a constructible set under a finite jet truncation of the ordinary coefficient arc space $(\mathbb{A}^d)_\infty$; its codimension is the codimension of that constructible set in the corresponding affine jet space. If a nonempty cylinder C_0 has constant finite contact $w = \text{ord}_t F_0(t^{\theta_i(g)} X_i(t))$, its *score* is

$$s_g(C_0) = \text{codim } C_0 + \sum_i \theta_i(g) - \frac{w}{m}.$$

The substitution uses the entire series F_0 . Its finite contact conditions depend on finitely many coefficients because every $\theta_i(g)$ is positive.

Lemma 8.11. *In Definition 8.10, assume that the diagonal action of J_0 is faithful. Let $k_0 \geq 1$ be an integer, let χ_0 be the character of $(dx_1 \wedge \dots \wedge dx_d)^{\otimes m}/F_0$, and assume $|\ker \chi_0| \leq k_0$. Let $I \leq J_0$ be cyclic of order e , let $N \geq 1$ be an integer, and let a_0 be real. Suppose every nonempty*

finite-contact cylinder in every J_0 -sector has score at least a_0 , and that one such cylinder C_0 in a sector $g_0 \in I$ has score a_0 and contact $w \leq N/e$. Then

$$|J_0| \leq k_0(4eN + 1)^d.$$

Only the stated numerical properties of the cylinders enter this lemma. In particular, it makes no identification of formal and algebraic function fields.

Proof. We choose a short character element, preserve the full equation's contact under two opposite shifts, and compare their scores.

Step 1. In this step, we choose an element outside the natural-character kernel with short coordinate phases. Put $Q = 4eN + 1$, and suppose that the claimed bound fails. Apply Lemma 8.9 with the identity map on J_0 , the subgroup $\ker \chi_0$, and the d coordinate characters. We obtain $\gamma \notin \ker \chi_0$ and signed coordinate phases v_i with $|v_i| < 1/Q$. Write $\theta_i = \theta_i(g_0)$. Since $g_0 \in I$, every θ_i is at least $1/e$.

Step 2. In this step, we preserve the contact of the full series under two opposite shifts. At least one monomial x^{α_0} in the support of F_0 has degree at most N : otherwise all its terms would have order greater than N/e on C_0 . Put $z_0 = \alpha_0 \cdot v$. For any two support exponents α, β , semi-invariance gives $(\alpha - \beta) \cdot v \in \mathbb{Z}$. If both have degree at most $2N$, its absolute value is less than $4N/Q < 1$, so it is zero. Hence every term of degree at most $2N$ changes by the same exponent z_0 under $x_i \mapsto t^{v_i} x_i$, and $|z_0| < N/Q$.

Consider both substitutions $x_i^\pm = t^{\pm v_i} x_i$. All coordinates retain positive order. Every term of degree greater than $2N$ has order greater than

$$2N \left(\frac{1}{e} - \frac{1}{Q} \right) > \frac{N}{e} + \frac{N}{Q} \geq w + |z_0|.$$

The lower-degree part is multiplied by $t^{\pm z_0}$, with all its cancellations preserved. Thus the full series has contact $w \pm z_0$ on the two shifted cylinders. If F_0 is a unit, one may take $\alpha_0 = 0$, and both contacts are zero.

Step 3. In this step, we compare the opposite score increments and contradict the chosen character value. The new sectors are those of $g_0 \gamma^{\pm 1}$. Let $\theta_i^\pm \in (0, 1]$ be their positive phases and set $n_i^\pm = \theta_i \pm v_i - \theta_i^\pm$. Because $\theta_i \geq 1/e > |v_i|$, each $n_i^\pm$ is zero or one. Multiplication of the ordinary coefficient arc $X_i(t)$ by $t^{n_i^\pm}$ identifies the shifted cylinder $C_0^\pm$ and adds $\sum_i n_i^\pm$ to its codimension. Consequently

$$s_{g_0 \gamma^{\pm 1}}(C_0^\pm) = a_0 \pm \left(\sum_i v_i - \frac{z_0}{m} \right).$$

Both scores are at least a_0 . Their opposite increments therefore vanish. But the phase of $\chi_0(\gamma)$ is $m \sum_i v_i - z_0$ modulo $\mathbb{Z}$, which would then be zero. This contradicts $\gamma \notin \ker \chi_0$. $\square$

Proposition 8.12. Fix positive integers m, k_0 and a real number $L_0 > 0$. Let a finite abelian group J act faithfully on a smooth algebraic germ $M \ni o$ of dimension $D \geq 2$, fixing o . Let F be a nonzero regular semi-invariant, possibly a unit, and let χ_J be the natural character on the fibre at o of $\mathcal{O}_M(mK_M + \text{div } F)$, generated by $\Omega_M^{\otimes m}/F$. Assume $|\ker \chi_J| \leq k_0$. Give M/J the actual effective crepant quotient boundary of $(M, \text{div}(F)/m)$, and suppose its exact closed-point minimum is $a = p/q > 0$, where p, q are coprime positive integers. Assume a primitive algebraic prime T centred at o satisfies

$$A_M(T) \leq L_0, \quad \frac{A_M(T) - T(F)/m}{e} = a, \quad e = |I_J(T)|.$$

If J admits a faithful representation by j characters, for a positive integer j , then

$$|J| \leq k_0 [4e \max\{1, \lceil m(L_0 + je) \rceil\} + 1]^{D+j}.$$

Here

$$e = |I_J(T) \cap \ker \chi_J| \frac{q}{\gcd(q, m)}, \quad 0 \leq T(F) < mL_0.$$

Divisorial ramification and pseudoreflections are allowed. An abelian extension of a cyclic group by a cyclic group admits $j \leq 2$.

The computing divisor controls the contact of the full boundary equation. The auxiliary representation will remove pseudoreflections while preserving its inertia and the natural character.

Proof. We first record the inertia and contact of the given computing prime. For an action without pseudoreflections, the completed quotient formula then supplies a cylinder to which the finite-contact packing lemma applies. Logarithmic auxiliary coordinates reduce the general action to this case while preserving its full inertia and natural character.

Step 1. In this step, we determine the inertia and contact of the specified computing prime. Apply Lemma 3.8 to $(M, \text{div}(F)/m)$ and the specified T . Its primitive restriction has discrepancy p/q , so that lemma gives the asserted formula for e . Regularity gives $T(F) \geq 0$, and the computing equality gives

$$T(F) = m(A_M(T) - ea) < mA_M(T) \leq mL_0.$$

Step 2. In this step, we obtain the completed contact formula for actions without pseudoreflections. Choose actual regular eigenparameters $x_1, \dots, x_D$ at o by projecting lifts of a cotangent eigenbasis. They identify the completed local ring with $k[[x_1, \dots, x_D]]$ equivariantly. This diagonal action is faithful: an element trivial on the cotangent space fixes these parameters, hence the completion, and hence the algebraic local ring. The natural character of $(dx_1 \wedge \dots \wedge dx_D)^{\otimes m}/F$ is χ_J , since changing a canonical generator by a unit does not change its character on the fibre at the fixed point.

Assume for now that this action has no pseudoreflections. For every subgroup $K \leq J$, the quotient map $M \rightarrow M/K$ is quasi-étale, and its actual crepant boundary is

$$\Delta_K = \frac{1}{m|K|} \text{div}_{M/K}(F^{|K|}).$$

The series $F^{|K|}$ is invariant and nonzero. Finite-group invariants commute with completion, so the completed quotient is $\text{Spec } k[[x_1, \dots, x_D]]^K$. For $\tau \in \mathbb{Q}_{>0}$, the completion comparison in [NS25a, Remark 3.1, pp. 452–453] identifies the closed-point minimal log discrepancy of the algebraic pair $(M/K, \tau\Delta_K)$ with that of its completion at o/K . We then apply Theorem 8.5 to the completed invariant ideal $(F^{|K|})^{\tau/(m|K|)}$ to compute this completed minimum, using [NS25a, Section 7.1, Theorem 7.9 and Remark 7.10, $c = 0$]. Together these two comparisons give the numerical identity

$$(8.3) \quad \text{mld}(M/K \ni o/K, \tau\Delta_K) = \inf_{g \in K, C_0} \left\{ \text{codim } C_0 + \sum_i \theta_i(g) - \frac{\tau}{m} w(C_0) \right\} \quad (\tau \in \mathbb{Q}_{>0}),$$

where C_0 ranges over nonempty finite-contact cylinders of the full completed equation, as in Definition 8.10. Indeed, the invariant power has contact $|K|w(C_0)$, and the positive phases in that definition are the exponents of the theorem. Each exact-contact stratum is a cylinder; replacing it by a nonempty cylinder inside it can only increase codimension. Thus the infimum is unchanged by allowing all such cylinders. The left-hand side is the minimal log discrepancy of the original algebraic pair; the comparison requires no identification of its function field with the fraction field of its completion.

Step 3. In this step, we obtain a cylinder of bounded contact and apply the finite-contact packing estimate. Set $I = I_J(T)$ and $c_T = T(F)$. The inertia group I is cyclic of order e , by its faithful conormal character. For every primitive prime centred at o on M , its inertia for I

is contained in its inertia for J . Finite comparison and $a > 0$ therefore bound every quotient discrepancy centred at o/I below by a . The restriction of T attains a , so

$$\text{mld}(M/I \ni o/I, \Delta_I) = a.$$

Take $\varepsilon = 1/(c_T + 2)$ and $\tau = 1 - \varepsilon$. Effectivity and the same prime give

$$a \leq \text{mld}(M/I \ni o/I, \tau \Delta_I) \leq a + \frac{\varepsilon c_T}{me} < a + \frac{1}{me}.$$

For $K = I$, every positive phase and every finite contact belongs to $e^{-1}\mathbb{Z}$. The right-hand side of (8.3) for this τ therefore lies in the discrete group $(me(c_T + 2))^{-1}\mathbb{Z}$, so its finite infimum is attained by a cylinder. Its unperturbed score lies in $(me)^{-1}\mathbb{Z}$, is at least a , and is smaller than $a + 1/(me)$. Since $a = (A_M(T) - c_T/m)/e$ lies in that same group, the unperturbed score equals a . Comparing the perturbed scores gives

$$a + \frac{\varepsilon}{m}w(C_0) \leq a + \frac{\varepsilon c_T}{me}, \quad w(C_0) \leq \frac{c_T}{e}.$$

With $N = \max\{1, \lceil mL_0 \rceil\}$, this is at most N/e . The full-group case of (8.3) bounds every J -sector score below by a . Lemma 8.11 proves

$$(8.4) \quad |J| \leq k_0 [4e \max\{1, \lceil mL_0 \rceil\} + 1]^D$$

whenever the action has no pseudoreflections. This also includes the unit case $c_T = 0$, when the selected contact is zero.

Step 4. In this step, we remove pseudoreflections by logarithmic auxiliary coordinates while preserving inertia and character. For the general action, choose characters $\beta_1, \dots, \beta_j$ whose direct sum is faithful, and set

$$M^\star = M \times \mathbb{A}_{s_1, \dots, s_j}^j, \quad F^\star = F \prod_i s_i^m.$$

Every nonidentity element acts nontrivially on the tangent space of M , as shown above, and on an auxiliary coordinate. Thus its fixed space on $M^\star$ has codimension at least two. The new action has no pseudoreflections.

The natural character is unchanged, as the identity

$$\frac{(\Omega_M \wedge ds_1 \wedge \dots \wedge ds_j)^{\otimes m}}{F \prod_i s_i^m} = \frac{\Omega_M^{\otimes m}}{F} \otimes \bigotimes_i \left(\frac{ds_i}{s_i} \right)^{\otimes m}$$

shows. Lemma 8.6 preserves the exact minimum of the actual quotient pair, including the original divisorial ramification, and extends T to a primitive algebraic prime $T^\star$ with the same full inertia and

$$A_{M^\star}(T^\star) \leq L_0 + je.$$

Apply the case already proved in (8.4) to this product, with dimension $D + j$ and discrepancy bound $L_0 + je$. It gives precisely the asserted bound for $|J|$.

Finally, an abelian extension of a cyclic group by a cyclic group is generated by a generator of its kernel and a lift of a generator of its quotient. Its invariant-factor decomposition therefore has at most two nontrivial cyclic factors. Their faithful characters give the asserted representation with $j \leq 2$, using one trivial character when the group is trivial. $\square$

Proposition 8.13 (Full-ramification ambient comparison). *Let a finite abelian group J act faithfully on a smooth algebraic germ $M \ni o$ and preserve a normal klt hypersurface $W = (f = 0) \subset M$. Let f and h be regular semi-invariants, with $h|_W \neq 0$, and let $B_W = \text{div}_W(h|_W)/m$ be effective, where $m \geq 1$ and mB_W is integral Cartier. Give W/J the actual effective crepant*

quotient boundary, using the actual divisorial ramification indices, and suppose its exact closed-point minimum at o/J is $a > 0$. Then every primitive algebraic prime T over M centred at o satisfies

$$\frac{A_M(T) - T(f) - T(h)/m}{|I_J(T)|} \geq a.$$

Equality is attained by an actual primitive algebraic ambient prime. The statement allows divisorial ramification and tangent pseudoreflections; it requires neither a faithful character on W nor a cyclic presentation of J .

A subgroup may fix the hypersurface pointwise while acting on its normal line. The comparison still uses the full inertia of its action on the ambient function field.

Proof. We first prove the assertion when the action on W is faithful, using logarithmic products and inversion of adjunction. We then remove the kernel of the hypersurface action through an explicit crepant intermediate quotient and compare the full ramification indices.

Put

$$C = \operatorname{div}(f) + \frac{1}{m} \operatorname{div}(h), \quad n = \dim M.$$

An eigen-volume form Ω on M gives natural log generators $\Omega^{\otimes m}/(f^m h)$ on M and $\operatorname{Res}(\Omega/f)^{\otimes m}/(h|_W)$ on W . Their characters need not be faithful. Every quotient boundary below is defined by the finite crepant equality. Lemma 3.8 gives its coefficient $1 - (1 - b)/u$ from an upstairs coefficient b and the actual ramification index u . These boundaries are effective. Their log classes are $\mathbb{Q}$ -Cartier: a group-order power of a natural local log generator is invariant and descends to a Cartier trivialization of the corresponding log multiple.

Step 1. In this step, we establish adjunction when the hypersurface action is faithful. Assume first that J acts faithfully on W . Choose $j \geq 1$ characters $\beta_1, \dots, \beta_j$ whose direct sum is a faithful representation of J , and form

$$M^\star = M \times \mathbb{A}_{s_1, \dots, s_j}^j, \quad W^\star = W \times \mathbb{A}^j,$$

with boundaries

$$C^\star = \operatorname{pr}_M^\star C + \sum_i \operatorname{div}(s_i), \quad B^\star = \operatorname{pr}_W^\star B_W + \sum_i \operatorname{div}(s_i).$$

Write $C_q^\star$ and $B_q^\star$ for their quotient boundaries. The hypersurface W is Gorenstein, so Lemma 8.6 applies to (W, B_W) and gives

$$\operatorname{mld}(W^\star/J \ni (o, 0)/J, B_q^\star) = a.$$

Each nonidentity element has a proper fixed locus on W and acts nontrivially on an auxiliary coordinate. Its fixed locus on $W^\star$, and likewise on $M^\star$, has codimension at least two. Both product quotient maps are therefore quasi-étale. The underlying $W^\star$ is klt, and finite discrepancy comparison with zero boundary makes its quotient klt. The same holds for $M^\star$. Only these underlying varieties are required to be klt; the auxiliary boundary has coefficient one.

We apply Theorem 8.4, namely [NS26a, Theorem 7.1], to these actual algebraic germs. To check the application for a nonlinear M , embed an invariant affine representative of $M^\star$ in a linear representation V , taking spans of group translates of algebra generators vanishing at $(o, 0)$. Its local defining ideal has semi-invariant generators $\ell_1, \dots, \ell_c$ with independent differentials: lift an eigenbasis of its conormal fibre by character projections and use Nakayama's lemma. Lift f and $H = h \prod_i s_i^m$ equivariantly to semi-invariants $\tilde{f}, \tilde{H}$ on V .

The representation V has no pseudoreflections. Indeed its tangent quotient $T_{(o, 0)}M^\star$ has fixed codimension at least two for each nonidentity element; the original tangent action on M is faithful by regular eigenparameters. The branch locus of $V \rightarrow V/J$ meets the quotients of the two complete intersections defined by $(\ell_1, \dots, \ell_c)$ and $(\ell_1, \dots, \ell_c, \tilde{f})$ in codimension at least two, by the product fixed-locus calculation. These quotients are klt as just checked. These assertions hold after restricting to an invariant neighbourhood of the marked point.

Set $d = |J|$. Apply the theorem first to M^* with the invariant ideal $((\tilde{f}^m \tilde{H})^d)^{1/(md)}$, and then to W^* with $(\tilde{H}^d)^{1/(md)}$. The respective restrictions are nonzero, and both applications give the same ambient ideal

$$(\ell_1^d \cdots \ell_c^d)^{1/d} (\tilde{f}^d)^{1/d} (\tilde{H}^d)^{1/(md)}.$$

Since the product quotients have no divisorial ramification, their ideal boundaries are exactly the crepant boundaries above. Thus

$$\text{mld}(M^*/J \ni (o, 0)/J, C_q^*) = a.$$

Step 2. In this step, we compare all original ambient primes and establish algebraic attainment. Let T be any primitive algebraic prime centred at o on M . The Gauss construction in the proof of Lemma 8.6 applies to this specified prime, independently of whether it computes a minimum. It gives a primitive algebraic extension T^* with

$$I_J(T^*) = I_J(T), \quad A_{M^*, C^*}(T^*) = A_{M, C}(T).$$

Finite comparison with the product quotient therefore gives $A_{M, C}(T)/|I_J(T)| \geq a$, the asserted inequality for every original ambient prime.

The positive minimum on the rational product quotient is attained, since its discrepancies lie in a fixed discrete subgroup of $\mathbb{Q}$. Lift a computing prime primitively to w on M^* . By the restriction argument in the product lemma, $w|_{k(M)} = \rho\nu$ with $\rho \geq 1$ integral and ν primitive algebraic divisorial, centred at o . Put $e_w = |I_J(w)|$ and $e_\nu = |I_J(\nu)|$. The embedded residue fields give $I_J(w) \subseteq I_J(\nu)$, and iteration of (8.1) gives

$$a = \frac{A_{M^*, C^*}(w)}{e_w} \geq \rho \frac{A_{M, C}(\nu)}{e_w} \geq \rho \frac{A_{M, C}(\nu)}{e_\nu} \geq \rho a \geq a.$$

The second inequality uses the positive lower bound just proved. Equality forces $\rho = 1$, $e_w = e_\nu$, and $A_{M, C}(\nu)/e_\nu = a$. This is attainment by an actual algebraic ambient prime over the given field.

Step 3. In this step, we remove the hypersurface-action kernel and recover the full original ramification comparison. Let K be the kernel of $J \rightarrow \text{Aut}(W)$. If W is singular at o , then $f \in \mathfrak{m}_o^2$, so its cotangent space equals that of M . Every element of K acts trivially on that space, hence on regular eigenparameters and on the local ring of M . Faithfulness on M then gives $K = 1$, the case already proved.

Suppose that W is smooth and $K \neq 1$. Extend f to regular simultaneous eigenparameters $(f, y_1, \dots, y_{n-1})$ for the abelian group J . The restrictions of the y_i to W are nonzero, so their characters are trivial on K . Thus K acts only in the normal coordinate f , and this character is faithful. It follows that K is cyclic. Set $s = |K|$ and $\pi_K : M \rightarrow M_0 = M/K$. The invariant completed local ring has regular parameters $f^s, y_1, \dots, y_{n-1}$. Regularity descends from completion, so M_0 is an actual smooth algebraic germ. Its hypersurface W_0 with equation $f_0 = f^s$ is isomorphic to W : taking invariants is exact and K acts trivially on W .

The restriction $h|_W \neq 0$ likewise forces the character of h on K to be trivial. Hence h descends to a regular semi-invariant h_0 on M_0 . The only ramification divisor of π_K near o is $(s-1)W$. Since $\pi_K^* \text{div}(f_0) = sW$, the finite canonical formula gives

$$K_M + C = \pi_K^*(K_{M_0} + C_0), \quad C_0 = \text{div}(f_0) + \frac{1}{m} \text{div}(h_0).$$

In particular the coefficient-one hypersurface boundary absorbs exactly the normal ramification.

The residual abelian group $J_0 = J/K$ acts faithfully on W_0 and on M_0 . Its hypersurface quotient is the original W/J with its prescribed crepant boundary and minimum a . The faithful case therefore applies to (M_0, W_0, h_0, J_0) . To return to the original full group, let T_0 be the primitive restriction of T to $k(M_0)$, and set $e_K = |I_K(T)|$. Lemma 3.8 and multiplication of ramification indices in the tower give

$$A_{M, C}(T) = e_K A_{M_0, C_0}(T_0), \quad |I_J(T)| = e_K |I_{J_0}(T_0)|.$$

Thus the required quotient discrepancy for T equals that for T_0 , which is at least a . Conversely, a primitive algebraic extension to $k(M)$ of an ambient computing divisor on M_0 is centred at o , the unique point over its image, and the same identities give equality. This proves both assertions with the original full inertia group. $\square$

9. BOUNDED AMBIENT COMPUTING DIVISORS

We retain the discrepancy and inertia conventions of Section 2. The first construction passes from a divisor over a hypersurface to a divisor over its smooth ambient germ, while preserving the boundary order and the full inertia group. The second applies this construction to the canonical cover of a terminal threefold pair.

Theorem 9.1 (Inversion of adjunction for a local complete intersection). *Let $X \subset A$ be a normal local complete intersection of codimension c in a smooth variety, $W \subsetneq X$ closed, and $\mathfrak{a} = \prod_i \mathfrak{a}_i^{q_i}$, $q_i > 0$, an $\mathbb{R}$ -ideal on A with nonzero restricted factors. Then*

$$\mathrm{mld}_W(X, \mathfrak{a}\mathcal{O}_X) = \mathrm{mld}_W(A, \mathcal{I}_X^c \mathfrak{a}).$$

For a normal Cartier divisor H in a normal lci variety X , the corresponding equality is

$$\mathrm{mld}_W(H, \mathfrak{a}\mathcal{O}_H) = \mathrm{mld}_W(X, H + \mathfrak{a}),$$

with $W \subsetneq H$ and nonzero restricted factors [EM06, Theorem 8.1 and Corollary 8.2].

We use this equality on normal surface sections and on smooth Cartier slices, always with a nonzero restricted equation.

Theorem 9.2 (Approximation compatible with substitution). *Let K be a field, x, y, z, t finite tuples, and $F(x, y, z), G(x)$ tuples of algebraic power series vanishing at the origin. Suppose formal tuples $\Phi(x), \Psi(t)$ vanish at the origin and $F(x, \Phi(x), \Psi(G(x))) = 0$. For every $N > 0$ there are algebraic power series $\tilde{\Phi}(x), \tilde{\Psi}(t)$, congruent to Φ, Ψ to order N , for which $F(x, \tilde{\Phi}(x), \tilde{\Psi}(G(x))) = 0$ [FQS16, Theorem 1.1].*

The fixed substitution in this theorem will preserve the two base variables of the conic presentation.

Theorem 9.3 (A bounded computing divisor on a terminal threefold). *Let $\Gamma \subset [0, 1]$ satisfy the descending chain condition. There is an integer $L(\Gamma) > 0$ such that every threefold germ $(X \ni x, B)$, with X terminal, $B \in \Gamma$, and $\mathrm{mld}(X \ni x, B) \geq 1$, has a prime divisor E centred at x satisfying*

$$A_{X,B}(E) = \mathrm{mld}(X \ni x, B), \quad A_X(E) \leq L(\Gamma)$$

[HLL25, Theorem 1.2].

This supplies some computing divisor. We will first identify the ordinary minimum to which it applies.

Lemma 9.4. *Let $\psi : (N \ni o') \rightarrow (M \ni o)$ be a pointed étale morphism of normal algebraic germs over k , where o', o are closed points. Restriction and primitive normalization give a bijection between primitive divisorial valuations centred at o' and those centred at o . The corresponding discrete valuation rings have ramification index one and identical residue fields. If K_M is $\mathbb{Q}$ -Cartier, the correspondence preserves bare discrepancies; it also preserves the orders of pulled-back rational functions. If a finite group acts equivariantly on the pointed morphism, the correspondence preserves decomposition groups, inertia groups and conormal characters.*

The assertion concerns the selected points. It does not identify the two function fields.

Proof. Shrink N so that the fibre over o consists of the chosen reduced point. Expose a primitive divisor E over M on a proper birational model $\hat{M} \rightarrow M$. Since E maps to o , its base change to N is

$$E \times_M N = E \times_{\mathrm{Spec} k} \{o'\} \simeq E.$$

The base-changed model is étale over $\widehat{M}$ and birational over N . The displayed divisor is therefore the unique extension centred at o' ; its ramification and residue degrees are one. Conversely, the restriction of a divisorial valuation through the finite function-field extension $k(N)/k(M)$ is divisorial. After exposing that restriction on a model of M , the same base change proves uniqueness and recovers the given valuation. Pullback of canonical divisors through étale maps proves the discrepancy assertion. All identifications are equivariant when ψ is equivariant. They identify the residue actions and the action on a uniformizer modulo its square, which proves the last assertion. $\square$

Lemma 9.5. *Let $M \ni o$ be a smooth algebraic fourfold germ, and let a finite abelian group J act faithfully on M , fixing o . Let f, h be regular semi-invariants, suppose $W = (f = 0)$ is normal and integral, and assume that the induced J -action on W is faithful and $h|_W \neq 0$. Fix an integer $m \geq 1$ and a real number $a > 0$, and suppose $(W, \operatorname{div}_W(h)/m)$ is klt. Let F be a primitive divisor over W , centred at o , and set*

$$I = I_J(F), \quad e = |I|, \quad P_0 = A_W(F), \quad b = F(h|_W).$$

Suppose there are I -equivariant pointed étale maps

$$N \longrightarrow M, \quad N \longrightarrow M_0 = \mathbb{A}_u^1 \times S,$$

where $S \ni s$ is a smooth threefold germ, the distinguished point of M_0 is $(0, s)$, and the pullback of f equals the pullback of $f_0 = u^2 + g$, with g regular on S . Assume I preserves the product projection and makes u an eigenfunction. Let F_0 be the divisor on $(f_0 = 0)$ corresponding to F by Lemma 9.4, and put $c = F_0(u) > 0$. Assume $P_0 - b/m = ea$ and that every primitive ambient divisor T_0 centred at o satisfies

$$\frac{A_M(T_0) - T_0(f) - T_0(h)/m}{|I_J(T_0)|} \geq a.$$

Then there is a primitive algebraic divisor T over $k(M)$, centred at o , such that

$$I_J(T) = I, \quad T(h) = b, \quad A_M(T) - T(f) - T(h)/m = ea,$$

and

$$A_M(T) \leq P_0 + \max\{2c, c + b\} + e.$$

A bound on the order of the square coordinate thus gives the required bound on the ambient discrepancy. The equation of the effective boundary is retained throughout the construction.

Proof. We deform the square coordinate in the completed valuation ring, restrict the resulting valuation to the algebraic ambient field, and use the assumed inequality to determine its primitive normalization and full inertia.

Step 1. In this step, we deform the square coordinate while preserving the full boundary order. Write the completed valuation ring of F_0 as $\kappa(F_0)[[t]]$, with I fixing the coefficient field and acting on t by its faithful conormal character λ . To obtain this presentation, lift a separating transcendence basis of the residue field through the invariant valuation ring, lift its finite separable residue extension by Hensel's lemma, and project a uniformizer onto its conormal eigenspace. The inertia is cyclic and the conormal character is faithful by Lemma 3.8.

Let n_0 be the least integer satisfying

$$n_0 > \max\{2c, c + b\}, \quad n_0 \equiv 2c \pmod{e}.$$

Thus $n_0 \leq \max\{2c, c + b\} + e$. Introduce a residue variable z transcendental over $\kappa(F_0)$, fixed by I , and keep the expansions of all functions from $k(S)$ unchanged. Replace the expansion u_F of u by

$$u_* = u_F \left(1 + \frac{t^{n_0} z}{u_F^2} \right)^{1/2}.$$

The root with constant term one exists because $n_0 > 2c$. It gives

$$u_*^2 + g = t^{n_0}z, \quad \text{ord}_t(u_* - u_F) = n_0 - c > b.$$

Every parameter of M_0 still has positive order. In the completed local ring of N , the pullback h_N of h is a power series in u and parameters from S . Changing only u by order greater than b changes this series only in orders greater than b . This assertion can be checked on a finite jet of degree greater than b , since all parameter orders are positive integers. Consequently the new order of h is exactly b , including any cancellation among its original terms.

Step 2. In this step, we restrict the deformation valuation to an algebraic ambient divisor. The finite map $(f_0 = 0) \rightarrow S$ has degree two. The restriction of F_0 to $k(S)$ has nonzero discrete value group $d_S\mathbb{Z}$ and residue transcendence degree two. The new element $f_0 = t^{n_0}z$ is transcendental over this valued base field, so substitution defines an injection of $k(M_0) = k(S)(u)$ into $\kappa(F_0)(z)((t))$. Choose $r \in k(S)$ of value d_S . The residue of $f_0^{d_S}/r^{n_0}$ is transcendental over the residue field of the base. The restricted valuation therefore has residue transcendence degree three and nonzero discrete value group. Its primitive normalization is an algebraic prime T_0 over M_0 . For example, it is exposed by normalizing a model of a positive-value generator and three independent residue parameters. Transport T_0 through N back to a primitive divisor T over M by Lemma 9.4. Write the unnormalized valuation on M_0 as $d \text{ord}_{T_0}$. This transport preserves d , discrepancy, and the computed order of the actual function h_N , even though h_N need not descend to M_0 .

Step 3. In this step, we determine the discrepancy, primitive normalization, and full inertia. Choose a local volume form η on S . On $(f_0 = 0)$ the residue form $\theta = \eta/(2u)$ has order $P_0 - 1$ at F_0 ; hence η has order $P_0 + c - 1$ in the completed valuation ring. The identity

$$2u_*\eta \wedge du_* = \eta \wedge df_0$$

shows that the ambient volume form has order $P_0 + n_0 - 1$. Indeed the independent residue differential dz in $df_0 = n_0 t^{n_0-1}z dt + t^{n_0}dz$ contributes n_0 , and division by u_* subtracts c . For the scaled primitive valuation this order is $dA_M(T) - 1$: differentiation of a primitive uniformizer of order d contributes $d - 1$, and the residue extension is separable. Thus

$$dA_M(T) = P_0 + n_0, \quad dT(f) = n_0, \quad dT(h) = b,$$

and in particular

$$d(A_M(T) - T(f) - T(h)/m) = ea.$$

The character of u_F is λ^c . The congruence for n_0 makes $t^{n_0}z/u_F^2$ invariant, so the selected root is equivariant. The group I fixes every residue of the new valuation. Consequently $I \subseteq I_J(T)$. Putting $e_T = |I_J(T)|$, the original ambient inequality gives

$$a \leq \frac{ea}{de_T} \leq a.$$

Since $a > 0$, $d \geq 1$ and $e_T \geq e$, equality forces $d = 1$ and $e_T = e$. This proves the inertia and contact assertions, and $A_M(T) = P_0 + n_0$ gives the bound. $\square$

Lemma 9.6. *Let $S \ni s$ be a smooth algebraic surface germ, let Q be a nondegenerate binary quadratic form, and let $g \neq 0$ be regular on S . Suppose $V = (Q + g = 0) \subset \mathbb{A}^2 \times S$ is normal. Let D be a primitive divisor over V , centred at $(0, s)$. Then its restriction to $k(S)$ is ρord_{D_S} for a primitive divisor D_S centred at s and an integer $\rho \geq 1$, and*

$$A_V(D) \geq \rho A_S(D_S).$$

If $Q = u^2 + v^2$, then

$$A_V(D) \geq \rho A_S(D_S) + |D(u) - D(v)|.$$

These estimates compare the given divisor with its base restriction; they do not require it to be monomial.

Proof. Over k , write $Q = xy$ after a constant linear change. Then $k(V) = k(S)(x)$. The restriction is nontrivial and discrete. The residue transcendence inequality for this extension of transcendence degree one, together with the divisorial residue degree two of D , shows that the restricted residue field has transcendence degree at least one. The rank-one Abhyankar inequality gives the opposite bound. Thus the restriction is divisorial and has the asserted form.

On a smooth model exposing D_S , write a local volume form on S as

$$\eta = t^{A_S(D_S)-1} \varepsilon dt \wedge dz,$$

where t is a uniformizer, z is a separating residue parameter, and ε is a unit. A canonical residue form on V is $\eta \wedge dx/x$. In a uniformizer of D , a differential of positive order loses at most one order in its uniformizer term, and a nonzero wedge contains at most one such term. The order of this residue form is therefore at least $\rho A_S(D_S) - 1$, proving the first inequality. For $Q = u^2 + v^2$, use instead

$$\frac{\eta \wedge du}{2v}, \quad -\frac{\eta \wedge dv}{2u}.$$

Their respective lower bounds give the two signed differences $D(u) - D(v)$ and $D(v) - D(u)$, hence the second inequality. $\square$

Proposition 9.7. *Fix an integer $m \geq 1$ and a real number $L_0 > 0$. Let a finite abelian group J act faithfully on a smooth algebraic fourfold germ $M \ni o$, fixing o , and faithfully on an isolated cDV hypersurface $W = (f = 0)$. Let f, h be regular semi-invariants with $h|_W \neq 0$, and suppose (W, B_W) is klt for $B_W = \text{div}_W(h)/m$. Assume the actual crepant quotient pair on W/J , including all divisorial ramification, has exact closed-point minimum $a > 0$. If a primitive algebraic prime F_W centred at o satisfies*

$$A_W(F_W) \leq L_0, \quad A_{W, B_W}(F_W) = ea, \quad e = |I_J(F_W)|,$$

then there is a primitive algebraic prime T over the original field $k(M)$, centred at o , such that

$$I_J(T) = I_J(F_W), \quad T(h) = F_W(h|_W), \quad A_M(T) - T(f) - T(h)/m = ea,$$

$$A_M(T) \leq C_m L_0 + 3e, \quad C_m = \max\{2m + 5, 31, m + 16\}.$$

The conclusion controls the divisor in the original ambient field and its inertia for the whole group. No invariance of the prescribed computing divisor under that whole group is assumed.

Proof. We first prepare the quadratic part by equivariant algebraic étale maps. A binary quadratic part is handled by Gauss valuations and normal deformation. In rank one, the reduced and pure-cube cubic terms give a bound on the square coordinate, whereas a double-line cubic gives a conic after nested approximation. Every constructed divisor is returned to the original field before the quotient comparison is applied.

Step 1. In this step, we record the original quotient comparisons that control normalization and inertia. Put

$$I = I_J(F_W), \quad P_0 = A_W(F_W), \quad b = F_W(h|_W).$$

Then I is cyclic of order e , and

$$P_0 - b/m = ea > 0, \quad 0 \leq b < mP_0 \leq mL_0.$$

Proposition 8.13 gives

$$(9.1) \quad \frac{A_M(T_0) - T_0(f) - T_0(h)/m}{|I_J(T_0)|} \geq a$$

for every primitive ambient prime T_0 centred at o . The corresponding intrinsic inequality on W follows from Lemma 3.8 and the quotient minimum. These comparisons are always used for the full original group J .

We will repeatedly use the following consequence. Suppose a constructed integer-valued valuation returns to M as $d_{\text{ret}} \text{ord}_T$, contains I in its inertia, preserves the homogeneous boundary contact b , and has homogeneous pair discrepancy at most $P_0 - b/m$. Then

$$ea \geq d_{\text{ret}}(A_M(T) - T(f) - T(h)/m) \geq d_{\text{ret}}|I_J(T)|a \geq ea.$$

Positivity forces $d_{\text{ret}} = 1$, $I_J(T) = I$, and equality of all the displayed discrepancies. The same argument applies to an intrinsic replacement of F_W . Thus neither primitive normalization nor extra inertia can be discarded in advance.

Step 2. In this step, we split the quadratic part through equivariant algebraic étale charts. The quadratic part q_2 of f is nonzero, since a general Du Val section has multiplicity two. Choose an I -stable nondegenerate subspace of dimension one if $\text{rank } q_2 = 1$, and of dimension two otherwise. For the latter choice, the quadratic form pairs character spaces whose product is the equation character. Distinct paired spaces supply a binary subspace. Among self-paired characters there are at most two, since I is cyclic; two vectors in one such space or one from each supply the remaining cases. Take an invariant orthogonal complement.

Character projection lifts these cotangent vectors to regular eigenparameters. Their map to affine four-space is étale. Let ξ be the chosen one or two quadratic coordinates and ζ the remaining coordinates. The locus $C_0 = (\partial f / \partial \xi = 0)$ is smooth and étale over the ζ -space, because its normal Hessian is invertible. After base change by C_0 , it is a critical section. With $\delta\xi = \xi - \xi|_{C_0}$ and $g = f|_{C_0}$, write

$$f - g = (\delta\xi)^T A \delta\xi, \quad A(o) = Q,$$

where Q is the matrix of the chosen nondegenerate form. The section ideal is generated by $\delta\xi$, so such a regular symmetric matrix A exists. Averaging its coefficients with the equation character makes the identity equivariant. The matrix $Q^{-1}A$ is Q -self-adjoint and equals the identity at o . Adjoin the solution R near the identity of

$$R^2 = Q^{-1}A.$$

The derivative in R is twice the identity there, so this is a pointed étale extension. Uniqueness in the completion implies that R is Q -self-adjoint and transforms by conjugation under I . Hence $R^T Q R = A$, and $\xi' = R\delta\xi$ gives $f = Q(\xi') + g$ exactly.

We obtain pointed I -equivariant étale maps

$$(9.2) \quad N \longrightarrow M, \quad N \longrightarrow M_0 = \mathbb{A}^{\dim \xi'} \times S, \quad f_0 = Q(\xi') + g(S).$$

Here S is smooth. If necessary, base change once more by the inverse image of the zero section to give the second map a section over $\xi' = 0$. All actions are realized on a common pointed neighborhood by taking the fibre product of its finitely many I -translates. Lemma 9.4 transports F_W to a primitive divisor F_0 on $W_0 = (f_0 = 0)$, preserving P_0 , residue field and conormal action. Write h_N for the actual pullback of h . We do not require h_N to descend to M_0 .

Step 3. In this step, we construct the bounded ambient divisor when the two quadratic factors are preserved. Suppose $\text{rank } q_2 \geq 2$. Write

$$F_0|_{k(S)} = \rho \text{ord}_{D_S}, \quad k_0 = \rho A_S(D_S) \leq P_0.$$

The last inequality is Lemma 9.6. The two isotropic lines of Q are permuted by I . If each is preserved, write $Q = xy$ with x, y eigenfunctions, and put $c_x = F_0(x)$, $c_y = F_0(y)$. If both $c_x, c_y > b + e$, choose the integer $d_x \in (b, b + e]$ congruent to c_x modulo e . On $k(W_0) = k(S)(x)$, give x Gauss weight d_x , keeping the base valuation. Its y -weight is $c_x + c_y - d_x > b$. The value group is discrete and the residue field has transcendence degree two, so its primitive normalization is an algebraic divisor. Its homogeneous bare discrepancy is k_0 : this follows from the residue form $\eta_S \wedge dx/x$ and the independent Gauss residue.

The section in (9.2) gives a regular base function $h_0 = h_N|_{x=y=0}$, with $h_N - h_0 \in (x, y)$. Both the old and new factor orders exceed b ; hence both contacts equal the unchanged base value

$F_0(h_0) = b$. The conormal character acts in degree n of the old graded field by its n -th power. The new x -weight has the same character modulo e , so I fixes the entire Gauss residue field. Return the primitive divisor through N to W . The intrinsic equality argument above forces primitivity, full inertia I , and bare discrepancy P_0 . After this replacement and exchange of x, y if necessary, we have an intrinsic divisor with x -order $c \leq b + e$, the same contact b , and the same inertia and bare discrepancy as F_W .

On the product fields use the independent variable $s_0 = f_0$:

$$k(M_0) = k(W_0)(s_0), \quad y_{M_0} = y_{W_0} + s_0/x.$$

Choose the least integer $n > c + b$ congruent modulo e to the equation character in the conormal grading. Thus $n \leq c + b + e \leq 2b + 2e$. The Gauss extension with s_0 -weight n has homogeneous bare discrepancy $P_0 + n$, since

$$\eta_S \wedge dx \wedge dy_{M_0} = \pm(\eta_S \wedge dx/x) \wedge ds_0.$$

All parameter values remain positive. The change of y has order $n - c > b$, so the formal series of h_N retains contact b . The weight congruence preserves I -inertia. The Gauss valuation is algebraic and divisorial, with one new independent residue parameter. Transporting it back and applying (9.1) gives

$$I_J(T) = I, \quad T(h) = b, \quad A_M(T) \leq P_0 + 2b + 2e.$$

Step 4. In this step, we treat a binary quadratic part whose factors are exchanged. In the other case take eigenvariables u, v with $Q = u^2 + v^2$. Their character ratio is the nontrivial order-two character of I ; thus e is even. After exchanging them, write

$$F_0(u) = c < F_0(v) = c + \delta_q, \quad \delta_q \equiv e/2 \pmod{e}.$$

The equation gives $F_0(g) = 2c$, and Lemma 9.6 gives $P_0 \geq k_0 + \delta_q$, with k_0 as above. If $c > b + e$, choose $c_0 \in (b, b + e]$ congruent to c , and put $d_0 = e/2 \leq \delta_q$. Keep the base valuation and give u, v Gauss weights $c_0, c_0 + d_0$, defining a homogeneous valuation v_G . Then

$$v_G(f_0) = 2c_0, \quad A_{M_0}(v_G) = k_0 + 2c_0 + d_0.$$

The first equality uses $F_0(g) = 2c > 2c_0$. Evaluation at the zero section again preserves contact b , and the weight congruences preserve I -inertia. The homogeneous pair discrepancy is at most $P_0 - b/m$. The original comparison therefore forces equality after return, and yields $A_M(T) \leq P_0 + 2b + 2e$. If $c \leq b + e$, apply Lemma 9.5, with base $\mathbb{A}_v^1 \times S$. Its bound is

$$A_M(T) \leq P_0 + \max\{2c, c + b\} + e \leq P_0 + 2b + 3e < (2m + 1)L_0 + 3e.$$

This completes every quadratic-rank-at-least-two case.

Step 5. In this step, we reduce quadratic rank one to the three possible nonzero cubic types. We now have $f_0 = u^2 + g(x, y, z)$, where S is a smooth threefold and g has order at least three. The critical shift used above starts in degree two, so the cubic g_3 is precisely the cubic on the original quadratic radical. The germ W_0 is still isolated cDV.

The cubic cannot vanish. Otherwise a general hyperplane with nonzero y -coefficient eliminates y as a linear combination of u, x, z . Give these variables weights $(2, 1, 1)$. The section equation has order at least four, equal to the sum of the weights. This produces an ambient divisor of nonpositive discrepancy for that coefficient-one equation. A general section is Du Val, hence normal and klt, contradicting Theorem 9.1.

For this use of the complex theorem, and its uses below, apply Lemma 3.11 to a common finite diagram consisting of the smooth germs, their Cartier sections, the equations and any divisor being tested. Realize it over a field embedded in $\mathbb{C}$. The explicit surface resolutions and the orders of canonical and Cartier pullbacks are preserved. Apply the theorem there and return these same discrepancy inequalities. No assertion about an unrelated complex divisor is needed.

A nonzero ternary cubic is reduced, a cube of a linear form, or L^2R for independent linear forms L, R . Indeed every repeated irreducible factor has degree one. We treat these three alternatives separately.

Step 6. In this step, we bound the square-coordinate order for reduced and pure-cube cubic terms. If g_3 is reduced, a general plane in the base cuts its projective cubic in three distinct smooth points. The restricted plane curve has three distinct tangent lines. One blowup separates its three smooth branches; for coefficient $2/3$, the crepant exceptional coefficient is one. Thus for this smooth Cartier plane $H_2 \subset S$,

$$(H_2, (2/3) \operatorname{div}(g|_{H_2})) \text{ is lc.}$$

If g_3 is a cube, choose a base coordinate so that $g_3 = x^3$, and write $g = x^3 + g_4 + g_5 + g_{\geq 6}$. At least one of the binary forms

$$Q_4 = g_4(0, y, z), \quad Q_3 = (\partial_x g_4)(0, y, z), \quad Q_5 = g_5(0, y, z)$$

is nonzero. If all vanished, x^2 would divide g_4 and x would divide g_5 . On a general hyperplane eliminating y , weights $(3, 2, 1)$ on (u, x, z) would then give equation order at least six and weight sum six. This contradicts a normal Du Val section by the same inversion argument.

Choose a base plane through the x -axis and a direction where the first nonzero form among Q_4, Q_3, Q_5 does not vanish. In its coordinates (x, t) , the respective weighted initial equations, weights and weighted degrees are

$$\begin{array}{l|l|l} x^3 + \alpha t^4 & (4, 3) & 12 \\ x^3 + \alpha x t^3 & (3, 2) & 9 \\ x^3 + \alpha t^5 & (5, 3) & 15 \end{array} \quad (\alpha \neq 0).$$

All omitted terms have strictly larger indicated weight. Their thresholds are respectively $7/12, 5/9, 8/15$. Here is the resolution calculation, including the higher terms. For $x^3 + \alpha t^r$, $r = 4, 5$, the sole compact Newton edge joins $(3, 0)$ to $(0, r)$. Its face equation is smooth on the torus and has no zero at the two endpoints of the weighted exceptional line. A regular toric refinement therefore gives simple normal crossings. The ratio on a positive ray (v_1, v_2) is

$$\frac{v_1 + v_2}{\min(3v_1, rv_2)},$$

whose minimum is $(3 + r)/(3r)$; strict components have ratio one. In the middle case the first edge has normal $(3, 2)$ and ratio $5/9$. If t^n is the first pure t -term, then $n \geq 5$ and the only additional edge has normal $(n - 3, 1)$ and ratio $(n - 2)/n \geq 3/5$. Its face is $t^3(\alpha x + \beta t^{n-3})$, smooth on the torus. If there is no pure term, x is a reduced component of ratio one. The first face is also smooth on the torus. All remaining exponents lie above these edges; regular toric refinement and linearity on each cone give the asserted threshold $5/9$. This calculation concerns the plane restriction, and assumes no Newton nondegeneracy of g . In particular, in every cube case

$$(H_2, (8/15) \operatorname{div}(g|_{H_2})) \text{ is lc.}$$

Set $\tau_0 = 2/3$ in the reduced case and $\tau_0 = 8/15$ in the cube case. The plane is smooth Cartier, its restricted equation is nonzero, and its centre is the closed point. Theorem 9.1, with the field realization already specified, gives for every primitive divisor D_S centred there

$$A_S(D_S) \geq D_S(H_2) + \tau_0 D_S(g) \geq \tau_0 D_S(g).$$

The double cover $W_0 \rightarrow S$ has branch pair $(S, \frac{1}{2} \operatorname{div} g)$: normality excludes a repeated divisorial factor of g , and integrality excludes a square in $k(S)$. Write $F_0|_{k(S)} = \rho \operatorname{ord}_{D_S}$, with $\rho \in \{1, 2\}$, and put $c = F_0(u)$. Lemma 3.8 gives

$$2c = \rho D_S(g), \quad P_0 = \rho A_S(D_S) - c \geq (2\tau_0 - 1)c.$$

Thus $c \leq 3P_0$ in the reduced case and $c \leq 15P_0$ in the cube case. The square-deformation lemma now gives respectively

$$A_M(T) \leq \max\{7, m+4\}L_0 + e, \quad A_M(T) \leq \max\{31, m+16\}L_0 + e.$$

In both cases it preserves the original contact and full inertia.

Step 7. In this step, we prepare the double-line cubic as an exact algebraic conic equation. It remains that $g_3 = L^2 R$. Its two factors have different multiplicities, so I preserves each. Choose eigenparameters (x, z, w) on the radical with $g_3 = x^2 z$; then

$$\chi_f = \chi_u^2 = \chi_x^2 \chi_z.$$

We put the actual equation into a conic form while keeping w as a base coordinate. At total degree $d \geq 4$, the terms to remove have a unique expression

$$x^2 A_{d-2}(x, z, w) + xz D_{d-2}(z, w).$$

The remaining terms are of the form $xB_{d-1}(w) + C_d(z, w)$. Replace z by $z - A_{d-2}$ and x by $x - D_{d-2}/2$. The differential of $x^2 z$ cancels the displayed terms. Every new error has degree at least $d+1$: the correction has degree $d-2 \geq 2$, and the terms quadratic in it have degree at least $2d-3$. The characters of A_{d-2}, D_{d-2} are respectively χ_z, χ_x . Iteration gives an equivariant formal change tangent to the identity, with w fixed, for which

$$(9.3) \quad g(\phi_x, \phi_z, w) = x^2 z + xB_{\text{for}}(w) + C_{\text{for}}(z, w),$$

where the last two orders are at least three and four.

We apply Theorem 9.2 to this precise substitution problem. Put $X = (x, z, w)$ and $v = (x^e, z^e, w^e)$. An eigenseries of character χ_0 has the unique expansion

$$\sum_{\substack{0 \leq \alpha_i < e \\ \chi(\bar{X}^\alpha) = \chi_0}} X^\alpha U_\alpha(v).$$

Use this expansion for ϕ_x, ϕ_z , and its two-variable version for $B_{\text{for}}, C_{\text{for}}$. The coefficient functions for the latter depend only on (v_z, v_w) . Replace these coefficient functions by their known constants plus indeterminates. The difference in (9.3) is an algebraic power series in X and those indeterminates. Expand it in the finite basis X^α over $k[[v]]$. Its coefficients are algebraic: projection onto an exponent class is averaging under the coordinatewise μ_e^3 action, followed by division by X^α . The quotient is invariant and algebraic over the rational field in v and the indeterminates.

Set this finite list of coefficients equal to zero. After subtracting their known constants, all unknown series vanish at the origin. The formal solution is the one just constructed. Apply the approximation theorem with the fixed projection $v \mapsto (v_z, v_w)$, to order at least five in v . Reassembling the eigenseries gives an exact algebraic solution

$$g(\phi_{x,\text{alg}}, \phi_{z,\text{alg}}, w) = x^2 z + xB(z, w) + C(z, w).$$

The retained jets give an identity tangent map and $\text{ord } B \geq 3, \text{ord } C \geq 4$. Characters are built into the expansions. The new B may depend on z , which is permitted.

These algebraic power series belong to the henselization of the polynomial local ring. Realize the finitely many of them, and the inverse change with invertible Jacobian, on common pointed étale neighborhoods. Taking the fibre product of their I -translates realizes the equivariance. We obtain new maps $N_c \rightarrow M$ and $N_c \rightarrow M_c$, with

$$M_c = \mathbb{A}_{u,x}^2 \times S_c, \quad f_c = u^2 + zx^2 + B(z, w)x + C(z, w),$$

where S_c is smooth and étale over the (z, w) -plane. The equation is exact. The boundary is still the original function $h_c = h|_{N_c}$ on the common neighborhood. Let F_c be the transported intrinsic divisor; it has bare discrepancy P_0 , contact b , and the same conormal action of I .

Step 8. In this step, we construct the quadratic extension of the conic and retain its differential and inertia data. Adjoin t_0 with $t_0^2 = z$. The base change S'_c of S_c is smooth and étale over the (t_0, w) -plane. On the actual extended rational fields put

$$U = 2t_0u, \quad Y_1 = 2t_0^2x + B(t_0^2, w), \quad R_c = 4t_0^2C - B^2, \quad f'_c = U^2 + Y_1^2 + R_c = 4t_0^2f_c.$$

The coordinates (U, Y_1) define the smooth ambient $M'_c = \mathbb{A}_{U, Y_1}^2 \times S'_c$. The function R_c is nonzero. Otherwise regularity of C and $4zC = B^2$ would imply $z \mid B$, and the branch polynomial would be $z(x + B/(2z))^2$. The original hypersurface would then be singular in codimension one, contrary to normality. The intrinsic quadratic extension also has degree two: if z became a square in $k(S_c)(x)(u)$, its square class would equal that of $zx^2 + Bx + C$. A quadratic polynomial in x could then have only even-order roots, forcing $B^2 = 4zC$, already excluded.

Let F' be a primitive extension of F_c , with ramification $\rho_c \in \{1, 2\}$. The group I lifts with the quadratic deck involution to a group of order $2e$. Its inertia I' at F' is cyclic of order $q_c = \rho_c e$, maps onto I , and intersects the deck group in order ρ_c . These are the inertia identities in the Galois valuation tower. The deck involution sends (t_0, U, Y_1, w) to $(-t_0, -U, Y_1, w)$.

The hypersurface $V' = (f'_c = 0)$ is normal. A constant linear change puts its equation in the form $XY + R_c = 0$; it is integral, and its singular locus has codimension at least two, while it is Cohen–Macaulay. The rational volume identity

$$(9.4) \quad \frac{du \wedge dx \wedge dz \wedge dw}{f_c} = 2 \frac{dU \wedge dY_1 \wedge dt_0 \wedge dw}{f'_c}$$

includes every power of t_0 . Taking residues and using tame ramification gives

$$P' = A_{V'}(F') = \rho_c P_0, \quad b' = \rho_c b.$$

Set

$$d_c = F'(t_0), \quad v_U = F'(U), \quad v_Y = F'(Y_1), \quad b_B = F'(B),$$

with $b_B = +\infty$ if $B = 0$. Original centering gives

$$v_U > d_c, \quad F'(Y_1 - B) > 2d_c, \quad F'(w) > 0.$$

Write the restriction to $k(S'_c)$ as $s_c \text{ord}_{D_c}$, and put $k_c = s_c A_{S'_c}(D_c)$. Lemma 9.6 and the smooth differential inequality on the base give

$$(9.5) \quad P' \geq k_c + |v_U - v_Y|, \quad k_c \geq d_c + F'(w) > d_c.$$

In particular $d_c < P'$. The square characters of U, Y_1 agree. Their characters either agree, or differ by the order-two character of I' ; in the second case $v_U - v_Y \equiv q_c/2 \pmod{q_c}$.

Step 9. In this step, we apply square deformation when a conic coordinate has small order. Put $c_c = F_c(u) = (v_U - d_c)/\rho_c$. If $b_B \leq 2d_c$, then $v_Y = b_B$, since $F'(Y_1 - B) > 2d_c$. Equation (9.5) gives $v_U < P' + d_c$, and in particular $c_c < 2P_0$. If $b_B > 2d_c$ and $\min(v_U, v_Y) \leq b' + 2d_c + q_c$, the same estimate gives

$$v_U \leq b' + 2d_c + q_c + P', \quad c_c < b + 2P_0 + e.$$

In either case the square-deformation lemma applies to $f_c = u^2 + (zx^2 + Bx + C)$, with base $\mathbb{A}_x^1 \times S_c$. Its original-field bound is

$$A_M(T) \leq P_0 + \max\{2c_c, c_c + b\} + e < 5P_0 + 2b + 3e < (2m + 5)L_0 + 3e.$$

Step 10. In this step, we treat large conic-coordinate orders while preserving the complete boundary contact. It remains that $b_B > 2d_c$ and $v_U, v_Y > b' + 2d_c + q_c$. Choose integers $a_U, a_Y \in (b' + 2d_c, b' + 2d_c + q_c]$, congruent to v_U, v_Y modulo q_c . Keep the base valuation $s_c \text{ord}_{D_c}$ and give U, Y_1 these Gauss weights, obtaining a homogeneous valuation v_c on $k(S'_c)(U, Y_1)$. It is a positive integer multiple $d_g \text{ord}_{T'}$ of a primitive algebraic ambient divisor: the base residue parameter and two independent Gauss residue parameters have transcendence degree three. The equation on F' gives $F'(R_c) \geq 2 \min(v_U, v_Y)$, so

$$n_c = v_c(f'_c) = 2 \min(a_U, a_Y), \quad p_c = A_{M'_c}(v_c) - n_c = k_c + |a_U - a_Y| \leq P'.$$

The last inequality follows from (9.5): the new difference is zero for equal characters and has absolute value $q_c/2$ otherwise. There is no cancellation between independent Gauss monomials in this calculation.

Every original parameter u, x, z, w has positive integral v_c -value. Indeed

$$v_c(u) = a_U - d_c > 0, \quad v_c(x) = v_c(Y_1 - B) - 2d_c > 0,$$

and the base values of $z = t_0^2, w$ are positive. To compare the actual boundary, choose an integer $j_0 > 2b$ and take the parameter jet $h_{j_0}(u, x, z, w)$ of h_c through degree j_0 . It is a polynomial semi-invariant, and its error belongs to the $(j_0 + 1)$ -st power of the maximal ideal of N_c . The error has order greater than $b' = \rho_c b$ for both F' and v_c , since $\rho_c \leq 2$ and the original parameter values are positive integers.

In the unchanged valued base field evaluate the polynomial at

$$h_* = h_{j_0}(0, -B/(2t_0^2), t_0^2, w).$$

All substituted arguments have nonnegative value, with the nonzero base arguments of positive value. For both F' and v_c , u differs from zero by order greater than b' , while x differs from $-B/(2t_0^2)$ by order greater than b' . Thus their evaluations of h_{j_0} have the same contact as h_* , namely b' . The jet error proves that this is also the contact of the actual h_c after étale transport. This argument keeps all mixed terms and all low-order cancellations.

The unchanged base graded field has its original conormal action. The two new weights preserve the coordinate characters modulo q_c , so I' fixes the Gauss residue field. Restrict T' primitively to M_c , with quadratic ramification $\rho_T \in \{1, 2\}$, and transport it through N_c to T over the original M . The inclusion of I' in inertia implies $\rho_T \geq \rho_c$ and $I \subseteq I_J(T)$. Set $e_T = |I_J(T)|$. The volume identity (9.4) and the boundary contact just proved give

$$A_M(T) - T(f) - T(h)/m = \frac{p_c - b'/m}{d_g \rho_T}.$$

Apply the original comparison to obtain

$$a \leq \frac{p_c - b'/m}{d_g \rho_T e_T} \leq \frac{\rho_c(P_0 - b/m)}{d_g \rho_c e} \leq a.$$

The first inequality ensures that the numerators are positive. Equality forces $d_g = 1$, $\rho_T = \rho_c$, $e_T = e$, and $p_c = P'$. Hence $I_J(T) = I$ and $T(h) = b$. Finally

$$T(f) = \frac{n_c - 2d_c}{\rho_c} \leq 2b + \frac{2d_c}{\rho_c} + 2e < 2b + 2P_0 + 2e,$$

and the computing equality gives $A_M(T) = P_0 + T(f) < 3P_0 + 2b + 2e$. This finishes the double-line case.

The binary cases give $(2m + 1)L_0 + 3e$; the double-line case gives $(2m + 5)L_0 + 3e$; the reduced and cube cases give respectively $\max\{7, m + 4\}L_0 + e$ and $\max\{31, m + 16\}L_0 + e$. These exhaust the nonzero quadratic and cubic alternatives. Their maximum is at most $C_m L_0 + 3e$. Each construction has already established the required original contact, full inertia and computing equality. $\square$

Proposition 9.8. *Fix an integer $m \geq 1$ and a real number $a > 0$. Put $C_m = \max\{2m + 5, 31, m + 16\}$. Let $(Y \ni P, B_Y)$ be an effective terminal threefold germ, with Y terminal and $m(K_Y + B_Y)$ integral Cartier. Let a finite cyclic group H act faithfully, preserve the pair, fix P , and have faithful natural m -log character. Suppose a primitive algebraic prime F centred at P satisfies*

$$A_{Y, B_Y}(F) = a, \quad I_H(F) = 1, \quad \frac{A_{Y, B_Y}(T)}{|I_H(T)|} \geq a$$

for every primitive prime T centred at P . Then $a > 1$. There is an integer $L(m)$, depending only on m , such that, for $\ell = \text{ind}_P(K_Y)$,

$$|H| \leq [4m\ell^2(C_m L(m) + 5) + 1]^6.$$

The group H is not required to preserve F . When $\ell \leq m$, write

$$P(m) = [4m^3(C_m L(m) + 5) + 1]^6.$$

We bound the whole group fixing the centre, even when it moves both the prescribed divisor and the bounded divisor selected below.

Proof. We identify the ordinary minimum, select a bounded computing divisor, and lift it to the canonical cover. We then choose an ambient computing divisor and apply Proposition 8.12.

Step 1. In this step, we identify the ordinary minimum and choose a computing divisor of bounded bare discrepancy. Every primitive divisor T centred at P satisfies

$$A_{Y,B_Y}(T) \geq a|I_H(T)| \geq a,$$

and F attains a . Thus the ordinary pair minimum equals a . Since F is exceptional over the terminal pair, $a > 1$. For a boundary component of coefficient β , blow up a general smooth codimension-two centre contained in that component. Its log discrepancy is $2 - \beta > 1$, so $\beta < 1$. Together with terminality for exceptional divisors, this shows that (Y, B_Y) is klt. Integrality of $m(K_Y + B_Y)$ implies integrality of mB_Y , so its coefficients belong to

$$\Gamma_m = \{0, 1/m, \dots, (m-1)/m\}.$$

Over $\mathbb{C}$, Theorem 9.3 applies: the underlying variety is terminal, the coefficient set is finite, and the ordinary minimum is greater than one. Set $L(m) = L(\Gamma_m)$, and abbreviate it by L in this proof. Lemma 3.12, together with the common realization in Lemma 3.11, supplies over the original field a primitive divisor E with

$$A_{Y,B_Y}(E) = a, \quad A_Y(E) \leq L.$$

This bounds $A_Y(E)$ for the chosen computing divisor E , not for the prescribed F . Applying the original inequality to E gives $a \geq a|I_H(E)|$, hence $I_H(E) = 1$.

Step 2. In this step, we retain the full canonical-cover extension and lift the chosen computing divisor. By Lemma 3.11, the actual crepant quotient of (Y, B_Y) by H has exact closed-point minimum a : the assumed inequality gives the lower bound and E gives equality. Proposition 8.8 supplies the connected canonical cover $W \rightarrow Y$ and the full abelian extension

$$1 \longrightarrow \mu_\ell \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1.$$

The inflated natural character has kernel exactly μ_ℓ . The group $\tilde{H}$ is generated by at most two elements and admits a faithful representation by at most two characters.

Choose a primitive extension $E^\#$ of E to $k(W)$. Its inertia for $\tilde{H}$ maps into $I_H(E) = 1$, so it is contained in μ_ℓ . Let its order be t . The inertia and ramification formula of Lemma 3.8 gives

$$1 \leq t \leq \ell, \quad A_W(E^\#) = tA_Y(E) \leq \ell L, \quad A_{W,B_W}(E^\#) = ta.$$

Here the bare equality uses the quasi-étale canonical cover; the pair equality uses its crepant boundary.

Step 3. In this step, we obtain an ambient computing divisor with bounded bare discrepancy. If W is smooth, take $M_0 = W$ and $T = E^\#$. Otherwise the cover is isolated cDV by Theorem 7.1 on the common field realization of Lemma 3.11. Apply Proposition 9.7 to its actual four-dimensional chart from Proposition 8.8. In both cases, putting

$$L_{\text{amb}} = \ell(C_m L + 3),$$

we obtain an ambient primitive divisor T with $A_{M_0}(T) \leq L_{\text{amb}}$, full inertia of order t , and quotient discrepancy a .

Let h_B be the regular equation of mB_W in that chart. For the application of the character estimate, use the single full equation

$$G_0 = h_B \quad \text{if } W \text{ is smooth,} \quad G_0 = f^m h_B \quad \text{otherwise.}$$

The ambient boundary is $\operatorname{div}(G_0)/m$. Its full quotient has exact minimum a , and the natural character of its m -log line has kernel of order ℓ , by Proposition 8.8. This retains the whole boundary equation and all ramification.

Step 4. In this step, we apply the character estimate to bound the whole fixed-centre group. The number $ma = mA_{Y,B_Y}(E)$ is an integer. In particular, a is a positive rational number, as required in Proposition 8.12. Apply that proposition with $k_0 = \ell$, ambient dimension $D \leq 4$, at most two faithful characters, and the computing inertia t . Dividing the resulting bound on $|\tilde{H}| = \ell|H|$ by ℓ gives

$$|H| \leq [4t \max\{1, \lceil m(L_{\text{amb}} + 2t) \rceil\} + 1]^6.$$

Since m, L, C_m, ℓ are integers and $t \leq \ell$,

$$\max\{1, \lceil m(L_{\text{amb}} + 2t) \rceil\} \leq m\ell(C_m L + 5).$$

Substitution proves the asserted estimate for the whole group H . If $\ell \leq m$, substituting m for ℓ gives the stated expression $P(m)$. $\square$

10. COMPLETE POINT-EXIT SURFACES

We retain the original models and natural character of Theorem 7.16. We construct a polarization on the complete point-exit surface, retain both full Cartier markings, and derive degree estimates for the normal and crepant configurations. We first state the results used below.

Theorem 10.1 (Effective relative generation). *Fix $n \geq 1$ and $a \geq 0$. There is an integer $b = b(n, a) > 0$ such that, for every lc pair (X, Δ) with effective rational boundary and $\dim X = n$, every projective surjection $\pi : X \rightarrow V$, and every π -nef Cartier divisor L with $aL - (K_X + \Delta)$ π -nef and π -log-big, the sheaf $\mathcal{O}_X(bL)$ is generated over V [Fuj09a, Theorem 2.2.4].*

Theorem 10.2 (Uniform regularity). *Fix integers $n \geq 0, r \geq 1$ and $P \in \mathbb{Q}[t]$. There is $b = b(n, r, P)$ such that, for every field k and every exact sequence of coherent sheaves on $\mathbb{P}_k^n$*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow 0$$

with Hilbert polynomial $P_{\mathcal{F}} = P$, the sheaf $\mathcal{K}$ is b -regular. If a coherent sheaf $\mathcal{G}$ is b -regular, it is j -regular for every $j \geq b$, $\mathcal{G}(b)$ is generated, and the multiplication maps $H^0(\mathcal{G}(j)) \otimes H^0(\mathcal{O}(1)) \rightarrow H^0(\mathcal{G}(j+1))$ are surjective for $j \geq b$ [Sta26, Lemmas 33.35.10–12 and 33.35.18, Tags 08A6–08A8 and 08AG].

Lemma 10.3 (Polynomial Euler characteristic). *Let X be a proper scheme over a field, $\mathcal{F}$ coherent, and $\mathcal{L}_1, \dots, \mathcal{L}_s$ invertible. The function*

$$(n_1, \dots, n_s) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_s^{n_s})$$

on $\mathbb{Z}^s$ is a numerical polynomial of total degree at most $\dim \operatorname{Supp} \mathcal{F}$ [Sta26, Lemma 33.45.1, Tag 0BEL].

Theorem 10.4 (Fundamental cycle of a rational surface singularity). *Let $S \ni p$ be a normal rational surface singularity and let Z be the fundamental cycle on its minimal resolution. Then*

$$p_a(Z) = 0, \quad \operatorname{mult}_p(S) = -Z^2, \quad \operatorname{embdim}_p(S) = \operatorname{mult}_p(S) + 1.$$

The exceptional components on a minimal good resolution are smooth rational curves [Art66]; see also [Ste13, Corollary 1.7, Definition 1.9 and Theorem 1.10].

Theorem 10.5 (Normal Gorenstein del Pezzo surfaces). *Let S be a normal projective Gorenstein surface over an algebraically closed field with $-K_S$ ample. Then either S has canonical singularities, or its minimal resolution is $\mathbb{P}_E(\mathcal{O}_E \oplus L)$, where E is a smooth elliptic curve and $\deg L < 0$, and S is obtained by contracting the negative section [HW81, Theorem 2.2]; see also [BM24, Proposition 2.9].*

Theorem 10.6 (A nonrational singularity on a negative surface). *Let S be a proper normal algebraic surface with a nonrational singularity, and suppose no positive multiple of K_S is effective. Let $f : Y \rightarrow S$ be the minimal resolution of its nonrational singularities. Then Y has a genus-zero fibration to a smooth projective curve C of positive genus, the exceptional curve of f is a section, and S has just one nonrational singularity. Moreover $h^1(S, \mathcal{O}_S) = 0$ and $\text{Pic}(S)$ is free of finite rank [Sch01, Theorem 2.2].*

Lemma 10.7 (Numerics of a terminal divisorial contraction). *Let $f : (Y \supset E) \rightarrow (X \ni p)$ be a proper birational contraction of terminal threefold germs with connected fibres, exceptional locus the prime divisor E mapping to p , and $-K_Y$ f -ample. Write $K_Y = f^*K_X + \beta E$, $\beta > 0$. In the terminal basket of Y , retain the fictitious quotient points Q of type $\frac{1}{r_Q}(1, -1, b_Q)$ at which E is not Cartier. Write $E \sim e_Q K_Y$ locally and choose $v_Q = \overline{e_Q b_Q}$ with $0 < v_Q \leq r_Q/2$. Put $R = \text{lcm}_Q r_Q$, with $R = 1$ for an empty basket. Then*

$$\beta E^3 + \sum_Q \frac{v_Q(r_Q - v_Q)}{r_Q} = 2, \quad \beta^2 R E^3 \in \mathbb{Z}.$$

Each fictitious index equals its actual source index except for a point of type $cAx/4$, whose basket has one index 4 and all remaining indices 2 [Kaw05, Section 2, proof of Theorem 1.1 and Lemma 2.3].

Theorem 10.8 (Normalization of a nonnormal del Pezzo surface). *Let E be a reduced connected nonnormal projective Gorenstein surface with $-K_E$ ample. Let $\nu : \coprod_i S_i \rightarrow E$ be its normalization. On each component let C_i be the full conductor divisor and set $H_i = \nu_i^*(-K_E)$. Then $K_{S_i} + C_i \sim -H_i$, and each triple (S_i, H_i, C_i) has one of the following forms:*

- (i) $S_i = \mathbb{P}^2$, H_i is a line, and C_i is a conic.
- (ii) $S_i = \mathbb{P}^2$, H_i is twice a line, and C_i is a line.
- (iii) S_i is the degree- a cone over a rational normal curve, $a \geq 2$, H_i is its hyperplane class, and C_i is linearly equivalent to twice a ruling.
- (iv) $S_i = \mathbb{F}_a$, $a \geq 0$, with a section B of self-intersection $-a$ and ruling fibre F , $H_i \sim B + (a+1)F$, and $C_i \sim B + F$.
- (v) $S_i = \mathbb{F}_a$, $a \geq 0$, with the same notation, $H_i \sim B + (a+2)F$, and $C_i = B$.

The conductor divisors retain their scheme multiplicities [Rei94, Theorem 1.1].

Theorem 10.9 (Tame conductor alternatives and checked gluing properties). *Let E be a reduced connected nonnormal projective Gorenstein del Pezzo surface, and let $H = -K_E$ be its ample polarization. Write D for the conductor subscheme downstairs and C for the full conductor subscheme on the normalization. The numerical alternatives for the connected configurations are the following (degrees are measured by H):*

- (A) $\text{Sing } E = \Gamma$ is irreducible of degree one, and C is a smooth conic mapping with degree two to Γ . The normalization is irreducible and E has ordinary double points in codimension one along Γ .
- (B) $\text{Sing } E = \Gamma$ is irreducible of degree two; the normalization has exactly two components, each with a smooth conic conductor mapping birationally to Γ . In fact each conic maps isomorphically to the same smooth curve $D = \Gamma$.
- (C) $\text{Sing } E = \Gamma$ is irreducible of degree one and the conductor on its irreducible normalization is a line pair.
- (D) E is a cyclic arrangement of $r \geq 2$ components, adjacent components meeting generically transversally along degree-one curves. Each normalization conductor is a line pair mapping birationally to the adjacent double curves. All nodes match at a unique common vertex; away from it the double curves are smooth projective lines.

(E) $\text{Sing } E = \Gamma$ is irreducible of degree one, and the conductor on each normalization component is a double line. The transverse singularity is a cusp for one double line, a tacnode for two, and r concurrent lines in $\mathbb{A}^{r-1}$, with no $r-1$ of them lying in a hyperplane, for $r \geq 3$.

In (B), (D), and the reducible cases of (E), each normalization piece maps isomorphically onto its reduced closed image. In (A), the conductor map is the linear projection of a smooth conic to a line. In (E), writing $\Gamma = D_{\text{red}} \simeq \mathbb{P}^1$, the full conductor structure sheaf is $\mathcal{O}_D \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(r-1)}$, with the second summand a square-zero ideal, and each component map is linear. These are the numerical alternatives of Lemma 4.1 and the indicated projective gluing properties proved in Theorem 1.5, Claim 4.3, Theorem 3.7(ii), and Proposition 4.9; Lemma 4.1 itself is only a first numerical approximation [Rei94, Lemma 4.1, Theorem 1.5, Claim 4.3, Theorem 3.7(ii), Proposition 4.9].

Theorem 10.10 (Degree of the intrinsic deformation line). *Let H be a surface germ along its reduced proper one-dimensional singular locus Δ , each component of Δ smooth. Suppose its singularities belong to the class qG of [Tzi05, Definition 2.10]: ordinary double crossings, pinch points, degenerate cusps of embedding dimension at most four, and smoothable slt quotients $(xy=0)/\mu_n(a, -a, 1)$, $\gcd(a, n) = 1$. For a component $C \subset \Delta$, let $\tilde{C}$ be the divisorial part of its inverse image in the normalization, and let L_C be the torsion-free quotient of $T_{qG}^1(H) \otimes \mathcal{O}_C$. Let p_C count pinch points on C , and let c_i count the degenerate cusps on C whose reduced exceptional cycle on the minimal semi-resolution has square $-i$. For each cusp $P \in C$ of square -3 or -4 , choose the parameters of the marked germ $P \in C \subset H$ as in [Tzi05, Lemma 2.11 and Corollary 2.12], and denote that marked type by $T_{u,v}^3$ or $T_{u,v,w}^4$. The parameters are positive integers or ∞ , with $1 \leq u, v \leq \infty$ for T^3 and $2 \leq u, v, w \leq \infty$ for T^4 . Then L_C is a line bundle and*

$$\deg L_C = \tilde{C}^2 + p_C + 2c_1 + 2c_2 + \sum_{T_{u,v}^3} \left(1 + \frac{u+v}{uv+u+v}\right) + \sum_{T_{u,v,w}^4} \frac{w(u+v)-4}{wuv-u-v}.$$

Each sum is over the indicated marked cusps on C . Infinite parameters are interpreted by the corresponding limits [Tzi05, Theorem 3.1].

Definition 10.11. An *original point-exit datum* is the entire construction of Theorem 7.16 when $h(E)$ is a point, retaining the original cover, prescribed computing prime, group H , natural character, boundary R , and contact function g . Put $M = \text{lcm}(1, \dots, m)$.

Proposition 10.12. *For an original point-exit datum, ME, MK_Y, mMR are Cartier near the complete exceptional surface E . The target Z is bare $\mathbb{Q}$ -Gorenstein and klt. The surface E is integral and Cohen–Macaulay, and Gorenstein outside a finite set. Let $j : E^\circ \hookrightarrow E$ be its Gorenstein open set. On E , dualizing powers mean their S_2 extensions $\omega_E^{[d]} = j_*(\omega_{E^\circ}^{\otimes d})$ for integers $d \geq 1$. In particular,*

$$\omega_E^{[M]} \simeq \mathcal{O}_Y(M(K_Y + E))|_E$$

is invertible. Write K_E for the resulting rational dualizing class. Set

$$\alpha = A_Z(E), \quad L = \mathcal{O}_Y(-ME)|_E, \quad v = L^2, \quad u = -K_E \cdot L.$$

The line bundle L is ample, $\alpha \geq a$, and

$$-MK_E \sim_{\mathbb{Q}} \alpha L, \quad u, v \in \mathbb{Z}_{>0}, \quad u \equiv v \pmod{2}, \quad \alpha = Mu/v.$$

There is an absolute even integer $b \geq 4$ such that the complete system $|L^b|$ embeds the same surface equivariantly. Moreover,

$$\alpha \leq bM, \quad h^0(E, L^n) = 1 + (vn^2 + un)/2 \quad (n \geq 0).$$

The full boundary marking $A_R = (mMR)|_E$ and the full zero scheme T_g of the leading coefficient of g^M are effective Cartier markings. If $\ell_g = \text{ord}_E(g)$, then the following identities hold, with the

last one natural and H -equivariant:

$$1 \leq \ell_g \leq a/c_{\text{ct}}(3, a), \quad T_g \sim \ell_g L, \quad \omega_E^{[mM]} \otimes \mathcal{O}_E(A_R) \otimes L^{ma} \simeq \mathcal{O}_E \otimes k_{\chi_M}.$$

For every bound $v \leq V$, the complete embeddings and full ideal sheaves of E, A_R, T_g are bounded in terms of m, a, V .

The polarization and both markings belong to the original complete surface. In particular, the numerical hypothesis in the last assertion is a bound for that surface's degree.

Proof. We establish Cartier adjunction and cohomology on E , construct the complete embedding, and then compute the two marked Hilbert polynomials. Lemma 3.11 lets us work over $\mathbb{C}$ with the prescribed diagram. The sheaves, their linearizations, cohomology, and intersection degrees used below are preserved by the field extensions in that lemma. The resulting closed embeddings and Cartier identities descend faithfully flatly.

Step 1. In this step, we establish Cartier adjunction and the polarization identities on the complete surface. Theorem 7.16 gives H - $\mathbb{Q}$ -factoriality and local canonical indices at most m . Apply Lemma 8.3 to the invariant integral Weil divisors E, K_Y , and mR . A canonical representative can be chosen invariant by pulling back a rational top form from the quotient. Since every index divides M , the divisors ME, MK_Y, mMR are Cartier on a neighbourhood of E .

At any closed point $P \in E$, apply Proposition 8.8 with the trivial group to obtain the actual local canonical cover $\pi_P : W_P \rightarrow Y$, of degree $r_P \leq m$. It is quasi-étale, Gorenstein terminal, and smooth or an isolated hypersurface in a smooth fourfold. Its full divisorial inverse image $E_P^\sharp$ has integer coefficients and generic multiplicities one; it may have several prime components. Lemma 8.3, now with canonical index one, makes it Cartier. Its local equation has a Cohen–Macaulay, generically reduced quotient, hence a reduced quotient. Thus $E_P^\sharp$ is precisely the full reduced inverse image of E .

The invariant ring of this quotient is the local ring of E : invariants are exact, and the contracted ideal is the ideal of the reduced prime E . A system of parameters of the invariant ring is a system of parameters of its finite covering ring. The latter is Cohen–Macaulay of dimension two. The Reynolds splitting, applied to the parameter Čech complex, therefore makes the former Cohen–Macaulay as well. This proves the assertion at every P .

Put $U = E \cap Y_{\text{reg}}$. The complement is finite, by Theorem 7.6. On U , ordinary Cartier adjunction gives $\omega_U = \omega_Y(E)|_U$. Both an invertible sheaf on the S_2 surface E and its restriction to the Gorenstein open set are determined across this finite complement. Consequently

$$\omega_E^{[M]} = \mathcal{O}_Y(M(K_Y + E))|_E.$$

This proves coherence and invertibility of the indicated S_2 extension, including at Gorenstein points outside U .

Write $B_Z = h_*R$. The quotient Z/H is $\mathbb{Q}$ -factorial by Theorem 7.16. Any invariant rational Weil divisor on Z is a rational pullback from this quotient: over a prime downstairs its coefficients are constant on the orbit of primes upstairs, and the finite pullback is the orbit sum multiplied by the ramification index. This applies to B_Z and to an invariant canonical representative. Thus both are $\mathbb{Q}$ -Cartier. Dropping the effective boundary from the klt pair (Z, B_Z) shows that Z is klt. Pair crepancy gives

$$(10.1) \quad K_Y = h^*K_Z + (\alpha - 1)E, \quad h^*B_Z = R + (\alpha - a)E, \quad \alpha \geq a.$$

A Cartier multiple of a divisor pulled back from Z restricts trivially to E . Restricting these equalities and using adjunction gives

$$(10.2) \quad -MK_E \sim_{\mathbb{Q}} \alpha L, \quad A_R \sim_{\mathbb{Q}} \lambda L, \quad \lambda = m(\alpha - a) \geq 0.$$

The line L is ample because $-E$ is h -ample.

Step 2. In this step, we prove the cohomology vanishing and construct the complete equivariant embedding. Relative vanishing in Theorem 7.8, applied to h and the terminal pair $(Y, (1-a)E+R)$, gives

$$R^i h_* \mathcal{O}_Y(-jE) = 0 \quad (i > 0, j \geq 0).$$

The adjoint difference is relatively equivalent to $-jE$, because the pair is crepant over Z . For $j > 0$ it is relatively ample. For $j = 0$ it is relatively nef and big because h is birational. There are no lc centres. This is an application to integral Weil divisors, so jE need not be Cartier.

The reduced ideal of E is $\mathcal{O}_Y(-E)$. Indeed, in a normal affine ring an element has positive order at the height-one prime of E precisely when it belongs to that prime ideal. Tensoring its ideal sequence by $\mathcal{O}_Y(-nME)$ and using the displayed vanishing yields

$$(10.3) \quad H^i(E, L^n) = 0 \quad (i > 0, n \geq 0), \quad H^0(E, \mathcal{O}_E) = \mathbb{C}.$$

The second equality also follows from integral projectivity.

Apply Theorem 10.1 to the Cartier divisor $-ME$, with dimension three and adjoint parameter one. A fixed multiple $-q_0ME$, with $q_0 \geq 2$ absolute, is relatively generated. Its restriction $J = L^{q_0}$ is globally generated. The resulting complete morphism $\phi : E \rightarrow \mathbb{P}^s$ is finite, since J is ample. For $\mathcal{F} = \phi_* \mathcal{O}_E$, (10.3) gives

$$H^1(\mathbb{P}^s, \mathcal{F}(1)) = 0, \quad H^2(\mathbb{P}^s, \mathcal{F}) = 0.$$

The higher groups vanish by support dimension. Thus $\mathcal{F}$ is 2-regular. The multiplication assertion in Theorem 10.2 shows that the full section ring of J is generated in degrees one and two. Its second Veronese is generated in degree one: an even-weight monomial in generators of weights one and two splits into weight-two blocks. Since an ample line recovers its scheme as the Proj of its section ring, $J^2 = L^b$ is very ample for the absolute even integer $b = 2q_0 \geq 4$. The divisor line L has its natural H -linearization, so the complete embedding is equivariant for the original action.

Step 3. In this step, we compute the Hilbert coefficients and bound the bare discrepancy. For every Cartier divisor D_0 on E , we have

$$(10.4) \quad \chi(E, \mathcal{O}_E(D_0)) = 1 + \frac{D_0^2 - K_E \cdot D_0}{2}.$$

To verify this without assuming normality, choose a general integral curve $C \in |D_0 + tbL|$ for t sufficiently large. It avoids $E \setminus U$ and is a Cartier divisor on the Gorenstein surface U . Curve adjunction and duality give $\omega_C = \omega_U(C)|_C$, and curve Riemann–Roch gives $\chi(C, \mathcal{O}_C(C)) = (C^2 - K_E \cdot C)/2$. The divisor sequence on E , together with (10.3), proves (10.4) for $D_0 + tbL$. Lemma 10.3 makes both sides polynomials in t . Equality for all large t proves equality at zero.

The positive number $v = L^2$ is an integer. Applying (10.4) to L shows that $u = 2\chi(E, L) - 2 - v$ is an integer of the same parity as v . (10.2) gives $u > 0$ and $\alpha = Mu/v$. Vanishing then proves the asserted formula for every $h^0(E, L^n)$. For a general integral $C \in |bL|$, adjunction gives

$$0 \leq p_a(C) = 1 + \frac{b^2v - bu}{2}.$$

Since $bv - u$ is an integer and $b \geq 4$, it follows that $u \leq bv$, hence $\alpha \leq bM$. The complete embedding has dimension $N = (b^2v + bu)/2 \leq b^2v$ and degree b^2v .

Step 4. In this step, we retain both full markings and their natural character and bound their presentations when the degree is bounded. The effective Cartier divisor mMR avoids the integral surface E ; its restriction is therefore the effective Cartier divisor A_R , with all multiplicities retained. Take the original function g supplied by Theorem 7.16, and write $\text{div}_Y(g) = \ell_g E + D$. If w is a local equation of ME , then g^M/w^{ℓ_g} is regular by normality and has divisor MD . Its nonzero restrictions to E glue to the leading section of L^{ℓ_g} . Natural linearizations make this section invariant. Its full zero scheme is the effective Cartier divisor $T_g \sim \ell_g L$. The bound on ℓ_g is the one in the marked-exit theorem.

Functorial adjunction on U , followed by the S_2 extension, gives

$$\omega_E^{[mM]} \otimes \mathcal{O}_E(A_R) \otimes L^{ma} = \mathcal{O}_Y(mM(K_Y + (1-a)E + R))|_E.$$

The right side is the M -th power of the original natural m -log line. Its original eigen-generator restricts with character χ^M , proving the equivariant identity.

Finally, the divisor sequences for the two full markings and (10.4) give, in the complete L^b -embedding,

$$\begin{aligned} P_E(t) &= 1 + \frac{b^2v}{2}t^2 + \frac{bu}{2}t, \\ P_{A_R}(t) &= b\lambda vt + \frac{\lambda u - \lambda^2v}{2}, \\ P_{T_g}(t) &= b\ell_g vt + \frac{\ell_g u - \ell_g^2v}{2}. \end{aligned}$$

When $v \leq V$, the positive integers v, u , with $u \leq bv$, have finitely many possibilities. The same holds for $\lambda = m(Mu/v - a)$ and for the positive integer $\ell_g \leq a/c_{\text{ct}}(3, a)$. Thus the ambient dimensions and all three Hilbert polynomials range over finite lists. Theorem 10.2 bounds each full ideal sheaf in these projective spaces. Their bounded generated twists give finite-dimensional coefficient spaces for their equations, hence bounded families of the complete marked schemes, depending only on m, a, V . $\square$

We next record the local and normal-surface estimates used in the volume alternative.

Lemma 10.13. *Let $r, d \geq 1$ be integers, and let $(\mathcal{A}, \mathfrak{n})$ be a complete Noetherian local k -algebra with residue field k and embedding dimension at most d . Suppose a cyclic group C_r of order r acts continuously by local k -algebra automorphisms on $\mathcal{A}$. Then*

$$\text{embdim}(\mathcal{A}^{C_r}) \leq \binom{r+d}{d} - 1.$$

The estimate uses the number of generators of the covering ring and the order of its cyclic group.

Proof. Put $e = \dim_k \mathfrak{n}/\mathfrak{n}^2 \leq d$. If $e = 0$, then $\mathcal{A} = k$. Otherwise lift a character basis of $\mathfrak{n}/\mathfrak{n}^2$ by the corresponding character projections. Completeness gives an equivariant surjection

$$k[[x_1, \dots, x_e]] \longrightarrow \mathcal{A}$$

for a diagonal action. Averaging makes the induced map on invariant rings surjective.

Identify characters with $\mathbb{Z}/r\mathbb{Z}$. An invariant monomial of degree greater than r has a nonconstant invariant factor of degree at most r : among the $r+1$ successive partial character sums of any r variable factors, two agree. Iteration expresses every invariant monomial as a product of invariant monomials of degrees $1, \dots, r$. There are at most $\binom{r+e}{e} - 1$ of these. They also generate invariant power series topologically. Fix one factorization of each monomial; a term of total degree j has at least j/r factors, so the resulting formal series in these generators converges. The induced surjection from their power-series ring bounds the embedding dimension by $\binom{r+e}{e} - 1 \leq \binom{r+d}{d} - 1$. $\square$

Lemma 10.14. *Let S be a normal projective surface, with K_S $\mathbb{Q}$ -Cartier, $-K_S$ ample, and $H^1(S, \mathcal{O}_S) = 0$. Let $t \geq 2$ be an integer such that $\text{embdim}_P(S) \leq t$ at every closed point. Then*

$$K_S^2 \leq \max\{t + 7, 2t\}.$$

The two terms control rational surfaces and surfaces with a nonrational singularity, respectively.

Proof. We bound the exceptional self-intersections on a minimal resolution, and then push the nef anticanonical class to a smooth ruled surface. The nonrational case additionally uses a local cohomological estimate for embedding dimension.

Step 1. In this step, we bound the anticanonical degree in the rational case. Suppose first that S is rational, and let $\pi : W \rightarrow S$ be its minimal resolution. Then $H^1(W, \mathcal{O}_W) = H^2(W, \mathcal{O}_W) = 0$. Also $H^2(S, \mathcal{O}_S) = 0$: by duality a nonzero section would give an effective canonical divisor, contradicting its negative intersection with $-K_S$. The Leray sequence therefore gives $H^0(S, R^1\pi_*\mathcal{O}_W) = 0$. This sheaf has finite support, so it vanishes and all singularities of S are rational.

For a fundamental cycle $Z_f = \sum_i n_i C_i$ over a singular point, write $C_i^2 = -b_i$. On the minimal resolution the C_i are smooth rational curves and $b_i \geq 2$. Theorem 10.4 and adjunction give

$$\text{mult}_P(S) = -Z_f^2 = 2 + \sum_i n_i(b_i - 2), \quad b_i \leq \text{mult}_P(S) \leq t - 1.$$

Write $K_W + D_W = \pi^*K_S$. The exceptional divisor D_W is effective: it has nonpositive intersection with every exceptional curve, and negative definiteness rules out a negative part. More explicitly, if $D_W = D_+ - D_-$ with disjoint effective parts and $D_- \neq 0$, then $D_W \cdot D_- = D_+ \cdot D_- - D_-^2 > 0$, a contradiction.

The divisor $P_W = -(K_W + D_W)$ is nef and big, with square K_S^2 . Under the contraction ρ of a smooth rational (-1) -curve C_0 , its pushforward P' is nef: for any effective curve C' , $P' \cdot C' = P_W \cdot \rho^*C' \geq 0$. Writing $c_0 = P_W \cdot C_0 \geq 0$ gives

$$P_W = \rho^*P' - c_0C_0, \quad (P')^2 = P_W^2 + c_0^2.$$

Thus pushing to a minimal smooth rational surface F produces $P = -(K_F + D_F)$ nef and big, $D_F \geq 0$, and $K_S^2 \leq P^2$.

If $F = \mathbb{P}^2$, then $P^2 \leq 9 \leq t + 7$. Otherwise $F = \mathbb{F}_n$. Let $\Gamma^2 = -n$ be its negative section and f a ruling fibre. Write $D_F = \beta\Gamma + G$, with $\beta \geq 0$ and $G \geq 0$ avoiding Γ . Each component of G has nef numerical class, so $P_0 = -(K_F + \beta\Gamma) = P + G$ is nef and big and $P^2 \leq P_0^2$. Its fibre degree is $2 - \beta > 0$, and

$$P_0^2 = 8 + \beta(2n - 4 - n\beta).$$

For $n \leq 2$ this is at most eight. For $n \geq 3$, nefness forces $\beta > 0$; otherwise $(-K_F) \cdot \Gamma = 2 - n < 0$. The strict transform of Γ on W is then a component of D_W . Its self-intersection is at most $-n$, so $n \leq t - 1$. Since $\beta(2 - \beta) \leq 1$,

$$P_0^2 = 8 + n\beta(2 - \beta) - 4\beta \leq 8 + n \leq t + 7.$$

Step 2. In this step, we bound the anticanonical degree when the surface has a nonrational singularity. Suppose that S has a nonrational singularity. No positive multiple of K_S is effective. By Theorem 10.6, its minimal smooth resolution W is ruled over a smooth curve B of genus $g > 0$. There is a unique horizontal exceptional curve C , a smooth section isomorphic to B ; the other exceptional curves are vertical and rational. The theorem first supplies the partial resolution of the nonrational singularities. Resolving its remaining rational points gives this smooth ruled model and the strict transform of its section. The image p of C is the unique nonrational singularity. Since $h^1(W, \mathcal{O}_W) = g$, $h^2(W, \mathcal{O}_W) = 0$, and $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$, Leray gives $p_g(S \ni p) = g$.

Put $n = -C^2 > 0$. Adjunction gives $K_W \cdot C = n + 2g - 2$, while every other minimal exceptional curve has nonnegative intersection with K_W . Every nonzero effective exceptional cycle D over p satisfies $p_a(D) \leq g$. Indeed, over an affine neighbourhood the group $H^2(W, \mathcal{O}_W(-D))$ vanishes by relative fibre dimension, so $H^1(W, \mathcal{O}_W) \rightarrow H^1(D, \mathcal{O}_D)$ is surjective, and $h^0(D, \mathcal{O}_D) \geq 1$. For the fundamental cycle Z_f , whose coefficient at C is at least one, this gives

$$-Z_f^2 = K_W \cdot Z_f + 2 - 2p_a(Z_f) \geq n.$$

Resolve the maximal ideal at p , with $\rho : W' \rightarrow W$ and $\mathfrak{m}_p\mathcal{O}_{W'} = \mathcal{O}_{W'}(-F_p)$ generated. The cycle F_p is anti-nef, and $F_p \geq \rho^*Z_f$. To see the latter assertion, ρ_*F_p is anti-nef and hence dominates Z_f ; negative definiteness on the ρ -exceptional curves then gives $F_p \geq \rho^*\rho_*F_p$. Two

general elements of a minimal reduction of $\mathfrak{m}_p$ compute its multiplicity as $-F_p^2$. Thus, putting $\mu = \text{mult}_p(S)$,

$$\mu = -F_p^2 \geq -Z_f^2 \geq n.$$

The middle inequality follows by intersecting $F_p - \rho^* Z_f \geq 0$ with the anti-nef sum $F_p + \rho^* Z_f$.

We give the required embedding-dimension comparison explicitly. Over the local affine base put $I_j = H^0(W', \mathcal{O}_{W'}(-jF_p))$ and $q_j = h^1(W', \mathcal{O}_{W'}(-jF_p))$. The maximal ideal is integrally closed, so $I_1 = \mathfrak{m}_p$. Euler characteristic in the divisor sequence for jF_p and adjunction on that cycle give

$$\text{length}(\mathcal{O}_{S,p}/I_j) = -\frac{j^2 F_p^2 + j K_{W'} \cdot F_p}{2} + g - q_j.$$

Using the value one at $j = 1$ and subtracting gives

$$\text{length}(I_1/I_2) = \mu + 1 - g + 2q_1 - q_2.$$

A general section of the generated line $\mathcal{O}_{W'}(-F_p)$ has zero divisor D' with no exceptional component. It is finite over an affine curve after shrinking the base, hence affine. The sequence

$$0 \longrightarrow \mathcal{O}_{W'}(-F_p) \longrightarrow \mathcal{O}_{W'}(-2F_p) \longrightarrow \mathcal{O}_{D'}(-2F_p) \longrightarrow 0$$

gives $q_2 \leq q_1$. Since $\mathfrak{m}_p^2 \subset I_2$,

$$(10.5) \quad \text{embdim}_p(S) \geq \mu + 1 - g \geq n + 1 - g.$$

As in the rational case, write $K_W + D_W = \pi^* K_S$. Nonnegative intersection of K_W with every exceptional curve implies $D_W \geq 0$. Contract the vertical (-1) -curves to a smooth $\mathbb{P}^1$ -bundle $F \rightarrow B$. The previous blowdown calculation produces a nef big $P = -(K_F + D_F)$ with $K_S^2 \leq P^2$. Let Γ be the image section, write $e = -\Gamma^2 \leq n$, and put $D_F = \beta\Gamma + G$, where G is a nonnegative rational sum of fibres. Then $P_0 = P + G$ is nef and big, $0 \leq \beta < 2$, and

$$\begin{aligned} P_0 &\equiv (2 - \beta)\Gamma + (e - 2g + 2)f, \\ 0 < P_0^2 &= (2 - \beta)(\beta e - 4g + 4) = e\beta(2 - \beta) - 4(g - 1)(2 - \beta) \leq e. \end{aligned}$$

Positivity implies $e > 2(g - 1)$. Combining this with (10.5) gives

$$\text{embdim}_p(S) \geq e - (g - 1) > e/2, \quad K_S^2 \leq P_0^2 \leq e < 2 \text{embdim}_p(S) \leq 2t.$$

Step 3. In this step, we show that the two cases exhaust the normal surfaces in the statement. If all singularities of S are rational, then its smooth resolution has irregularity zero. It has no effective positive pluricanonical divisor, since such a divisor would push to one on S . The classification of smooth projective surfaces therefore makes this resolution rational. The first case applies. Otherwise the second case applies, proving the claimed maximum. $\square$

Lemma 10.15. *Let Y be a normal threefold over k , smooth in codimension two. Let $E \subset Y$ be an integral Cohen–Macaulay effective $\mathbb{Q}$ -Cartier divisor, and let $R \geq 0$ be a rational $\mathbb{Q}$ -Cartier divisor with $E \not\subset \text{Supp } R$. Let $a \in \mathbb{Q} \cap (0, 1]$, and assume that $(Y, (1 - a)E + R)$ is terminal near E . If E is nonnormal, then $a > 1/2$.*

Thus the small-minimum range forces normality of the complete exceptional surface.

Proof. The Cohen–Macaulay surface satisfies S_2 . If it is nonnormal, it is nonregular at the generic point η of an integral curve C . The local ring $\mathcal{O}_{Y,\eta}$ is regular of dimension two, and an equation of E has maximal-ideal order $k \geq 2$. Blowing up a smooth dense open subset of C in Y_{reg} gives a primitive divisor F with $A_Y(F) = 2$ and $\text{ord}_F(E) = k$. Terminality and effectivity give

$$1 < A_{Y,(1-a)E+R}(F) = 2 - (1 - a)k - \text{ord}_F(R) \leq 2a.$$

$\square$

Lemma 10.16. *For an original point-exit datum, put $t_m = \binom{m+4}{4} - 1$, and retain M, L, v, α from Proposition 10.12. Then:*

- (i) If E is normal, then $v \leq M^2 \max\{t_m + 7, 2t_m\}/a^2$.
- (ii) If $\alpha > 1$, then $\alpha - 1 \geq 1/(2M)$ and $v \leq 4M^3$.
- (iii) If $a \leq 1/2$, then E is normal.

These estimates leave only nonnormal surfaces with $1/2 < a \leq \alpha \leq 1$.

Proof. We treat the normal surface and positive-discrepancy contraction separately, then invoke the existing normality criterion.

Step 1. In this step, we bound the degree when E is normal. The local Cartier inverse divisor $E_P^\sharp$ constructed in the proof of Proposition 10.12 has embedding dimension at most four. Its invariant local ring is $\mathcal{O}_{E,P}$, under a cyclic group of order $r_P \leq m$. Completion commutes with these invariants by the Reynolds splitting. Lemma 10.13 therefore gives $\text{embdim}_P(E) \leq t_m$. If E is normal, apply Lemma 10.14 using (10.3). Squaring $-MK_E \sim_{\mathbb{Q}} \alpha L$ yields

$$v = \frac{M^2 K_E^2}{\alpha^2} \leq \frac{M^2 \max\{t_m + 7, 2t_m\}}{a^2}.$$

Step 2. In this step, we bound the degree and discrepancy gap when $\alpha > 1$. Suppose $\alpha > 1$, and put $\beta = \alpha - 1$. Every exceptional prime over Z other than E has pair log discrepancy greater than one by Theorem 7.16. The effective $\mathbb{Q}$ -Cartier divisor B_Z only increases bare discrepancy. Thus Z is terminal, including the divisor E , whose bare discrepancy is $\alpha > 1$. (10.1) makes $-K_Y$ relatively ample. Normality of the target gives connected fibres. The exceptional locus is E : a contracted curve outside E would have nonnegative intersection with E , contrary to relative ampleness of $-E$.

Lemma 10.7 now applies. Write R_b for the least common multiple of its basket indices. Its index rule and the source index bound give $R_b \mid M$. Since $v = M^2 E^3 > 0$, its two formulas imply

$$0 < \beta E^3 \leq 2, \quad 1 \leq \beta^2 R_b E^3 \leq 2\beta R_b.$$

Hence $\beta \geq 1/(2R_b) \geq 1/(2M)$ and $v \leq 4M^2 R_b \leq 4M^3$. Finally, (iii) follows from Lemma 10.15 and the Cohen–Macaulayness established above. $\square$

The crepant calculation takes place on the complete divisor in a Gorenstein terminal threefold. We first establish the local bound needed when the normalization of an integral divisor is a cone.

Lemma 10.17. *Let T be an integral nonnormal projective Gorenstein surface, and let $\nu : S \rightarrow T$ be its normalization. Suppose S is the projective cone over a rational normal curve of degree $d \geq 3$, and its full conductor divisor C is linearly equivalent to twice a ruling. If Q is the vertex and $P = \nu(Q)$, then*

$$\text{embdim}_P(T) \geq d.$$

Proof. We compute the first-order conductor ideal on the normalization and then use the cotangent sequence downstairs.

Step 1. In this step, we compute the first-order classes of the full conductor at the cone vertex. Put $A = \mathcal{O}_{T,P}$, let $B = (\nu_* \mathcal{O}_S)_P$ be its finite semilocal normalization algebra, and write

$$J = \{b \in B : bB \subset A\} \subset A$$

for the conductor. It is also an ideal of B . Over the inverse image of an affine open subset of T , consider a section f of the divisorial ideal $\mathcal{O}_S(-C)$. Its product with any normalization function belongs to $\mathcal{O}_T$ in codimension one, by the definition of C . These products extend across the remaining finite set because T is S_2 . Taking the normalization function to be one first shows that f itself extends; taking local module generators then shows that it belongs to the conductor. The opposite inclusion follows from the codimension-one definition. Thus the full conductor ideal is $\mathcal{O}_S(-C)$, including at Q .

Write the cone as $S = \text{Proj } \mathbb{C}[x, y, z]$, with weights $\deg x = \deg y = 1$ and $\deg z = d$. The degree- d monomials in x, y , together with z , give its cone embedding, and a ruling has class

$\mathcal{O}_S(1)$. Since $2 < d$, the effective divisor C is defined by a nonzero binary quadratic $q(x, y)$. Over $\mathbb{C}$, this is either a product of two distinct linear forms or the square of a linear form. These give respectively the line-pair and double-line conductor schemes, with their full multiplicities.

For a nonnegative integer j , let $\mathbb{C}[x, y]_j$ denote the degree- j homogeneous forms. On the vertex chart, with $\mathfrak{n}$ the ideal of positive Veronese degree, we have

$$R := \mathcal{O}_{S,Q} = \left(\bigoplus_{n \geq 0} \mathbb{C}[x, y]_{nd} \right)_{\mathfrak{n}}, \quad JR = \left(\bigoplus_{n \geq 1} q \mathbb{C}[x, y]_{nd-2} \right)_{\mathfrak{n}}.$$

The space $q \mathbb{C}[x, y]_{d-2}$ has dimension $d - 1$. Its elements have Veronese degree one, whereas the square of the positive-degree ideal has no degree-one part. Localization at the vertex does not change this cotangent space. Consequently these $d - 1$ classes are independent in $\mathfrak{m}_R/\mathfrak{m}_R^2$.

Step 2. In this step, we transfer those conductor classes to the downstairs cotangent space. The local map $A \rightarrow R$ sends $\mathfrak{m}_A^2$ into $\mathfrak{m}_R^2$. The displayed conductor classes lift to J . Indeed, write a representative in JR as b/s , with $b \in J$ and $s \in B$ a unit at Q . Since $b \in \mathfrak{m}_R$,

$$b/s - s(Q)^{-1}b \in \mathfrak{m}_R^2.$$

Thus scalar multiples of elements of J realize the same first-order classes, and

$$\dim_{\mathbb{C}} J/(J \cap \mathfrak{m}_A^2) \geq d - 1.$$

The local conductor ring A/J has dimension one, because a conductor ruling through Q maps finitely onto a curve through P . Its cotangent space therefore has dimension at least one. The exact sequence

$$0 \longrightarrow J/(J \cap \mathfrak{m}_A^2) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_{A/J}/\mathfrak{m}_{A/J}^2 \longrightarrow 0$$

now gives $\text{embdim}_P(T) \geq (d - 1) + 1 = d$. □

We now treat the integral crepant divisor.

Lemma 10.18. *Let W be a Gorenstein terminal threefold, and let $T \subset W$ be an integral nonnormal projective Cartier divisor. Suppose ω_W is trivial on a neighbourhood of T , and $-K_T$ is ample. Then*

$$1 \leq (-K_T)^2 \leq 7.$$

The bound includes the correction at every singular ambient point on the conductor.

Proof. We reduce the high-degree case to two scroll gluings and compute the determinant of the normalization's tangent map along the complete conductor. Put $d = (-K_T)^2$. It is a positive integer. The surface is Gorenstein by Cartier adjunction. By Theorem 7.1, the ambient germ has embedding dimension at most four, and so does T .

Step 1. In this step, we reduce the high-degree integral case to the two scroll conductor formats. The normalization table in Theorem 10.8 and the integral cases of Theorem 10.9 give the following alternatives when $d > 4$. If the normalization $\nu : S \rightarrow T$ is a rational normal cone, the table makes $\nu^*(-K_T)$ its hyperplane class. Its square is d by the finite birational projection formula, and the full conductor is linearly equivalent to twice a ruling. Together with the integral projective Gorenstein hypotheses on T and $d > 4$, these are precisely the assumptions of Lemma 10.17. It gives a point of T with embedding dimension at least $d > 4$, contrary to the ambient bound. The remaining normalizations are

S	conductor on S	conductor map
$\mathbb{F}_{d-2}$	$B + f$	$f \simeq B$, with the intersection identified
$\mathbb{F}_{d-4}$	B	$B \longrightarrow C \simeq \mathbb{P}^1$ of degree two.

Here B is the negative section and f a ruling fibre. Indeed, the plane and smooth-conic low-degree rows of the normalization table have degree at most four; its two remaining scroll rows have exactly the displayed conductors. In both formats the reduced conductor image C is $\mathbb{P}^1$.

Step 2. In this step, we compute the correction from singular ambient points along the conductor. Each conductor germ in these scroll formats is a hypersurface $F(x, y, z) = 0$ of embedding dimension three. At a singular ambient point, a local equation w of T has a nonzero cotangent class: quotienting the four-dimensional ambient cotangent space gives the three-dimensional cotangent space of T . Thus the actual ambient germ has a presentation

$$W = (Q = 0), \quad T = (Q = w = 0), \quad Q = F(x, y, z) + wB_0(x, y, z, w).$$

Along $C = (y = z = w = 0)$, put $\gamma(x) = B_0(x, 0, 0, 0)$. It is not identically zero, because W has isolated singularities. Let $\epsilon_P = \text{ord}_P \gamma \geq 0$. At a smooth ambient point use the dummy equation $Q = w$ and set $\epsilon_P = 0$. The residue generator, where $\gamma \neq 0$, is

$$\frac{dx \wedge dy \wedge dz}{\partial Q / \partial w}.$$

Hence pairing a tangent determinant with ω_W contributes a pole of order ϵ_P . This order is unchanged by a unit change of the local canonical generator. It is positive exactly at singular ambient points of C . The tangent determinants below are rational maps from vector bundles on the normalization curves; their definition does not require a locally free tangent sheaf on singular W .

Step 3. In this step, we bound the degree when the conductor identifies a section with a fibre. Let $S = \mathbb{F}_n$, $n = d - 2$, and let $\iota : f \rightarrow B$ be the conductor identification. Identify C with f . The quotient

$$\mathcal{Q}_C = (T_S|_f \oplus \iota^*(T_S|_B))/T_C$$

is a vector bundle of rank three, where the tangent line is embedded by $v \mapsto (v, -d\iota(v))$. Its degree is

$$\deg \mathcal{Q}_C = 2 + (2 - n) - 2 = 4 - d.$$

At a generic conductor point its map to the ambient tangent space is an isomorphism. Its determinant therefore gives a nonzero rational section of $(\det \mathcal{Q}_C)^\vee \otimes \omega_W^\vee|_C$. At an ordinary double point, the equation $F = yz$ shows that this section has order $-\epsilon_P$.

At the image P_0 of $f \cap B$, choose normalization parameters s, t on the two conductor axes. The conductor identifies s on one axis with t on the other. Its pushout ring consists of series whose restrictions agree; it is generated by

$$x = s + t, \quad y = st, \quad z = s^2t, \quad z^2 + y^3 - xyz = 0.$$

For example the common pure powers are symmetric in s, t , and every mixed term reduces using $s^2 - xs + y = 0$, proving the asserted ring description. Differentiation on the two axes gives the tangent image

$$\langle \partial_x, x\partial_y, x^2\partial_z \rangle.$$

The common tangent line is exactly the specified kernel, and the quotient determinant has order three. Its order at P_0 is consequently $3 - \epsilon_{P_0}$. Since $\omega_W|_C$ is trivial, degrees give

$$d - 4 = 3 - \sum_{P \in C} \epsilon_P, \quad d = 7 - \sum_{P \in C} \epsilon_P \leq 7.$$

Step 4. In this step, we bound the degree when the conductor section maps two-to-one to its image. Let $S = \mathbb{F}_n$, $n = d - 4$, and let $\rho_B : B \rightarrow C$ be the conductor double cover, with involution τ and reduced ramification divisor R_B of degree two. On B form

$$\mathcal{Q}_B = (T_S|_B \oplus \tau^*(T_S|_B))/T_B, \quad v \mapsto (v, -d\tau(v)).$$

It has rank three and degree $2(2 - n) - 2 = 10 - 2d$. At an unramified point the determinant has the ordinary double-point order $-\rho_B^*(\epsilon_P P)$.

At a ramification point take coordinates s, t with $B = (s = 0)$ and $\tau(t) = -t$. The conductor pushout is

$$\mathbb{C}[[t^2]] + s\mathbb{C}[[s, t]] = \mathbb{C}[[t^2, s, st]], \quad (x, y, z) = (t^2, s, st), \quad z^2 - xy^2 = 0.$$

In the two summands the tangent images are

$$(0, 1, t), (2t, 0, 0), (0, 1, -t), (-2t, 0, 0).$$

Their kernel is the displayed copy of T_B ; a basis of the rank-three quotient has determinant a nonzero constant times t^2 . Each of the two ramification points therefore contributes order two. The ambient residue contributes $-2\epsilon_P$ at a ramified point over P . The complete divisor of the rational determinant section is

$$2R_B - \rho_B^* \left(\sum_{P \in C} \epsilon_P P \right).$$

Its degree is both $2d - 10$ and $4 - 2 \sum_P \epsilon_P$. Thus again $d = 7 - \sum_P \epsilon_P \leq 7$. This accounts for both ramification points and every singular ambient point. $\square$

We record the corresponding calculation for full reducible conductor configurations. Its hypotheses specify the gluings; the conclusion does not require a choice of one component.

Lemma 10.19. *Let W be a Gorenstein terminal threefold, and let $T \subset W$ be a reduced connected nonnormal projective Cartier divisor with $-K_T$ ample and ω_W trivial near T . Let $s \geq 2$ be the number of its irreducible components. Write $\nu : \coprod_{i=1}^s S_i \rightarrow T$ for its normalization, C_i for the full conductor on S_i , and $H_i = \nu_i^*(-K_T)$. Assume each S_i maps isomorphically onto its reduced closed image, and assume one of the following conductor configurations:*

- (i) $s = 2$, and the C_i are smooth conics for H_i , identified isomorphically with a single curve $C \simeq \mathbb{P}^1$.
- (ii) Each C_i is a pair of distinct lines for H_i . These lines are identified isomorphically with those on the adjacent components in a cycle, including the possibility $s = 2$. The two intersection points on each pair of identified lines correspond, so all nodes map to one point. The full conductor maps give this cyclic pushout.
- (iii) Every C_i is a double line for H_i , with its full nonreduced conductor structure.

Then $(-K_T)^2 \leq 7s$.

The estimates are respectively 12, $7s$, and $3s$.

Proof. Put $d = (-K_T)^2 = \sum_i H_i^2$. We bound the normalization pieces and use the actual Cartier embedding to constrain the degrees of the conductor normal bundles.

Step 1. In this step, we bound component embedding dimensions and obtain the intrinsic deformation-line inequality. Theorem 7.1 bounds the local embedding dimension of W by four. Passing to the reduced closed components shows that each S_i has the same upper bound. In Theorem 10.8, a cone of degree n over a rational normal curve has vertex embedding dimension $n + 1$. Hence every such component has $n \leq 3$. A ruling line on this cone has self-intersection $1/n$, because its n -fold multiple is numerically the hyperplane class, of square n .

The Cartier embedding gives a natural morphism

$$\mathcal{O}_T(T) \longrightarrow \mathcal{T}_T^1 := \mathcal{E}xt_{\mathcal{O}_T}^1(\Omega_T, \mathcal{O}_T).$$

Locally it sends a change of the Cartier equation to the corresponding first-order deformation. At a generic ordinary double point it sends the normal generator to the smoothing parameter in $xy = 0$, so it is nonzero there. For a double curve C on which the torsion-free quotient of $\mathcal{T}_T^1|_C$ is a line bundle F_C , adjunction therefore gives

$$(10.6) \quad K_T \cdot C = \deg \mathcal{O}_T(T)|_C \leq \deg F_C.$$

This is a morphism supplied by the given ambient embedding, including its isolated singularities.

Step 2. In this step, we bound the degree for two components glued along smooth conics. In (i), the conductor pushout is an ordinary double crossing along C . Changing the two transverse equations in $xy = 0$ identifies its deformation line as

$$F_C = N_{C_1/S_1} \otimes N_{C_2/S_2}.$$

Since $(-K_T) \cdot C = 2$, (10.6) gives $C_1^2 + C_2^2 \geq -2$. The normalization table gives the following possible pairs (H_i^2, C_i^2) :

$$(1, 4), \quad (4, 1), \quad (2, 2), \quad (n+2, 2-n) \ (n=0, 1), \quad (n+4, -n) \ (n \geq 0).$$

Their sums are at most five. Consequently $d \leq 10 - (C_1^2 + C_2^2) \leq 12 \leq 7s$.

Step 3. In this step, we bound the degree for a cyclic gluing of line-pair conductors. In (ii), let u_i be the sum of the individual self-intersections of the two conductor lines on S_i . The relevant rows of the normalization table give

$$(H_i^2, u_i) = (1, 2), \quad (n, 2/n) \ (2 \leq n \leq 3), \quad (n+2, -n) \ (n \geq 0).$$

Thus $H_i^2 + u_i \leq 4$.

We check the singularities along the full double locus. Away from its common vertex P the pushout is an ordinary double crossing. At P , a normalization branch is either smooth with two transverse conductor axes or a cyclic cone with its two toric boundary generators. Resolve the cone vertices, and blow up the crossing of the conductor axes on every smooth branch. The strict conductor curves are now disjoint on each branch, and the exceptional rational curve meets them in two distinct points. Glue by the same cyclic identifications. This gives a semismooth semiresolution whose exceptional curves form a rational cycle; contracting superfluous curves gives a cycle or a rational nodal curve. Thus P is a degenerate cusp. Its embedding dimension is at most four, as it is a point of T .

The double curves $C_j \simeq \mathbb{P}^1$ are proper and smooth, each containing this one cusp and no pinch point. These are the hypotheses of Theorem 10.10. The germs are Gorenstein local complete intersections, so their ordinary and qG first-order deformation sheaves agree. The theorem therefore computes the same torsion-free line F_{C_j} as in (10.6).

The cusp contribution in that formula is at most two. For type $T_{u,v}^3$, $1 + (u+v)/(uv+u+v) \leq 2$. For type $T_{u,v,w}^4$ it is at most one, since for finite $u, v, w \geq 2$,

$$wuv - u - v - (w(u+v) - 4) \geq 0;$$

the expression is zero at $u = v = 2$ and increases with either variable, and the infinite-parameter limits preserve the inequality. The other cusp contributions are exactly two. Hence

$$\deg F_{C_j} \leq \tilde{C}_j^2 + 2.$$

Since $(-K_T) \cdot C_j = 1$, (10.6) gives $\tilde{C}_j^2 \geq -3$. The two preimage lines of C_j lie on distinct normalization components, also when $s = 2$. Thus their square is the sum of their individual self-intersections, without an intersection cross term. Every conductor line occurs once, so

$$\sum_i u_i = \sum_j \tilde{C}_j^2 \geq -3s, \quad d = \sum_i H_i^2 \leq 4s - \sum_i u_i \leq 7s.$$

Step 4. In this step, we bound the degree with the full double-line conductors retained. In (iii), the normalization table permits only $\mathbb{P}^2$ with its line polarization and a rational normal cone with its hyperplane polarization. Their degrees are respectively one and $n \leq 3$, by the ambient embedding-dimension bound. Therefore $d \leq 3s$. This estimate uses the full double-line case and requires no ordinary-double-crossing hypothesis. $\square$

Lemma 10.20. *Let an original point-exit datum be given with E nonnormal and $\alpha = 1$. Put $o_Z = h(E)$ and $q_Z = \text{ind}_{o_Z}(K_Z)$. Let $\pi_Z : Z^\sharp \rightarrow Z$ be the connected local canonical cover at o_Z , and let W be the normalization of the component of $Y \times_Z Z^\sharp$ dominating $Z^\sharp$. Write $\pi : W \rightarrow Y$ and $h^\sharp : W \rightarrow Z^\sharp$ for the induced maps, let T be the full reduced divisorial inverse image of E , and let s be its number of irreducible components. Then:*

(i) *W is Gorenstein terminal, ω_W is trivial near T , and T is a reduced connected nonnormal projective Gorenstein Cartier divisor with $-K_T$ ample.*

(ii) *$s \leq q_Z$ and $(-K_T)^2 = q_Z E^3$.*

(iii) $E^3 \leq 7$ and $v \leq 7M^2$.

The degree of the target cover cancels in the estimate for the original complete surface.

Proof. We construct the crepant canonical base change, identify the full conductor configurations of its inverse divisor, and divide the resulting degree estimate by the cover degree.

Step 1. In this step, we construct the crepant canonical base change and identify its full inverse divisor. (10.1) gives $K_Y = h^*K_Z$. Since Y is terminal, Z is canonical: E has log discrepancy one, and every other exceptional prime has log discrepancy greater than one. Apply Theorem 3.9 with zero boundary and multiplier one. This gives the specified cover of degree q_Z , one point $o_Z^\#$ over o_Z , and a trivial canonical line on $Z^\#$.

The normalized base change is finite of degree q_Z over Y , and $h^\#$ is projective birational. We check ramification at the exceptional prime as well as away from it. A root equation defining the canonical cover has divisor $q_Z K_Z$. Its pullback has divisor $q_Z h^* K_Z = q_Z K_Y$; hence its order at every codimension-one point of Y is divisible by q_Z . The Kummer extension is unramified at these discrete valuations in characteristic zero. Consequently

$$K_W = \pi^* K_Y = (h^\#)^* K_{Z^\#}.$$

Finite discrepancy comparison from Lemma 3.8 shows that W is terminal: an exceptional prime upstairs restricts to an exceptional prime downstairs, and its bare log discrepancy is multiplied by the positive ramification index. Its canonical line is the trivial pulled-back line, so W is Gorenstein.

Absence of divisorial ramification gives $\pi^* E = T$, including the multiplicity one on every component. This is an integral $\mathbb{Q}$ -Cartier Weil divisor on the index-one terminal threefold W , so Lemma 8.3 makes it Cartier. It is the reduced full inverse divisor, and adjunction gives

$$\omega_T = \mathcal{O}_W(T)|_T, \quad \mathcal{O}_W(-MT)|_T = \pi^* L.$$

The second line is ample by finiteness, so $-K_T$ is ample. The support of T is the full fibre over $o_Z^\#$. This fibre is connected because $h^\#$ is proper birational with normal target. Thus T is connected and projective, and Cartier adjunction makes it Gorenstein.

Over Y_{reg} the pulled-back root equation, after dividing by the q_Z -th power of a local canonical trivialization, is the root of a unit. Hence π is étale there. The nonnormal S_2 surface E has a curve of nonnormal points whose generic point lies in this smooth locus. If T were normal over that curve, normality would descend through the étale surjection, contradicting nonnormality of E . Therefore T is nonnormal.

Every component of T dominates E , and their generic degrees sum to q_Z . In particular $s \leq q_Z$. The projection formula and adjunction give

$$d := (-K_T)^2 = T^3 = q_Z E^3.$$

Step 2. In this step, we identify all conductor configurations of the full inverse divisor and bound their degrees. If $s = 1$, Lemma 10.18 gives $d \leq 7$. Suppose $s \geq 2$, and apply Theorem 10.9 to the whole surface T . Its cases (A)–(E) are, respectively, Reid's cases (A), (B), (C_1) , (C_r) , (D). Cases (A) and (C) have irreducible normalization, so only (B), (D), and the $r = s \geq 2$ part of (E) occur here. In each of these reducible cases, every normalization piece is isomorphic to its reduced closed image in T .

In (B), the two smooth conic conductors map isomorphically to the same smooth curve. This is case (i) of Lemma 10.19. In (D), the line pairs form a cycle with $r = s \geq 2$, and all their nodes map to the unique common vertex. Since the normalization pieces are closed components, each conductor line maps isomorphically to its image, including at the vertex. Thus every double curve is a smooth projective line, and the marked intersection points agree under the identifications. These are the full cyclic conductor maps in case (ii). Indeed, for a normalization ring $A \subset B$ and conductor ideal I , the conductor square is the identity $A = B \times_{B/I} (A/I)$. Here the conductor is reduced; away from the vertex this identity gives the ordinary double crossings. The S_2 property

of T extends their functions uniquely across the vertex, so there is no additional modification supported there. When $s = 2$, the two components meet along two distinct double curves, and both curves enter the cyclic calculation.

Finally, in (E) the conductor on every normalization piece is the full double line. If D_T denotes the downstairs conductor and $\Gamma = (D_T)_{\text{red}} \simeq \mathbb{P}^1$, the tame scheme structure is

$$\mathcal{O}_{D_T} \simeq \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-1)^{\oplus(s-1)},$$

with square-zero second summand and linear component maps. This is case (iii) of the same lemma, with the nonreduced conductor retained on every piece. Lemma 10.19 therefore gives $d \leq 7s$ in all the reducible cases.

Step 3. In this step, we divide by the cover degree to bound the original complete surface. For every $s \geq 1$ we have proved $d \leq 7s$. Hence

$$E^3 = \frac{d}{qz} \leq \frac{7s}{qz} \leq 7, \quad v = M^2 E^3 \leq 7M^2.$$

Only this numerical calculation uses the auxiliary cover. The polarization, markings, and natural character in Proposition 10.12 remain those on the original E . $\square$

Proposition 10.21. *For the data of Proposition 10.12, put*

$$t_m = \binom{m+4}{4} - 1, \quad V_\star = \max \left\{ 1, 7M^2, 4M^3, \frac{M^2 \max(t_m + 7, 2t_m)}{a^2} \right\}.$$

Either $v \leq V_\star$, or E is nonnormal and $1/2 < a \leq \alpha < 1$.

Proof. Lemma 3.11 permits us to use the same marked diagram over $\mathbb{C}$, preserving m, a, M, α, v and normality of E . Suppose $v > V_\star$. The normal-surface estimate in Lemma 10.16(i) forces E to be nonnormal, and part (iii) then gives $a > 1/2$. Part (ii) excludes $\alpha > 1$, since $V_\star \geq 4M^3$. If $\alpha = 1$, Lemma 10.20 gives $v \leq 7M^2 \leq V_\star$, again a contradiction. Thus $\alpha < 1$. Finally, Proposition 10.12 supplies $a \leq \alpha$, giving exactly $1/2 < a \leq \alpha < 1$. The same field comparison returns this alternative for the original marked surface over k . $\square$

11. POINT EXITS

We retain the original point-exit data of Definition 10.11. Thus E is the complete exceptional surface, H is its original stabilizer, and χ is the faithful natural character. We use the polarization L and the bare discrepancy α of Proposition 10.12. There are two estimates: one uses the bounded presentation of the marked surface, and the other uses the denominator of α .

Theorem 11.1 (Bounded generators of a polynomial kernel). *Let F be a field, $D = F[x_1, \dots, x_N]$, and $A \in D^{m \times n}$ have entries of degree at most d , where $m, n, d \geq 1$. The D -module $\{y \in D^n : Ay = 0\}$ is generated by vectors of degree at most $(2md)^{2^N}$ [Asc04, Theorem 3.2].*

Theorem 11.2 (The noncanonical threefold gap). *Let X be a normal $\mathbb{Q}$ -Gorenstein klt threefold. Define $\text{mld}(X)$ as the infimum of log discrepancies over all prime divisors over X , with arbitrary centres. If $\text{mld}(X) < 1$, then $\text{mld}(X) \leq 12/13$ [LX21, Theorem 1.4].*

Theorem 11.3 (Terminal residue lemma). *Let $r \geq 2$ and a_1, a_2, a_3, a_4, e be integers. Set $q = \gcd(e, r) = \gcd(a_4, r)$, and assume $\gcd(a_i, r) = 1$ for $i = 1, 2, 3$. For an integer u , let $\bar{u}$ be its least nonnegative residue modulo r . Suppose*

$$\sum_{i=1}^4 \overline{ka_i} = \overline{ke} + k + r \quad \text{for every } 1 \leq k < r.$$

If $q > 1$, then $a_4 \equiv e \pmod{r}$ and, after permuting a_1, a_2, a_3 , one has $a_1 \equiv 1$ and $a_2 + a_3 \equiv 0 \pmod{r}$. If $q = 1$, the multiset $\{a_1, a_2, a_3, a_4, -e, -1\}$ splits into three pairs whose sums are zero modulo r [Jia19, Theorem 2.10].

Theorem 11.4 (Analytic Cartier inversion of adjunction). *Let X be normal analytic, $X_0 \subset X$ an effective Cartier divisor, and $\Delta \geq 0$ an $\mathbb{R}$ -divisor with no irreducible component in common with X_0 , so that the restriction is defined. Assume $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Then $(X, X_0 + \Delta)$ is plt near X_0 if and only if $(X_0, \Delta|_{X_0})$ is klt [Kol21, Proposition 30].*

Theorem 11.5 (Analytic Weierstrass preparation). *Let $f \in \mathbb{C}\{x_1, \dots, x_{n-1}, t\}$ satisfy $\text{ord}_t f(0, t) = d < \infty$. There are a unique monic polynomial*

$$P = t^d + a_1(x)t^{d-1} + \dots + a_d(x), \quad a_i(0) = 0,$$

with convergent coefficients, and a unit u in the same convergent power-series ring, such that $f = uP$ [Sch24, Theorem 2.8].

We use Theorem 11.1 for canonical modules of the bounded surface presentations, before passing to any local cover.

Lemma 11.6. *Let $m \geq 1$ be an integer. Let $(Z \ni z, B)$ be an algebraic klt threefold germ with K_Z $\mathbb{Q}$ -Cartier. Let a cyclic group H act faithfully, fix z , and preserve B . Suppose mB is integral and $m(K_Z + B)$ has a natural eigen-generator of faithful character χ . Put $r = \text{ind}_z(K_Z)$, and let $W \rightarrow Z$ be the canonical cover of Theorem 3.9 with boundary zero and multiplier one. Then H lifts to a faithful abelian action fitting into*

$$1 \longrightarrow \Delta = \mu_r \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1.$$

The natural pulled m -log character is the inflation of χ , and its kernel is exactly Δ .

The degree r is not assumed bounded in this lemma.

Proof. Choose an H -eigen-generator of $\mathcal{O}_Z(rK_Z)$ by projecting a local generator to its fibre character at z . The projection remains a generator near z . In the reflexive canonical algebra defining the cover, a lift of an element of H is specified by an r -th root of its scalar on this generator. The lifts form the displayed extension. Natural pullback commutes with the deck scalars, so Δ is central. A lift of a generator of H , together with Δ , generates $\tilde{H}$; hence it is abelian. An element acting trivially on W acts trivially on Z and then is a trivial deck transformation, proving faithfulness. Quasi-étaleness identifies the pulled natural log line with the log line of the pulled pair. Deck transformations fix the pulled generator, and a lift of $h \in H$ multiplies it by $\chi(h)$. Faithfulness of χ gives the asserted kernel. $\square$

Lemma 11.7. *Let an original point-exit datum be given. Let $P \in E$ be closed and let $J \leq H$ fix P . On the local canonical cover $\rho : (W \ni P^\#) \rightarrow (Y \ni P)$, write $\tilde{J}$ for the lifted group and $E^\# = \rho^{-1}(E)_{\text{red}}$, $R^\# = \rho^*R$. Then $E^\#$ and $mR^\#$ are Cartier, with regular semi-invariant equations z and u . The full scheme $E^\#$ is reduced and Cohen–Macaulay, and is a local complete intersection. Its components have discrepancy a and inertia one for $\tilde{J}$. If θ_W generates the canonical sheaf, the natural log generator is, up to a unit,*

$$\frac{\theta_W^m}{z^{m-ma}u}.$$

Every primitive prime T with centre in $E^\#$ satisfies

$$\frac{A_{W, (1-a)E^\# + R^\#}(T)}{|I_{\tilde{J}}(T)|} \geq a.$$

These assertions include $a = 1$.

The equations retain every branch of $E^\#$ and the full boundary multiplicities.

Proof. Proposition 8.8 supplies the actual cover and its smooth or isolated cDV chart. Its use here requires only the positive exact minimum of the local quotient pair, not that this minimum equal a . The cover degree is at most m by Theorem 7.16. Quasi-étaleness makes the Weil pullback of E reduced. Both this divisor and $mR^\#$ are integral and $\mathbb{Q}$ -Cartier. Lemma 8.3,

applied on the Gorenstein terminal W , makes them Cartier. Character projections give equations z, u . Since W is a local complete intersection and z is a nonzerodivisor, $E^\#$ is a Cohen–Macaulay local complete intersection. Its generic multiplicities are one, so it is reduced.

A component of $E^\#$ is unramified over the generic point of E , hence has discrepancy a . An element of its full inertia projects to $I_H(E) = 1$. The remaining deck inertia is also trivial by quasi-étaleness. For any prime T in the assertion, primitive restriction and Lemma 3.8 give

$$\frac{A_{W, (1-a)E^\# + R^\#}(T)}{|I_J(T)|} = \frac{A_{Y, (1-a)E + R}(T_Y)}{|I_J(T_Y)|} \geq \frac{A_{Y, (1-a)E + R}(T_Y)}{|I_H(T_Y)|} \geq a.$$

The last inequality is Theorem 7.16. The displayed generator follows by inserting the two Cartier equations into the natural log line. When $a = 1$, its z -exponent is zero; the Cartier assertion for $E^\#$ was proved independently of that exponent. $\square$

Lemma 11.8. *Let $N \geq 2$ and $D \geq 1$ be integers. Let a finite cyclic group H , of order h_0 , act on $\mathbb{P}_{\mathbb{C}}^N$, and let $E \subset \mathbb{P}_{\mathbb{C}}^N$ be an invariant integral surface not contained in a hyperplane. Suppose the action on E is faithful. Let $A_1, \dots, A_s \subset E$ be invariant closed subschemes. Assume that the homogeneous ideals of $E, A_1, \dots, A_s$ are generated in degrees at most D . Let $\chi : H \rightarrow \mathbb{C}^*$ be faithful, choose a primitive h_0 -th root ζ , and choose $\sigma \in H$ with $\chi(\sigma) = \zeta$. For every integer $Q > 2D$ such that $h_0 > 2Q^N$, there are an integer j with $h_0/2 < j < h_0$ and a diagonal torus action $\tau : \mathbb{C}^* \rightarrow \text{Aut}(\mathbb{P}^N)$ such that*

- (i) τ preserves E and every A_i as schemes;
- (ii) $\tau(\zeta) = \sigma^j$, and $\chi(\sigma^j) = \zeta^j$;
- (iii) the diagonal weights of τ can be chosen in $(-h_0/Q, h_0/Q)$.

On a minimum-weight affine chart the coordinate weights belong to $[0, 2h_0/Q)$. That chart meets E , and its fixed locus on E is nonempty and has dimension at most one.

This construction preserves the defining ideals of the markings, including their multiplicities.

Proof. Lift the projective action of σ to a matrix. Its h_0 -th power is scalar, so rescaling the matrix makes that power the identity. Diagonalize, and subtract one common weight. Write the resulting weights as $0, c_1, \dots, c_N$ modulo h_0 .

Apply Lemma 8.9 to the identity map of H , its subgroup $H[2]$ of elements of order at most two, and the N coordinate characters relative to the first one. Since $|H[2]| \leq 2$, it gives an element outside $H[2]$ whose relative character phases have absolute value less than $1/Q$. Replace this element by its inverse if necessary. It then has the form σ^j with $h_0/2 < j < h_0$. Choose integer representatives $b_0 = 0, b_1, \dots, b_N$ of jc_i in $(-h_0/Q, h_0/Q)$, and let τ have these weights.

Each invariant homogeneous ideal has generators that are H -eigenvectors of degree at most D : take eigenvector bases in its homogeneous pieces of these degrees. Two monomials in such an eigenvector have equal τ -weights modulo h_0 . Their integer weights differ by less than $2Dh_0/Q < h_0$, and therefore are equal. Each generator is τ -homogeneous. This proves scheme-theoretic preservation, and the choice of weights gives $\tau(\zeta) = \sigma^j$.

Choose a coordinate of minimum weight. Its affine chart meets E , since E is not contained in a hyperplane. The affine weights are the differences from this minimum, hence lie in $[0, 2h_0/Q)$. Taking the limit as the torus parameter tends to zero gives a fixed point of the closed invariant affine surface. A fixed component of dimension two would be all of E , because E is integral. This would make σ^j act trivially on E , contrary to faithfulness and $0 < j < h_0$. $\square$

Theorem 11.9. *For every $m \geq 1$, rational $a \in (0, 1]$ with ma integral, and $V \geq 1$, there is an integer $C_{\text{vol}}(m, a, V)$ such that every original point-exit datum with $L^2 \leq V$ satisfies*

$$|H| \leq C_{\text{vol}}(m, a, V).$$

The bound concerns the full original stabilizer and retains all boundary and contact markings.

The bounded embeddings of Proposition 10.12 allow Lemma 11.8 to be applied to the complete surface and its full markings. The remaining estimate concerns the natural dualizing weight and the normal direction on an actual local canonical cover.

Proof. We bound the coarse dualizing weight, lift a short power of the original action to one local cover, and deform a surface valuation in its actual normal direction. Its quotient discrepancy will contradict the original comparison. We first work over $\mathbb{C}$. Put $k_0 = ma$ and $M = \text{lcm}(1, \dots, m)$.

Step 1. In this step, we bound the coarse dualizing weight from the marked surface presentation. By Proposition 10.12, the complete embedding of E , together with the ideals of A_R and T_g , has dimension at most N_0 and equations of degree at most D , with these bounds depending only on m, a, V . Enlarge D to a positive integer. Suppose a diagonal torus preserves this presentation and, on a minimum-weight chart, the coordinate weights lie in $[0, 2s_0)$. There is an integer $K = K(N, D, M) \geq 1$ such that the weight κ of the invertible sheaf $\omega_E^{[M]}$ at a fixed point, or a generic fixed curve, satisfies

$$(11.1) \quad |\kappa| \leq Ks_0.$$

Here is the presentation argument for this assertion. Write the affine coordinate ring as $B = \mathbb{C}[x_1, \dots, x_N]/I$. Bounded-degree eigen-equations give a bounded presentation. Theorem 11.1 bounds both degrees and numbers of generators of every subsequent kernel: the number is bounded by the dimension of the space of polynomial vectors up to the degree bound. Kernels over B are computed by adding columns for the generators of I . Repeating this operation gives a partial free resolution long enough to compute $\omega_B = \text{Ext}^{N-2}(B, \mathbb{C}[x])$, with its differential-form twist. Tensor products and the same kernel operation then present

$$\text{Hom}_B(\text{Hom}_B(\omega_B^{\otimes M}, \omega_B), \omega_B).$$

This is $\omega_B^{[M]}$. Indeed it agrees with that line on the Gorenstein open set. A canonical dual is torsion-free and has depth at least two at a surface point: express it as a kernel between finite sums of ω_B ; the image is torsion-free and has depth at least one, so the depth lemma applies. The double canonical dual is therefore the unique S_2 extension across the finite complement.

All presentations can be taken homogeneous for the torus by projecting their generators to weight spaces. This does not increase polynomial degrees. A degree- d monomial has weight in $[0, 2ds_0)$, and the finitely many dual, tensor and kernel operations add or subtract boundedly many shifts. All resulting generator weights are thus bounded in absolute value by Ks_0 . A generator nonzero in the one-dimensional fibre has exactly its weight, proving (11.1). At a generic fixed curve, localize the weight-zero coefficient ring to its function field; polynomial degrees do not increase and the coefficient field has weight zero.

Step 2. In this step, we construct a torus preserving the complete surface and both markings. Set $Q = \lceil 64m(K + D + 10) \rceil$, $h_0 = |H|$, and suppose $h_0 > 2Q^N$. Lemma 11.8 gives a torus preserving E, A_R, T_g and an element $\gamma = \sigma^j$, with $h_0/2 < j < h_0$ and $\chi(\gamma) = \zeta_{h_0}^j$. Put $s_0 = h_0/Q$. Choose a component of the attracting fixed locus. Let ξ be its generic point if it is a curve, and its closed point otherwise. Its coefficient field K_0 is respectively the curve function field or $\mathbb{C}$. The transverse surface ring is positively graded over K_0 , of dimension $d = 1$ or 2 . To see this, its degree-zero ring is reduced, being a subring of an integral ring; localization at the generic point of the chosen fixed component makes it a field. The positive coordinate weights are less than $2s_0$.

The invertible ideal of A_R has a homogeneous local generator of weight μ_A with

$$0 \leq \mu_A < 2Ds_0.$$

Choose a bounded-degree equation whose image generates the ideal modulo the maximal ideal; it has this weight. A unit marking has weight zero.

Step 3. In this step, we lift the grading to the actual local canonical cover and identify its natural weights. Put $J = \langle \gamma \rangle$. If ξ is closed, use Lemma 11.7 on the actual canonical cover

$W \rightarrow (Y \ni \xi)$, of degree $r_0 \leq m$. Write $E^\# = (z = 0)$ and $mR^\# = (u = 0)$. After shrinking, the cover is μ_{r_0} -étale on the punctured E : a terminal threefold is smooth outside its isolated singular points, by Theorem 7.6. If ξ is a generic curve point, Y is smooth there; use its smooth transverse surface over K_0 , the identity cover, and the same equations. Set $r_0 = 1$ in that case.

For a closed ξ , positive real scaling identifies the punctured analytic cone with its link times a radial interval. The finite punctured cover consequently extends along the cone. Lift paths of the coarse torus through it. A period of the universal covering group of $\mathbb{C}^*$ acts by a deck transformation, of order $s \mid r_0$. Thus the lifted action factors through a torus mapping to the coarse torus by the s -th power. The punctured $E^\#$ is connected: its Cohen–Macaulay local ring has depth two, and an idempotent on the puncture extends to the local ring. All deck transformations are therefore the given μ_{r_0} . Hartogs extension for this S_2 ring extends the lifted action and its inverse across the vertex. Path lifting supplies the group law.

The lifted weights on the cotangent space are positive. Finiteness over the positively graded base excludes a negative-degree integral element by its monic equation; the degree-zero part is the finite residue algebra at the vertex, hence $\mathbb{C}$. Choose homogeneous analytic generators by circle averaging and Nakayama's lemma. Positive weights make each homogeneous relation polynomial. This gives a positive grading on the completed local $E^\#$. At a curve point put $s = 1$ and use the existing grading.

Evaluate the lifted torus at ζ_{sh_0} . Its action on $E^\#$ is the restriction of an actual lift η of γ , after changing that lift by a deck transformation. This restriction is faithful on $\langle \eta \rangle$, since each component of $E^\#$ has inertia one. In particular, $\chi(\eta) = \zeta_{h_0}^j$. If w_i are weights of homogeneous generators of $E^\#$, put

$$g_0 = \gcd(sh_0, w_1, w_2, \dots), \quad |\langle \eta \rangle| = sh_0/g_0.$$

Let β be the weight of its dualizing generator and μ the weight of the ideal of $u|_{E^\#}$. Natural étale adjunction on the puncture, extended by S_2 , gives

$$(11.2) \quad M\beta = s\kappa, \quad M\mu = s\mu_A, \quad |\beta| \leq sKs_0, \quad 0 \leq \mu < 2sDs_0.$$

For the second identity, the M -th power of the invertible ideal of u is the pullback of the ideal of A_R . Its root ideal is torus invariant: it agrees with its transforms in codimension one, and an invertible ideal on an S_2 scheme is determined there. Homogeneous generators differ by units of weight zero. This proves the identity with the full multiplicities.

Step 4. In this step, we bound a Jacobian-minor order on the chosen normalization component. Choose a component of $E^\#$. Its grading defines an integer-valued, possibly nonprimitive weight valuation w , with residue transcendence degree $d - 1$ over K_0 . In homogeneous parameter expansions, a top form of weight b has order $b - 1$: it is $T^{b-1}dT \wedge \vartheta$ for a nonzero residue form ϑ . Roots of a uniformizer may be used to write these expansions; the final valuation will be restricted to the original algebraic field.

Choose d coarse coordinates with independent differentials on E . Their wedge on the cover has weight less than $2dss_0 \leq 4ss_0$. Dividing by the dualizing generator gives an element of the image of $\bigwedge^d \Omega_{E^\# / K_0} \rightarrow \omega_{E^\#}$ of order less than $4ss_0 - \beta$. For a reduced local complete intersection this image is the ideal of maximal Jacobian minors: the residue formula identifies a coordinate differential wedge with the complementary minor times a dualizing generator. This calculation is valid on each component.

Use the actual smooth ambient chart of W . Its equations are $q_1, \dots, q_c$, where $c = 0$ or 1 , and the equations of $E^\#$ are $(q_1, \dots, q_c, z)$. There is a corresponding Jacobian minor J_0 with

$$(11.3) \quad 0 \leq e_0 = w(J_0) \leq s(K + 4)s_0.$$

Call the other d ambient coordinates the free coordinates. Choose their expansions and a coefficient field equivariantly for η . One obtains the latter by lifting a separating residue transcendence basis and then Hensel lifting the finite separable residue extension. The torus fixes the residue field.

Step 5. In this step, we use the actual normal character to construct an algebraic prime contradicting the original discrepancy comparison. Let n_{act} be the exponent modulo sh_0 of η on z . The residue identity for the dualizing generator and Lemma 11.7 give

$$k_0 n_{\text{act}} \equiv sj - m\beta + \mu \pmod{sh_0}.$$

Choose $l \in \{0, \dots, k_0 - 1\}$ congruent modulo k_0 to $(k_0 n_{\text{act}} - sj + m\beta - \mu)/(sh_0)$, and put

$$n = \frac{sj - m\beta + \mu + lsh_0}{k_0}.$$

Then n is an integer congruent to n_{act} modulo sh_0 , and $g_0 \mid n$. Equations (11.2) and (11.3), and $k_0 \leq m$, show

$$n \geq ss_0 \left(\frac{Q}{2m} - K \right) > \max\{2s(K+4)s_0, s(K+4+2D)s_0\}.$$

In particular $n > 2e_0$ and $n > e_0 + \mu$.

Keep the free-coordinate expansions fixed, introduce a new residue variable Z_0 fixed by η , and solve

$$q_1 = \dots = q_c = 0, \quad z = T^n Z_0$$

for the other $c+1$ coordinates. Newton iteration gives a solution in the completed valuation field. At the original solution the error has order n , and the inverse Jacobian changes coordinates only in orders at least $n - e_0$. The new error has order at least $2(n - e_0) > n$; iteration converges and preserves the leading determinant. Uniqueness in this ball and the congruence for n make the solution η -equivariant. All coordinates still have positive order, and $n - e_0 > \mu$ preserves $w(u) = \mu$.

This defines an injection of the actual field $K_0(W)$. The free coordinates are algebraically independent, by the nonzero Jacobian minor; the independent variable Z_0 adds the remaining transcendence degree through z . The resulting kernel in the integral local ring of W has height zero, hence is zero. Its value group is $d_0\mathbb{Z}$ for a positive integer d_0 , and its residue field has transcendence degree d over K_0 : the surface grading contributes $d-1$, and a value-zero quotient involving a power of z contributes a transcendental power of Z_0 . The primitive normalization is therefore an algebraic divisorial valuation $P^\#$. Explicitly, choose a positive-value function and lifts of a residue transcendence basis, normalize the resulting algebraic model, and take the codimension-one centre. At a curve point this model spreads over a dense open subset of the curve.

The residue formula for W is the free differential wedge times dz , divided by J_0 . On $E^\#$ the free wedge has order $e_0 + \beta - 1$. It was kept fixed; the new residue differential contributes n , and the unchanged Jacobian subtracts e_0 . The homogeneous bare log discrepancy is consequently $\beta + n$. With the exact boundary orders it gives

$$\beta + an - \mu/m = \frac{s(j + lh_0)}{m} > 0.$$

All expansion exponents are divisible by g_0 , and Newton iteration preserves that divisibility. Thus $d_0 \geq g_0$. The constructed residue field is fixed by η , so $\langle \eta \rangle \subset I_{\tilde{J}}(P^\#)$. We obtain

$$\frac{A_{W, (1-a)E^\# + R^\#}(P^\#)}{|I_{\tilde{J}}(P^\#)|} \leq \frac{s(j + lh_0)}{md_0} \frac{g_0}{sh_0} \leq \frac{j/h_0 + l}{m} < \frac{k_0}{m} = a.$$

This contradicts Lemma 11.7. At a generic curve point the same finite-crepant comparison is the identity-cover version, and its centre is contained in E . The contradiction takes place in the original function field; the intermediate analytic grading supplies no substitute minimum.

We have proved $|H| \leq 2Q^N$. Maximize this expression over $2 \leq N \leq \max\{2, N_0\}$ to define $C_{\text{vol}}(m, a, V)$. Finally Lemma 3.11 transfers the complete marked diagram and its all-prime comparison to $\mathbb{C}$ and back, preserving L^2 , the finite group and its natural character. This proves the asserted bound over k . $\square$

Lemma 11.10. *Let $Z \ni z$ be an algebraic $\mathbb{Q}$ -Gorenstein klt threefold germ. Suppose that exactly one exceptional prime divisor E over a neighbourhood of z has log discrepancy at most one, that its centre is z , and that $1/2 < \alpha = A_Z(E) < 1$. Write $\alpha = u_0/v_0$ in lowest terms and put $r = \text{ind}_z(K_Z)$. Let $\rho : (W \ni o) \rightarrow (Z \ni z)$ be the canonical cover supplied by Theorem 3.9 with boundary zero and multiplier one, and let Δ be its deck group. Then W is Gorenstein terminal. Every primitive extension F of E to $\mathbb{C}(W)$ satisfies*

$$|I_\Delta(F)| = v_0, \quad A_W(F) = u_0.$$

The terminality assertion uses the interval $(1/2, 1)$ and uniqueness over a neighbourhood.

Proof. Theorem 3.9 makes W klt and K_W Cartier. Its exceptional log discrepancies are positive integers. If W were not terminal, there would be an exceptional primitive prime T with $A_W(T) = 1$. Its primitive restriction T_Z is exceptional over Z : a finite morphism preserves the dimension of a centre, and an exceptional centre has codimension at least two. By Lemma 3.8,

$$A_Z(T_Z) = \frac{1}{|I_\Delta(T)|} \leq 1.$$

Uniqueness forces $T_Z = E$, so α is the reciprocal of a positive integer. No such reciprocal lies strictly between $1/2$ and 1 . Thus W is terminal. Applying the inertia and discrepancy formulas of Theorem 3.9 to E , again with multiplier one, gives the two displayed identities. $\square$

Lemma 11.11. *For every integer $d \geq 1$ and every real $\epsilon > 0$, there is an integer $I(d, \epsilon) \geq 1$ with the following property. If $v_1, \dots, v_d \in (0, 1]$ and $R \geq 2$ is an integer satisfying*

$$\sum_{i=1}^d (1 + (n-1)v_i - \lceil nv_i \rceil) \geq \epsilon \quad (2 \leq n \leq R),$$

then $R \leq I(d, \epsilon)$.

Strict positivity is necessary: a zero coordinate would contribute one to every sum.

Proof. Suppose that no bound exists. Choose $R_\ell \rightarrow \infty$ and vectors $v^{(\ell)} \in (0, 1]^d$ satisfying the inequalities. Pass to a subsequence with $v^{(\ell)} \rightarrow \bar{v} \in [0, 1]^d$. Choose $\delta > 0$ smaller than $\epsilon/(2d)$ and than $\min(\bar{v}_i, 1 - \bar{v}_i)$ for every coordinate in $(0, 1)$.

There is a positive integer n_0 such that each $n_0 \bar{v}_i$ is within δ of an integer and is an integer whenever $\bar{v}_i$ is rational. To obtain it, first take a common multiple of the denominators of the rational coordinates. Apply the finite-box argument to multiples of this integer in the torus of the remaining coordinates, and subtract two multiples in the same box.

If $0 < \bar{v}_i < 1$, our choice of δ makes $(n_0 + 1)\bar{v}_i$ nonintegral and gives

$$|\{(n_0 + 1)\bar{v}_i\} - \bar{v}_i| < \delta.$$

Consequently

$$\lim_{\ell \rightarrow \infty} (1 + n_0 v_i^{(\ell)} - \lceil (n_0 + 1)v_i^{(\ell)} \rceil) = \{(n_0 + 1)\bar{v}_i\} - \bar{v}_i.$$

If $\bar{v}_i = 0$, strict positivity gives $0 < (n_0 + 1)v_i^{(\ell)} < 1$ for large ℓ , so the same limit is zero. If $\bar{v}_i = 1$, the bound $v_i^{(\ell)} \leq 1$ also makes the limit zero. The sum of the limits is less than $d\delta < \epsilon$. For large ℓ , this contradicts the assumed inequality at $n = n_0 + 1 \leq R_\ell$. $\square$

We apply Theorem 11.2 only after identifying the minimum on a whole neighbourhood with the discrepancy of its unique low prime. Both the identity for every residue and the gcd conditions will be checked before applying Theorem 11.3.

We use Theorem 11.4 with the actual normal hypersurface in a smooth ambient germ and with zero extra boundary. Uniqueness in Theorem 11.5 makes the preparation compatible with a diagonal finite group action when the distinguished variable is an eigenparameter.

Set-up 11.12. Let $W = (f = 0) \subset (\mathbb{C}^4, 0)$ be an isolated terminal hypersurface, and let μ_r act diagonally, freely on $W \setminus \{0\}$, so that $W \rightarrow X = W/\mu_r$ is the canonical cover. Suppose exactly one exceptional prime over a neighbourhood of $0 \in X$ has discrepancy at most one, and its discrepancy is $\alpha \in (1/2, 1)$. Write the coordinate characters as $c_1, \dots, c_4$ and the equation character as e , choosing the generator so that $\sum c_i - e \equiv 1 \pmod{r}$. We use one of the forms

$$\begin{aligned} \text{cA :} & \quad f = x_1x_2 + g(x_3, x_4), \quad g \in (x_3, x_4)^2, \\ \text{odd :} & \quad f = x_1^2 + x_2^2 + g(x_3, x_4), \quad g \in (x_3, x_4)^3, \quad c_1 \not\equiv c_2, \\ \text{rank one :} & \quad f = x_1^2 + g(x_2, x_3, x_4), \quad g \in (x_2, x_3, x_4)^3. \end{aligned}$$

Set

$$\mathcal{L} = \mathbb{Z}^4 + \mathbb{Z}(c_1, c_2, c_3, c_4)/r, \quad \Phi(w) = \sum_i w_i - w(f), \quad \mathcal{U} = (\mathcal{L} \cap [0, 1]^4) \setminus \{0, 1\}^4.$$

Here $w(f)$ is the least weight of a nonzero monomial of the convergent equation. For $1 \leq j < r$, write $v_j = (\{jc_1/r\}, \dots, \{jc_4/r\})$.

The domain $\mathcal{U}$ excludes every integral corner. The complement of a vector in it remains in the same domain.

Lemma 11.13. *In Setup 11.12, every positive $w \in \mathcal{L}$ has $\Phi(w) > 0$. There is at most one primitive positive vector β with $\Phi(\beta) \leq 1$; when it exists, $\Phi(\beta) = \alpha$. Every other positive vector in $\mathcal{L}$ has cost greater than one. Every vector in $\mathcal{U}$ with a zero coordinate also has cost greater than one. Moreover,*

$$(11.4) \quad \gcd(c_i, r) \mid \gcd(e, r), \quad \gcd(c_i, c_j, r) = 1 \quad (i \neq j).$$

The low vector, when present, gives an actual exceptional prime; no nondegeneracy assumption on f is required. The local adjunction argument follows the method in [HL25a, proof of Theorem 6.4, pp. 430–431]; we prove the admissibility and uniqueness of the weights used here.

Proof. We first verify the geometry of a positive weight, then uniqueness and exact cost, and finally the boundary of the cube.

Step 1. In this step, we verify the gcd conditions and compare positive weights with actual exceptional primes. A subgroup fixing a coordinate axis must fix the equation character, since otherwise the whole axis lies in W . A subgroup fixing two coordinates fixes a plane whose intersection with W has positive dimension. Freeness off the origin excludes the latter, and proves (11.4). It also excludes ambient pseudoreflections.

The actual Cartier hypersurface W is normal and klt, and the ambient canonical divisor is Cartier. Thus Theorem 11.4 gives a plt pair $(\mathbb{C}^4, W)$. Finite crepancy gives a plt pair $(\mathbb{C}^4/\mu_r, X)$, with zero different on X , since the restricted cover is quasi-étale. If $w = tw_0$, with w_0 primitive in $\mathcal{L}$, its toric ray modification has exceptional divisor D_0 and satisfies

$$K_{M_{w_0}} + X_{w_0} + (1 - \Phi(w_0))D_0 = \pi^*(K_{\mathbb{C}^4/\mu_r} + X).$$

This follows by subtracting the equation weight from the toric canonical coefficient. Plt implies $\Phi(w) > 0$. If $\Phi(w) \leq 1$, the displayed boundary is effective and plt. Each component D of $D_0 \cap X_{w_0}$ is an exceptional prime over the origin. At its generic point the cyclic uniformizing cover of the plt surface slice gives

$$(11.5) \quad A_X(D) = \frac{1 - s(1 - \Phi(w_0))}{l} \leq \Phi(w_0) \leq 1, \quad s, l \in \mathbb{Z}_{>0}.$$

Here s is the multiplicity of the restricted exceptional divisor upstairs and l is the restriction ramification. This is the different calculation of Theorems 6.4 and 6.5; its numerator is positive by klt. The inequality follows from $(l - 1)\Phi(w_0) + (s - 1)(1 - \Phi(w_0)) \geq 0$. Whenever the initial component has both $x_i, x_j \neq 0$, restriction of the ambient divisors of the invariant functions gives

$$(11.6) \quad \frac{\text{ord}_D(x_i^r)}{\text{ord}_D(x_j^r)} = \frac{w_i}{w_j}.$$

The common restriction multiplicity cancels; reducedness of the initial equation is not needed for this identity.

Step 2. In this step, we prove uniqueness of the positive low-weight ray. If the initial polynomial at a low positive weight is one monomial involving two variables, its coordinate components give two distinct primes in (11.5); the diagonal group preserves each component. This is impossible. If it is a pure power x_i^p , then $p \geq 2$. Every other monomial x^u has excess degree $d_u = \langle u, w \rangle - pw_i \in \mathbb{Z}_{>0}$. At $\gamma = w + \mathbf{e}_i$, positivity forces $\min\{p, d_u + u_i\} = 1$; otherwise its cost is at most zero. Thus $h = \sum_{d_u=1, u_i=0} a_u x^u \neq 0$ is the initial polynomial there and $\Phi(\gamma) = \Phi(w)$. At $\eta = pw + \mathbf{e}_i$ the initial polynomial is $x_i^p + h$ and $0 < \Phi(\eta) = p\Phi(w) - (p-1) \leq 1$. A component of $h = 0$ has x_i and some x_j , $j \neq i$, nonzero generically. The equation $x_i^p + h$ has a component meeting the full coordinate torus. Their ratios in (11.6) are $(w_i + 1)/w_j$ and $(pw_i + 1)/(pw_j)$, which differ. This again gives two low primes. Hence the initial polynomial of every positive low weight is nonmonomial.

A nonmonomial Laurent polynomial is a nonunit and has a torus zero. Its weighted projectivization supplies a torus component. Uniqueness of the low prime and (11.6), for every pair of coordinates, imply that all positive low weights lie on one ray.

Step 3. In this step, we identify the exact cost of the primitive low-weight vector. For a low weight in cA form, $w_1 + w_2 - w(f)$ is a nonnegative integer. Since the other two weights are positive and the cost is at most one, that integer is zero. The term $x_1 x_2$ is initial. For a square form, if the distinguished square is not initial, put $d_1 = 2w_1 - w(g) \in \mathbb{Z}_{>0}$. The vector $2w - d_1 \mathbf{e}_1$ is positive, has cost $2\Phi(w) - d_1 \in (0, 1]$, and is not proportional to w . This contradicts the preceding uniqueness. Thus the square is initial as well.

At every torus point the derivative in x_1 of the initial equation is x_2 , respectively $2x_1$, and is nonzero. The primitive toric ray model is smooth along the open orbit of its exceptional divisor. Its strict transform is therefore smooth and transverse there, so $s = l = 1$ in (11.5). Its prime has discrepancy $\Phi(w_0) = \alpha$. A nonprimitive multiple has cost at least $2\alpha > 1$, proving the positive claims.

Step 4. In this step, we exclude low weights on the boundary of the specified cube. For $w \in \mathcal{U}$ with a zero coordinate, (11.4) gives exactly one zero and three strictly fractional coordinates. Freeness gives a nonzero axis restriction of f , so $w(f) = 0$. Also $\Phi(w) \equiv j/r \pmod{\mathbb{Z}}$ for its nonzero residue class, so $\Phi(w)$ cannot equal one. In cA form the first two characters are units: their axes lie on W . Their weights sum to one when the equation residue is zero, and the remaining positive tail makes the cost greater than one. In an odd form, or if a tail weight vanishes in a rank-one form, a nonzero square character forces the represented subgroup to have order two. All three nonfixed coordinates then have weight $1/2$, so the cost is $3/2$.

It remains to exclude $w_1 = 0$ in rank one with $t = \Phi(w) < 1$. The initial term is x_1^2 , and $d_1 = w(g)$ is a positive integer. At $w + \mathbf{e}_1$, positivity gives $t + 1 - \min(2, d_1) > 0$, so $d_1 = 1$. At $2w + \mathbf{e}_1$ the cost is $2t - 1 \in (0, 1)$. The initial equations are respectively $\text{in}_w g$ and $x_1^2 + \text{in}_w g$. They give different coordinate-order ratios, exactly as above, and hence two low primes. This is impossible.

All primes used here are primes over the original algebraic germ. Indeed a positive weight has a finite initial polynomial; only finitely many jets of the equivariant parameters and equation determine the generic exceptional components and the displayed orders. Algebraic parameters with the same jets, projected to the same characters, give these components on the normalized weighted blowup of an algebraic ideal. Thus the analytic coordinates do not change the uniqueness hypothesis being used. $\square$

Theorem 11.14. *There is an absolute integer Q_{enc} with the following property. Let $Z \ni z$ be an algebraic $\mathbb{Q}$ -Gorenstein klt threefold germ. Suppose exactly one exceptional prime divisor E over a neighbourhood of z has log discrepancy at most one, its centre is z , and*

$$1/2 < \alpha = A_Z(E) < 1.$$

Then the reduced denominator of α is at most Q_{enc} . No $\mathbb{Q}$ -factoriality is assumed.

Lemma 11.10 gives the terminal canonical cover needed for the local weight argument. The compactness estimate used in that argument is Lemma 11.11, on its strictly positive domain. We use the residue organization of [HL25b, Sections 2–4]. The argument below proves the needed arithmetic on $\mathcal{U}$, including the empty-cube case.

Proof. We pass to the terminal canonical cover, establish the allowed weight domain, and reduce the denominator estimate to compactness and a finite list of residue calculations. Put

$$Q_{\text{enc}} = \max\{30, 1 + I(3, 1/14), 1 + I(5, 1/14)\},$$

where I is supplied by Lemma 11.11.

Step 1. In this step, we obtain the terminal canonical cover, apply the global gap, and treat the smooth-cover case. Lemma 11.10 gives a terminal canonical cover $W \rightarrow Z$ of degree $r = \text{ind}_z(K_Z)$. The germ Z is terminal away from z . Theorem 7.6 makes the terminal threefold singularities isolated, so after shrinking it is smooth there, and the canonical cover is free off its origin. Theorem 7.1 says that W is smooth or an isolated cDV hypersurface. On this neighbourhood the global bare minimum is α : every other exceptional prime has discrepancy greater than one, and primes on Z have discrepancy one. Hence Theorem 11.2 gives

$$(11.7) \quad \frac{1}{2} < \alpha \leq \frac{12}{13}.$$

If W is smooth, diagonalize its cyclic action. All three characters c_i are units modulo r , since the action is free off the origin. The smooth-quotient formula, namely Theorem 8.5 with the unit ideal, gives a vector

$$v = (\{j_0 c_1/r\}, \{j_0 c_2/r\}, \{j_0 c_3/r\}) \in (0, 1)^3, \quad \sum_i v_i = \alpha.$$

The same formula on the algebraic germ follows from its toric resolution; the finitely many fractional rays and their discrepancies are unchanged by completion. Its primitive ray computes the unique low prime. Any other fractional vector has coordinate sum greater than one: a different primitive low ray would give a second prime, and a nonprimitive multiple has cost at least $2\alpha > 1$. Set $q = r/\text{gcd}(r, j_0)$. For $2 \leq n \leq q-1$, all nv_i are nonintegral and their residue class differs from that of v . Thus

$$\sum_{i=1}^3 (1 + (n-1)v_i - \lceil nv_i \rceil) = \sum_i \{nj_0 c_i/r\} - \alpha > 1 - \alpha \geq 1/13.$$

Lemma 11.11 gives $q \leq 1 + I(3, 1/14)$; the cases $q \leq 2$ satisfy this bound as well. The denominator of α divides q .

Step 2. In this step, we choose equivariant hypersurface coordinates and retain both possibilities for low weights in the cube. Suppose W is singular. Its quadratic part is nonzero by the cDV condition. Take eigenparameters and split that quadratic form equivariantly. A nonzero cross pairing gives $x_1 x_2$. Two square directions of the same character also give such a pair after a linear change. If neither occurs, there are at most two square directions, since $2c = e$ has at most two solutions modulo r . Two distinct square characters give the odd form, and one gives the rank-one form of Setup 11.12.

The splitting is convergent. For a square direction, Theorem 11.5 writes the equation, up to a unit, as $x^2 + 2h(y)x + k(y)$. Uniqueness makes h an eigenfunction of the character of x , and k of twice that character; the unit is invariant. Replacing x by $x + h(y)$ splits the square. For a nondegenerate binary block, solve its two partial derivatives for its critical point by the analytic implicit function theorem, translate to that point, and complete the nondegenerate quadratic form. Each operation is equivariant by uniqueness, or by projection to its finite character. The radical remainder has order at least three when the entire quadratic part is removed. These changes preserve the actual quotient germ and its canonical character; we introduce weights only

after making them. Thus Lemma 11.13 applies in the displayed coordinates without a Newton nondegeneracy assumption.

If $r \leq 30$, the denominator already divides r , by Theorem 3.9 with multiplier one. Assume henceforth $r > 30$. If $\mathcal{U}$ contains a low weight, call it β . By Lemma 11.13, it is positive, primitive and has cost α . Put $\Psi = \{\beta, 1 - \beta\}$. If there is no low weight in $\mathcal{U}$, put $\Psi = \emptyset$. This second case is retained throughout the argument; no existence of a low vector in the cube is assumed.

Step 3. In this step, we compare complementary vectors and derive the residue identity outside the exceptional set. For cA write $A = w_1 + w_2$, $T = w_3 + w_4$. The first two characters are units, so $0 < A < 2$ on $\mathcal{U}$. Semi-invariance and the term x_1x_2 give $w(f) \in \{A, A - 1\}$, omitting a negative value. For $w, 1 - w \notin \Psi$, their costs exceed one. If $w(f) = A$, then $T > 1$; the complement has tail sum $2 - T < 1$ and must have degree $(2 - A) - 1$. Thus $A \leq 1$. Conversely, the degree $A - 1$ gives $T < 1$. At $A = 1$, a zero-weight monomial requires a zero tail coordinate; the other tail is strictly fractional, giving the same strict inequality.

For a square form put $A = w_1$, $T = w_2 + w_3 + w_4$. Its equation degree is $2A$ or $2A - 1$. A value $2A - 2$ would require both $A = 1$ and a zero tail coordinate, contrary to (11.4). Outside Ψ , complementarity gives

$$w(f) = 2A \implies A \leq 1/2, T > 1 + A, \quad w(f) = 2A - 1 \implies T < 1 + A.$$

In the borderline case $A = 1/2$ and degree zero, the represented element fixes one tail axis and has order two by (11.4); the vector has three coordinates $1/2$ and one zero. This verifies the strict inequality there as well. For the fractional vectors v_j , these alternatives and $\sum c_i - e \equiv 1$ give

$$(11.8) \quad \sum_i \{jc_i/r\} = \{je/r\} + j/r + 1 \quad \text{if } v_j \notin \Psi.$$

Indeed in cA the two cases have respectively $A = \{je/r\}$, $T = 1 + j/r$ and $A = 1 + \{je/r\}$, $T = j/r$. In a square form they have respectively $2A = \{je/r\}$, $T - A = 1 + j/r$ and $2A = 1 + \{je/r\}$, $T - A = j/r$. In particular a nonexceptional cA test with tail sum less than one forces equation degree $A - 1$; a nonexceptional square test with first coordinate greater than $1/2$ forces degree $2A - 1$.

Step 4. In this step, we determine the gcd alternatives and isolate bounded-denominator cases. In cA set $d = \gcd(e, r)$, $h = r/d$. If neither tail character has gcd d , all coordinates of v_h and v_{r-h} are nonzero. They are complementary, whereas (11.8) would make their coordinate sums add to three, rather than four. Thus one of them is β , unless some tail has gcd d . If $v_h = \beta$, its cost modulo integers is $1/d$, so $\alpha = 1/d$, which is impossible. If $v_{r-h} = \beta$, then $\alpha = 1 - 1/d$, and (11.7) gives $d \leq 13$. This is already a bounded-denominator case. Otherwise, after switching tails,

$$(11.9) \quad \gcd(c_1, r) = \gcd(c_2, r) = \gcd(c_3, r) = 1, \quad \gcd(c_4, r) = \gcd(e, r).$$

The remaining tail is a unit by the pairwise gcd condition.

For a square form put $s = \gcd(c_1, r)$ and $d = \gcd(e, r)$; then $d = s$ or $2s$. If $s \geq 2$ and $d = 2s$, the same complementary test at $h = r/(2s)$ has first coordinate $1/2$ and all tail coordinates nonzero. It is contradictory unless $\alpha = 1/(2s)$, which is impossible, or $\alpha = 1 - 1/(2s)$, whose denominator is at most thirteen. Otherwise $s \geq 2$ implies $d = s$ and all tails are units. If $s = 1$, then either r is odd and every character is a unit, or r is even and exactly one tail has gcd two, all others being units. In the latter case, if all tails were odd, the vector $(1/2, 1/2, 1/2, 1/2)$ would have cost one, contradicting Lemma 11.13. Thus, after a permutation, three characters are units and the fourth has the same gcd with r as e .

Step 5. In this step, we apply five-coordinate compactness and verify the residue identity for every remaining residue. Suppose $\beta = v_{j_0}$ is strictly fractional and

$$\sum_i \beta_i = \{j_0 e/r\} + \alpha, \quad 0 < \{j_0 e/r\} < 1.$$

The canonical congruence gives $\alpha = j_0/r$. Every other noncomplementary residue has residue cost $1 + j/r$ by (11.8). The complement has residue cost $3 - \alpha$. Both exceed $\alpha + 1/14$, by (11.7). Apply Lemma 11.11 to the five strictly positive numbers

$$(\beta_1, \beta_2, \beta_3, \beta_4, 1 - \{j_0 e/r\}), \quad \text{whose sum is } 1 + \alpha.$$

For $2 \leq n < q = r/\gcd(r, j_0)$, the first four positive representatives are at least the corresponding fractional parts, and the fifth is exactly $1 - \{n j_0 e/r\}$. Subtracting the original sum shows that the compactness expression is at least $1/14$. Hence $q \leq 1 + I(5, 1/14)$, and the denominator of α divides q .

Outside this case, a cA low vector, if present, has

$$(11.10) \quad \beta = v_{j_0}, \quad \beta_1 + \beta_2 = 1 + \{j_0 e/r\} > 1, \quad \beta_3 + \beta_4 = \alpha = j_0/r.$$

Indeed the cross term is initial by Lemma 11.13, and its two tail coordinates cannot be one. The equation residue cannot vanish, since (11.9) would then make one tail zero. For a square form the low vector has either

$$(11.11) \quad \beta = v_{j_0}, \quad 2\beta_1 = 1 + \{j_0 e/r\} > 1, \quad \sum_{i>1} \beta_i = \beta_1 + \alpha, \quad \alpha = j_0/r,$$

or

$$(11.12) \quad \beta = (1, b_2, b_3, b_4), \quad \beta(f) = 2, \quad b_2 + b_3 + b_4 = 1 + \alpha.$$

To check these alternatives, the distinguished square is initial by the same lemma. A fractional first coordinate less than $1/2$ is the compactness case just treated. An equation residue zero is excluded by the gcd patterns: it makes the first coordinate zero in the nonunit-first case, or a tail zero in the even-unit case. A coordinate one in a tail would instead give the order-two vector with the other three coordinates $1/2$, of cost greater than one. Thus a nonfractional low vector has only its first coordinate equal to one, and its initial square has degree two.

In (11.10) and (11.11), the residue identity holds at β by the displayed formulas, and at its complement by subtraction, since the equation residue is nonzero. In (11.12), both exceptional residue vectors have first coordinate zero; their costs are $1 + \alpha$ and $2 - \alpha$, so the identity holds there too. Consequently, after the bounded denominator cases already settled, (11.8) holds for every $1 \leq j < r$. If $\Psi = \emptyset$, it already did so. Every hypothesis of Theorem 11.3 has now been verified.

Step 6. In this step, we enumerate the character types supplied by the terminal residue lemma. For cA, the relations $c_1 + c_2 = e$, $c_3 + c_4 = 1$, together with that theorem, give, up to switching the two pairs, the following types modulo r :

	$(c_1, c_2, c_3, c_4; e)$	conditions
<i>T</i>	$(c, -c, 1, 0; 0)$	$\gcd(c, r) = 1$
<i>I</i>	$(c, 1, -c, c+1; c+1)$	$\gcd(c, r) = \gcd(c+1, r) = 1$
<i>II</i>	$(c, -c-1, -c, c+1; -1)$	$\gcd(c, r) = \gcd(c+1, r) = 1$
<i>III</i>	$(1, c, -c, c+1; c+1)$	$\gcd(c, r) = 1, \quad \gcd(c+1, r) > 1.$

In type III, $c+1 \not\equiv 0$. To obtain the list when $\gcd(e, r) > 1$, the nonunit tail equals e , and the three units consist of one and an opposite pair. This gives T or III. When $\gcd(e, r) = 1$, pairing $-e$ with -1 gives II. Otherwise $-e$ pairs with a tail and -1 with one of the first two characters, giving I.

For a square form, if the first character is not a unit, the terminal residue lemma gives $c_1 = e = 0$ and tails $(c, -c, 1)$. Otherwise, in even index the nonunit tail is $e = 2c_1$; choosing

which unit is one gives types A and B below. In odd index, pairing $-e$ with -1 gives C; pairing it with a tail and then pairing -1 with a tail or with c_1 gives D or E. Thus the rank-one list is

	$(c_1, c_2, c_3, c_4; e)$	parity of r
A	$(c, -c, 1, 2c; 2c)$	even
B	$(1, c, -c, 2; 2)$	even
C	$((r-1)/2, (r+1)/2, c, -c; -1)$	odd
D	$(c, -c, 2c, 1; 2c)$	odd
E	$(1, c, -c, 2; 2)$	odd
F	$(0, c, -c, 1; 0)$	arbitrary,

where $\gcd(c, r) = 1$. In odd form the two squared characters are distinct. A nonunit squared character would be zero, and the other would be $r/2$, violating the pairwise gcd condition for $r > 2$. They are therefore units. The terminal residue lemma gives

$$(c_1, c_2, c_3, c_4; e) = (1, 1 + r/2, r/2 - 1, 2; 2), \quad 4 \mid r.$$

If the remaining unit were one instead, the two squared characters would be opposite, forcing $r \mid 4$.

Step 7. In this step, we exclude the remaining cA character types in large index. In types I–III choose $1 \leq c \leq r-2$. Type T is terminal, which is impossible here. In fact write its equation as $xy + g(u, z)$ with characters $(c, -c, 1, 0)$. Freeness forces $g(0, z) \neq 0$; a change of the fixed coordinate makes $(u = 0)$ the Du Val surface $xy + z^t = 0$, $t \geq 2$. It is a quotient of a smooth surface by inverse cyclic weights, hence klt. Theorem 11.4 makes this Cartier pair plt upstairs, and finite crepancy makes its image pair (X, D) plt. The logarithmic residue generator has character $1 - 1 = 0$, so $K_X + D$ is Cartier. Every exceptional prime centred at the origin consequently has positive integral pair discrepancy and positive order on D , hence bare discrepancy greater than one.

In type I choose $b(c+1) \equiv 1 \pmod{r}$, $1 \leq b < r$. Then

$$v_b = (r - b + 1, b, b - 1, 1)/r.$$

Its first-pair sum exceeds one, so it is not $1 - \beta$ by (11.10). Unless it is β , the tail test forces degree $1/r$ in $g \in (x_3, x_4)^2$, which is impossible. Thus $\beta = v_b$ and $b/r = \alpha > 1/2$. Set $j = 2b - r > 0$. If $j \geq 4$, the vector v_j has tail numerators $(j-2, 2)$ and first-pair sum $1 + 2/r$. It is outside Ψ and forces degree $2/r$, although every tail monomial has degree at least $4/r$. If $j = 2$, the congruence for b gives $2c \equiv 0$, contradicting c being a unit. If $j = 1$, then $r = 2b - 1$ and $c = 1$; at v_{b+1} the tail numerators are $(b-2, 3)$ and the forced degree is $3/r$. If $j = 3$, then $r = 2b - 3$ and $3c \equiv -1$; at v_{b+3} those numerators are $(b, 3)$, again with forced degree $3/r$. For $r > 30$ both tests are outside Ψ , have tail sum less than one, and cannot have a degree- $3/r$ monomial of ordinary degree at least two.

In type II, v_{r-1} has tail numerators $(c, r - c - 1)$ and first-pair sum $1 + 1/r$. It cannot be the complement of β . If it is β , then $\alpha = (r-1)/r$, contrary to (11.7) for $r > 13$. Otherwise the forced degree is $1/r$, impossible. In type III, the same test either gives $r \leq 13$, or forces a pure power x_3^l , $l \geq 2$, with

$$lc = r - c - 1.$$

Thus $r > 2c + 2$ for $r > 4$. The test v_{r-2} either is β , giving $r \leq 26$ by the gap, or forces a pure power x_3^s with $2sc = r - 2c - 2$. Both tests have first-pair sum greater than one, so neither is the complementary exception. Minimality of the two powers at the two tests gives $l = s$, and the two equations then imply $r = 0$. This excludes all cA types for $r > 30$.

Step 8. In this step, we exclude the remaining odd character type in large index. For the displayed odd type, $v_{r-2} = (r-2, r-2, 2, r-4)/r$. It is not the complement in (11.11). If it is β , the gap gives $r \leq 26$. Otherwise it forces degree $(r-4)/r$. Both squares have larger degree,

and a monomial of g involving x_4 has larger degree. Hence x_3^l occurs with $2l = r - 4$. Since $4 \mid r$, its character is

$$(r/2 - 2)(r/2 - 1) \equiv 2 + r/2 \not\equiv 2 \pmod{r},$$

a contradiction.

Step 9. In this step, we exclude the remaining rank-one types and conclude the denominator bound. All tests below have first coordinate greater than $1/2$. They cannot be $1 - \beta$, by (11.11) and (11.12). Their equation degree, unless they equal β , is twice the first coordinate minus one.

In type A choose $j_t c \equiv r/2 + t$ for $t = 1, 2, 3$. The test v_{j_1} must be β , since otherwise it forces degree $2/r$ in a cubic-order remainder. The next test has tail numerators $(r/2 - 2, j_2, 4)$ and forces degree $4/r$. Since $r > 30$, only a pure power of x_3 can attain it. Its exponent is at least three, so $j_2 = 1$ and x_3^4 occurs. Thus $c \equiv r/2 + 2$ and $2c \equiv 4$. At the third test the tail numerators are $(r/2 - 3, j_3, 6)$. An attaining monomial of degree $6/r$ is again a pure power of x_3 . Its degree at least three gives $j_3 \leq 2$, while the already present fourth power gives $4j_3 \geq 6$. Hence $j_3 = 2$. But doubling $j_3 c \equiv r/2 + 3$ gives $4j_3 \equiv 6 \pmod{r}$, contradicting $r > 30$.

Types B and E have the common list $(1, c, -c, 2; 2)$. Put $d = 2$ for B and $d = 1$ for E, and let $j_0 = (r + d)/2$, $j_1 = j_0 + 1$, $j_2 = j_0 + 2$. The first test must be β , since it would otherwise force degree d/r in a cubic-order remainder. Put $s = d + 2$, so $s = 4$ or 3 . At j_1 the forced numerator is s . The last tail has numerator s itself and cannot occur in an attaining monomial. The other two tail numerators are opposite and one exceeds $r/2 > s$. Thus an attaining monomial is a pure power of the smaller tail, of exponent at least three. It follows that $j_1 c \equiv \epsilon \pmod{r}$, with $\epsilon \in \{1, -1\}$, and its exponent is s . Since $2j_1 \equiv s$, we have $sc \equiv 2\epsilon \pmod{r}$. At j_2 the forced numerator is $s + 2$. The same argument gives $j_2 c \equiv \delta$, with $0 < |\delta| \leq 2$ for B and $|\delta| = 1$ for E. Subtracting yields $c \equiv \delta - \epsilon$, so

$$s(\delta - \epsilon) \equiv 2\epsilon \pmod{r}.$$

The difference has absolute value at most fourteen, less than r ; equality as integers is impossible for $s = 4$ or 3 . This treats both signs and every residue alternative in these two types.

In type C the test v_{r-1} either is β , giving $r \leq 13$, or forces degree $1/r$ in a cubic-order remainder. In type D choose $j_t c \equiv (r + 2t - 1)/2$ for $t = 1, 2, 3$. The first test must be β . The next two have tail numerators

$$((r - 3)/2, 3, j_2), \quad ((r - 5)/2, 5, j_3),$$

and force degrees $3/r$ and $5/r$. Only a pure power of x_4 can attain either degree. The first gives $j_2 = 1$ and the least exponent three. Minimality at the second gives the same least exponent, hence $3j_3 = 5$, which is impossible. This includes the type with the first character c and tails $(-c, 2c, 1)$.

Finally, in type F use the vectors

$$w_j = (1, \{jc/r\}, \{-jc/r\}, j/r) \in \mathcal{U}.$$

Their equation degree is one or two. Unless $w_j = \beta$, their cost greater than one forces $w_j(g) = 1$, since their tail sum is $1 + j/r$. Choose b_1, b_2 with $cb_i \equiv \lceil r/2 \rceil - i$, $i = 1, 2$. For $r > 30$ the two opposite pairs $\{b_i, r - b_i\}$ are distinct and all their first two tail weights exceed $1/3$. At most one pair contains the residue class of β ; choose the other. The degree-one monomial of g at each of its two tests, having ordinary degree at least three, forces respectively $j/r \leq 1/3$ and $1 - j/r \leq 1/3$, a contradiction.

The tests use only $\mathcal{U}$. If this cube had no low vector, each assertion that a test must equal β was already a contradiction. Thus no additional assumption about weights outside the cube was used. All types have been excluded for $r > 30$ except for the bounded-denominator cases above. These cases give a denominator at most thirteen or at most $1 + I(5, 1/14)$. Together with the smooth-cover bound, they prove the stated Q_{enc} . $\square$

Lemma 11.15. *Let an original point-exit datum satisfy $1/2 < \alpha = A_Z(E) < 1$, and write $\alpha = u_0/v_0$ in lowest terms. Put $z = h(E)$, $B_Z = h_*R$, and let $\pi : Z \rightarrow Q = Z/H$ be the quotient near z , with boundary B_Q defined by $K_Z + B_Z = \pi^*(K_Q + B_Q)$. Let $\rho : (W \ni o) \rightarrow (Z \ni z)$ be the canonical cover supplied by Theorem 3.9 with multiplier one, and put $B_W = \rho^*B_Z$. Then E is the unique exceptional prime over a neighbourhood of z with bare discrepancy at most one, and*

$$\text{mld}(Q \ni \pi(z), B_Q) = a.$$

Every primitive extension F of E to $\mathbb{C}(W)$ satisfies

$$A_W(F) = u_0, \quad A_{W, B_W}(F) = v_0 a, \quad |I_\Delta(F)| = v_0,$$

where Δ is the canonical deck group.

Here α is a bare discrepancy; the minimum of the original quotient pair remains a .

Proof. By Theorem 7.16, the pair on Z satisfies

$$K_Y + (1 - a)E + R = h^*(K_Z + B_Z),$$

and every exceptional prime other than E has pair discrepancy greater than one. Proposition 10.12 makes K_Z $\mathbb{Q}$ -Cartier, so B_Z is separately $\mathbb{Q}$ -Cartier. Since $B_Z \geq 0$, every other exceptional prime P satisfies

$$A_Z(P) = A_{Z, B_Z}(P) + P(B_Z) > 1.$$

Together with $A_Z(E) = \alpha < 1$, this proves bare uniqueness and justifies applying Lemma 11.10.

For a primitive prime P centred at z , properness gives a centre in $h^{-1}(z)$. This fibre is supported on E . Indeed $-E$ is h -ample. A contracted curve not contained in E has nonnegative intersection with E , contradicting that ampleness. Every positive-dimensional fibre component is therefore contained in E . The fibres are connected, so an additional isolated point outside E is also impossible. Thus the comparison in Theorem 7.16, with discrepancies and inertia computed in the same function field, gives

$$\frac{A_{Z, B_Z}(P)}{|I_H(P)|} \geq a.$$

The original prime E attains equality because $A_{Z, B_Z}(E) = a$ and $I_H(E) = 1$. The quotient formula in Lemma 3.11 now gives the asserted exact minimum. Its boundary is the actual finite crepant boundary, including the ramification terms of Lemma 3.8.

The canonical cover is quasi-étale. The last two identities of Lemma 11.10 give $A_W(F) = u_0$ and $|I_\Delta(F)| = v_0$. Finite crepancy for $(W, B_W) \rightarrow (Z, B_Z)$, using Lemma 3.8, gives $A_{W, B_W}(F) = v_0 A_{Z, B_Z}(E) = v_0 a$. $\square$

Proposition 11.16. *Let an original point-exit datum satisfy $1/2 < \alpha = A_Z(E) < 1$, and write $\alpha = u_0/v_0$, where $u_0 < v_0$ are coprime positive integers. With $C_m = \max\{2m + 5, 31, m + 16\}$, one has*

$$|H| \leq [4mv_0(C_m u_0 + 5v_0) + 1]^6 \leq [4mv_0^2(C_m + 5) + 1]^6.$$

No bound on L^2 or on $\text{ind}_{h(E)} K_Z$ is assumed.

The lift in Lemma 11.15 has prescribed bare discrepancy u_0 . The character estimate must retain the canonical deck group until the final cancellation.

Proof. We retain the prescribed lift through the full canonical extension, apply the ambient computing-divisor bound, and cancel the canonical deck degree only at the end.

Step 1. In this step, we retain the full canonical-cover group and the original quotient minimum. Use $B_Z, Q, B_Q, W, B_W, \Delta$ from Lemma 11.15, and put $r = \text{ind}_z(K_Z)$. Theorem 7.16 supplies on Z the original natural m -log generator with faithful H -character. Lemma 11.6 therefore gives

$$1 \longrightarrow \Delta = \mu_r \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1, \quad |\ker \tilde{\chi}| = r.$$

Its quotient pair is the same (Q, B_Q) , of exact minimum a . The canonical cover W is terminal by Lemma 11.10. For a primitive lift F of E , the target-quotient lemma gives

$$A_W(F) = u_0, \quad A_{W, B_W}(F) = v_0 a, \quad |I_\Delta(F)| = v_0.$$

An element of $I_{\tilde{H}}(F)$ projects to an element acting trivially on $\kappa(E) \subset \kappa(F)$, hence to $I_H(E) = 1$. Consequently

$$I_{\tilde{H}}(F) = I_\Delta(F), \quad |I_{\tilde{H}}(F)| = v_0.$$

The full group need not preserve F .

Step 2. In this step, we construct the bounded ambient prime with its original full inertia. The integral $\mathbb{Q}$ -Cartier divisor mB_W is Cartier by Lemma 8.3, since W is Gorenstein terminal. Choose a regular semi-invariant equation h_B by a character projection. If W is smooth, take $M_0 = W$, $T = F$, and $P_B = h_B$. Its ambient quotient is already the prescribed quotient pair.

If W is singular, Theorem 7.1 gives an isolated cDV hypersurface. It has an actual algebraic equivariant presentation in a smooth fourfold M_0 . Indeed embed the affine germ in a representation by a finite invariant set of algebra generators, and choose eigen-equations with independent linear parts to cut the smooth minimal ambient germ. The remaining principal equation can be chosen semi-invariant by projecting its generator. Write it as f , lift h_B semi-invariantly from W , and put $P_B = f^m h_B$. The action on M_0 is faithful, since its restriction to W is faithful.

Apply Proposition 9.7 with the full group $\tilde{H}$, intrinsic prime F , bare bound u_0 , and inertia v_0 . All its hypotheses refer to the actual quotient pair established above. It gives an algebraic ambient prime T with

$$|I_{\tilde{H}}(T)| = v_0, \quad \frac{A_{M_0}(T) - T(P_B)/m}{v_0} = a, \quad A_{M_0}(T) \leq C_m u_0 + 3v_0.$$

Proposition 8.13 gives the lower bound a for every ambient prime, so the exact ambient quotient minimum is a . The smooth case satisfies the same displayed bounds with $T = F$. The natural generator $\Omega_{M_0}^m/P_B$ has character $\tilde{\chi}$, by residue in the hypersurface case and by the identity in the smooth case; its kernel still has order r .

Step 3. In this step, we apply the character estimate and cancel the canonical deck degree. An abelian extension of two cyclic groups has a faithful representation by at most two characters. Apply Proposition 8.12 with $k_0 = r$, $e = v_0$, $L_0 = C_m u_0 + 3v_0$, $j = 2$, and ambient dimension at most four. Since these quantities are positive integers where required, it yields

$$|\tilde{H}| \leq r[4mv_0(C_m u_0 + 5v_0) + 1]^6.$$

Now $|\tilde{H}| = r|H|$. Cancel r and use $u_0 < v_0$ to obtain both asserted inequalities. $\square$

Theorem 11.17. *Fix $m \geq 1$ and a rational $a \in (0, 1]$ with ma integral. There is an integer $C_{\text{pt}}(m, a) > 0$ such that, for every original cover datum and every point contraction supplied by Theorem 7.16, the full original stabilizer satisfies*

$$|H| \leq C_{\text{pt}}(m, a).$$

The assertion holds over every algebraically closed field of characteristic zero.

The volume alternative and the fractional discrepancy estimate exhaust the point contractions.

Proof. We first work over $\mathbb{C}$. Let $V_\star$ be the constant of Proposition 10.21, put $C_m = \max\{2m + 5, 31, m + 16\}$, and set

$$C_{\text{pt}}(m, a) = \max \left\{ C_{\text{vol}}(m, a, V_\star), [4mQ_{\text{enc}}^2(C_m + 5) + 1]^6 \right\}.$$

If $L^2 \leq V_\star$, Theorem 11.9 gives the first bound. Otherwise Proposition 10.21 gives $1/2 < a \leq \alpha < 1$. By Lemma 11.15, the bare target satisfies every hypothesis of Theorem 11.14. The reduced denominator v_0 of α is therefore at most Q_{enc} . Proposition 11.16 gives the second bound.

For an arbitrary algebraically closed field k of characteristic zero, apply Lemma 3.11 to the entire marked point-exit diagram. Its complex realization retains the original finite group, natural character, computing prime, terminal pair, polarization, and full markings. The same lemma preserves the comparison over all primes by identifying it with the minimum of the actual quotient pair. Hence the complex estimate applies to this realization. The group H and its order are unchanged, which proves the bound over k . $\square$

12. VALUATIONS AT CURVE POLES

We retain the original cover, its natural character, and the marked curve exit. We construct valuations near a point of the normalization and at the generic point of a fibre component. Every discrepancy below includes the full inverse image of the marked divisor and the full remaining boundary.

For a complete non-Archimedean valued field K , the Tate algebra $K\langle X \rangle$ consists of the series $\sum_{j \geq 0} a_j X^j$ with $|a_j| \rightarrow 0$, and its Gauss norm is $\max_j |a_j|$. A series of norm at most one reduces coefficientwise to a polynomial over the residue field.

Theorem 12.1 (One-variable preparation over a complete valued field). *Let K be a complete nontrivially valued non-Archimedean field and $g \in K\langle X \rangle$ have Gauss norm one, with reduction of degree s . Then g is a unit times a monic polynomial of degree s and norm one. The valuation has a unique extension to each algebraic extension of K , and finite extensions are complete [Bos14, Section 2.2, Definition 6, Theorem 8 and Corollary 9; Appendix A, Theorems 1 and 3].*

Preparation counts the roots in the chosen closed disc. We will descend their full multiset before bounding the degree of a root.

Set-up 12.2. Let original cover data of minimum $0 < a \leq 1$ and a curve exit $h : Y \rightarrow Z$ from Theorem 7.16 be given. Write $B_Y = (1 - a)E + R$, $\nu : S \rightarrow E$, and $c_0 = c_{\text{ct}}(3, a)$. Let $J \leq H$ fix a point $P \in S$. At $P_0 = \nu(P)$, take the actual canonical cover W , the abelian extension $\tilde{J}$, and ambient germ M_0 from Proposition 8.8. Their local canonical degree satisfies $r \leq m$. The full inverse-image divisors $E^\#$ and $mR^\#$ are Cartier; let Z_E, U be their full regular equations, lifted equivariantly to M_0 . In the singular case let f be the equation of W . The natural tensor is

$$\frac{\Omega^m}{f^m Z_E^{m(1-a)} U},$$

with f omitted in the smooth case. Its character is the inflation of the original $\chi|_J$. Choose a point $P^\#$ above P on the normalization of $E^\#$, and an element η of its stabilizer mapping to $\gamma \in J$. All ambient eigenparameters are chosen for the full abelian extension. For a positive number ϵ , suppose their phases at η have absolute value less than ϵ , and every character of γ on the coarse cotangent space $\mathfrak{m}_{S,P}/\mathfrak{m}_{S,P}^2$ has a representative in $(0, 2\epsilon)$.

The full Cartier equations in this setup follow from Lemma 8.3. Indeed, E and mR are invariant integral Weil divisors on the H - $\mathbb{Q}$ -factorial model of Theorem 7.16, so their pullbacks are integral $\mathbb{Q}$ -Cartier divisors on W . Apply the lemma there, where the canonical index is one. The full divisor $E^\#$ is reduced: the cover is etale in codimension one, and its Cartier divisor on the Cohen–Macaulay germ W has no embedded components. Character projection chooses equivariant equations and their ambient lifts.

Put

$$\mathcal{D}_0 = (1 - a) \operatorname{div}(Z_E) + \frac{1}{m} \operatorname{div}(U)$$

on M_0 , adding $\operatorname{div}(f)$ when W is singular. This notation always refers to the full equations in the setup. Theorem 7.16 and Lemma 3.8 give a minimum at least a for the quotient by J at P_0 : for every prime centred there, its J -inertia is contained in its H -inertia. Proposition 8.8

therefore gives

$$(12.1) \quad \frac{A_{M_0, \mathcal{D}_0}(F)}{|I_{\tilde{J}}(F)|} \geq a$$

for every primitive algebraic ambient prime centred at the closed point.

We use the same normalization of a valuation in both closed-point constructions.

Lemma 12.3. *In Setup 12.2, let $w = \lambda \operatorname{ord}_F$, with $\lambda > 0$ rational and F a primitive algebraic prime over M_0 centred at the closed point. Let $\sigma \in \tilde{J}$ preserve w , and suppose that its action on each homogeneous element of value d in the associated graded field is multiplication by $\exp(2\pi i d)$. Write $\bar{\sigma}$ for its image in J . Then*

$$\sigma \in I_{\tilde{J}}(F), \quad \lambda \geq \frac{1}{|I_{\tilde{J}}(F)|}, \quad \lambda A_{M_0, \mathcal{D}_0}(F) \geq a.$$

The conormal phase of σ is $\lambda \bmod \mathbb{Z}$, and

$$m\lambda A_{M_0, \mathcal{D}_0}(F) \equiv \operatorname{phase} \chi(\bar{\sigma}) \pmod{\mathbb{Z}}.$$

The hypothesis identifies the scale of the valuation with the conormal character; it is stronger than merely preserving the valuation.

Proof. The degree-zero action is trivial, so σ belongs to inertia. A uniformizer for F has value λ , giving the stated conormal phase. By Lemma 3.8, this character has order $\operatorname{ord}(\sigma)$. If $\sigma \neq 1$, the positive number λ is at least $1/\operatorname{ord}(\sigma)$; if $\sigma = 1$, it is a positive integer. The lower bound follows from (12.1). The same lemma applied to the natural tensor gives the congruence, since $m A_{M_0, \mathcal{D}_0}(F)$ is an integer. $\square$

The next lemma isolates the use of the positive cotangent characters on the coarse normalization. Let A denote the completed normal local ring at $P^\#$, and let $R_{\text{br}} \subset A$ be its reduced formal branch ring. Write $\mathfrak{n}$ for the maximal ideal of A , and $e(R_{\text{br}})$ for the Hilbert–Samuel multiplicity of the branch.

Lemma 12.4. *In Setup 12.2, let $x_1, \dots, x_d$, $d \in \{3, 4\}$, be the ambient eigenparameters, with signed phases μ_i at η . Suppose $e(R_{\text{br}}) \leq B$ for a positive integer B . Put $L = B(m+1)$, $T = B(dm+1)$, and $I_+ = (x_i : \mu_i > 0)A$. If $(2L+m)\epsilon < 1$, then*

$$e_{\mathfrak{n}}(A) \leq B, \quad \dim_{\mathbb{C}} \mathfrak{n}/\mathfrak{n}^2 \leq B+1, \quad \mathfrak{n}^T \subset I_+,$$

and

$$\operatorname{length}(A/I_+) \leq \binom{B+T}{B+1} \leq (T+1)^{B+1}.$$

At least two of the μ_i are positive. If exactly two are positive, their coordinates u_1, u_2 induce a finite free inclusion $\mathbb{C}[[u_1, u_2]] \subset A$ of rank $\operatorname{length}(A/(u_1, u_2))$.

Thus positive parameters can be found without requiring positive cotangent characters on the covering normalization.

Proof. We first bound the normalization and then use deck-invariant powers to eliminate the nonpositive coordinates.

Step 1. In this step, we bound the multiplicity, colength, and embedding dimension of the normalization. The finite birational extension $R_{\text{br}} \subset A$ has cokernel of dimension at most one. If $I = (x_1, \dots, x_d)A$, multiplicity additivity gives

$$e_I(A) = e(R_{\text{br}}) \leq B.$$

The normal surface ring A is Cohen–Macaulay. A general two-generated reduction of I is a parameter ideal of colength $e_I(A)$. Consequently

$$\operatorname{length}(A/I) \leq B, \quad \mathfrak{n}^B \subset I, \quad e_{\mathfrak{n}}(A) \leq B.$$

A general two-generated reduction of $\mathfrak{n}$ removes two independent cotangent classes. Its quotient has length $e_{\mathfrak{n}}(A)$ and has at least $1 + \dim_{\mathbb{C}}(\mathfrak{n}/\mathfrak{n}^2) - 2$ independent constant and linear classes. This proves the embedding-dimension bound.

Step 2. In this step, we use deck-invariant powers to eliminate nonpositive coordinates and bound the positive-parameter quotient. The canonical deck group has order $r \leq m$. Since the parameters diagonalize the full extension, each x_i^r descends to a regular function on Y , hence on S . For $\mu_i \leq 0$, its phase is $r\mu_i$. Every nonconstant monomial of degree at most L in eigenlifts of the coarse cotangent basis has phase in $(0, 2L\epsilon)$. Its difference from $r\mu_i$ lies strictly between zero and $(2L + r)\epsilon < 1$. Character projection in the truncated local ring therefore gives

$$x_i^r \in \mathfrak{m}_{S,P}^{L+1} A \subset \mathfrak{n}^{L+1} \quad (\mu_i \leq 0).$$

In $\bar{A} = A/I_+$, let $\bar{\mathfrak{n}}$ be the maximal ideal and let I_- be generated by the remaining coordinate images. Then

$$\bar{\mathfrak{n}}^B \subset I_-, \quad x_i^r \in \bar{\mathfrak{n}}^{B(m+1)+1} \subset I_-^{m+1}.$$

As $r \leq m$, every remaining x_i^m also lies in I_-^{m+1} . Every monomial of degree $dm + 1$ in at most d generators contains one such factor. Hence $I_-^{dm+1} = I_-^{dm+2}$, and Nakayama's lemma makes these ideals zero. It follows that $\mathfrak{n}^T \subset I_+$. Counting monomials of degree less than T in at most $B + 1$ cotangent generators proves the two length bounds.

An ideal generated by fewer than two elements cannot be primary in a two-dimensional local ring. If exactly two positive coordinates occur, they form a parameter sequence. Completeness makes A finite over $\mathbb{C}[[u_1, u_2]]$, the map is injective by dimension, and Cohen–Macaulayness makes this finite module free. Reduction modulo (u_1, u_2) gives its rank. $\square$

The contact function in Theorem 7.16, pulled through the canonical cover using Lemma 3.8, shows that $(W, c_0 E^\#)$ is lc. Indeed, the coefficient of $E^\#$ in the pulled contact pair is $1 - a + c_0 \ell \geq c_0$, and every other coefficient is nonnegative. This observation also applies when $a = 1$.

Proposition 12.5. *In Setup 12.2, suppose W is singular. Put*

$$B_4 = \lceil 4/c_0 \rceil, \quad L_4 = B_4(m+1), \quad T_4 = B_4(4m+1), \quad K_4 = (T_4 + 1)^{B_4+1}, \quad E_4 = B_4 T_4^2, \\ K_\star = \max(K_4, E_4), \quad P_\star = (K_\star!)^3.$$

Assume

$$(2L_4 + m)\epsilon < 1, \quad 4\epsilon < a, \quad E_4\epsilon < 1, \quad 4P_\star\epsilon < 1.$$

There is a primitive algebraic prime F over the original ambient germ M_0 , centred at its closed point, and a rational $\lambda > 0$, such that

$$\eta^{P_\star} \in I_{\tilde{F}}(F), \quad \Lambda = \lambda(A_{M_0}(F) - F(f) - (1-a)F(Z_E) - F(U)/m)$$

satisfies

$$a \leq \Lambda < a + 6P_\star\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma^{P_\star}) \pmod{\mathbb{Z}}.$$

The conormal phase of $\eta^{P_\star}$ at F is $\lambda \bmod \mathbb{Z}$, and $\lambda \geq 1/|I_{\tilde{F}}(F)|$.

The power in this assertion controls three separate operations: fixing a valuation, fixing its residue field, and fixing the conormal phase.

Proof. We bound the multiplicity of the full Cartier divisor, construct a bounded set of surface valuations, and then construct the ambient prime with its original differential. The two-positive case uses a finite completed parameter map; the three-positive case uses Theorem 12.1 to obtain the required valuation bounds. Write $\mu_i \in (-\epsilon, \epsilon)$ for the signed phases of the chosen ambient eigenparameters at η .

Step 1. In this step, we bound the multiplicity of the full Cartier inverse divisor. Theorem 7.1 [Rei83, Theorem (1.1), pp. 138–139] identifies W as an isolated compound Du Val hypersurface. In particular its multiplicity is two and its general hyperplane section is Du Val. The ordinary blowup $b: \widehat{W} \rightarrow W$ has exceptional Cartier divisor Q equal to the projective tangent quadric.

We check normality of $\widehat{W}$, including a double tangent plane. For a quadratic part of rank at least two, each reduced component of the quadric is generically smooth, so the strict transform is regular at its generic point. For rank one, completing the square writes the equation as

$$\xi_0^2 + g_0(\xi_1, \xi_2, \xi_3) = 0, \quad \text{ord}(g_0) \geq 3.$$

The cubic term $(g_0)_3$ is nonzero: a general hyperplane section is Du Val, and its rank-one quadratic case has type D or E , whose equation has a nonzero cubic term on the radical of the quadratic part. On a blowup chart the equation is

$$(\xi'_0)^2 + s_0(g_0)_3(1, \xi'_2, \xi'_3) + s_0^2(g_0)_4(1, \xi'_2, \xi'_3) + \cdots = 0.$$

Its derivative in s_0 is nonzero at a general point of the exceptional plane. The hypersurface charts satisfy S_2 , so this codimension-one regularity proves normality. Adjunction gives

$$K_{\widehat{W}} = b^*K_W + Q.$$

Write $Q = \sum_i n_i Q_i$ and let d_i be the degree of the reduced component Q_i . Then

$$n_i \in \{1, 2\}, \quad \sum_i n_i d_i = 2, \quad A_W(Q_i) = 1 + n_i.$$

For an effective Cartier divisor D' through the point, two general hyperplanes cut its multiplicity and, on the blowup, meet each Q_i in d_i smooth transverse points. The hyperplanes can avoid the strict transform of D' , the other exceptional components, and the singular locus. On each resulting normalized curve branch the order of D' is $\text{ord}_{Q_i}(D')$. Computing the intersection length on these branches gives

$$e_o(D') = \sum_i d_i \text{ord}_{Q_i}(D').$$

Here the finite-length cokernel of the curve ring in its normalization has equal kernel and cokernel lengths under multiplication by the equation of D' , so it contributes nothing to this equality. If (W, cD') is lc, then

$$e_o(D') \leq \frac{\sum_i d_i(1 + n_i)}{c} = \frac{2 + \sum_i d_i}{c} \leq \frac{4}{c}.$$

Apply this with $D' = E^\#$ and $c = c_0$. Multiplicity additivity bounds the selected reduced branch by the multiplicity of the full divisor. Lemma 12.4 therefore applies with $B = B_4$.

Step 2. In this step, we determine the sign alternatives and construct the surface valuations when exactly two phases are positive. At least two ambient phases are positive. Four positive phases would define a positive monomial valuation on M_0 with full boundary discrepancy at most their sum, less than $4\epsilon < a$. This contradicts Lemma 12.3 with $\sigma = \eta$. If exactly two phases are positive, call the coordinates u_1, u_2 . The completed normal branch is finite free of rank at most K_4 over $\mathbb{C}[[u_1, u_2]]$. Give $\mathbb{C}(u_1, u_2)$ the monomial valuation v_0 with values μ_{u_1}, μ_{u_2} . Its value group is $\delta_0\mathbb{Z}$, and its residue field is $\mathbb{C}(\tau)$, where τ is a primitive value-zero Laurent monomial. The same valuation on $\text{Frac } \mathbb{C}[[u_1, u_2]]$ has this residue field, since a leading form is a finite polynomial. It has at most K_4 extensions to $\text{Frac } A$, each with ramification and residue degree at most K_4 . Indeed, representatives of distinct value cosets multiplied by lifts of residue bases are linearly independent over the base field. Integrality and the primary ideal $(u_1, u_2)A$ make every extension centred at $P^\#$. The extensions form an η -stable set. Their restrictions to the algebraic surface field give the valuations used below.

Step 3. In this step, we bound the equation degree and finite rank associated with three positive coordinates. Suppose the positive coordinates are z_1, z_2, z_3 , and denote the remaining coordinate by y . Put $I = (z_1, z_2, z_3)A$. Since $\mathfrak{n}^{T_4} \subset I$, comparison of the quadratic leading terms of the length functions gives

$$e_I(A) \leq T_4^2 e_{\mathfrak{n}}(A) \leq E_4.$$

Completeness and the primary property make A finite over the image of $\mathbb{C}[[z_1, z_2, z_3]]$. Its prime kernel has height one and is generated by an irreducible series g_+ . Write

$$R_+ = \mathbb{C}[[z_1, z_2, z_3]]/(g_+), \quad h_+ = \text{ord}(g_+), \quad q_+ = [\text{Frac } A : \text{Frac } R_+].$$

Multiplicity additivity for this finite rank- q_+ module gives

$$q_+ h_+ = e_I(A) \leq E_4.$$

Character projection makes g_+ an η -semi-invariant without changing its ideal or its order.

Choose an ordinary degree- h_+ monomial of g_+ , and let b_+ be its positive weight. Then $0 < b_+ < h_+ \epsilon \leq E_4 \epsilon < 1$. All support monomials have positive weights congruent to b_+ modulo integers, so b_+ is the least weight. Its initial form $(g_+)_{b_+}$ is a polynomial. Among the variables occurring in that polynomial choose one with largest weight and call it z ; call the other two original positive coordinates u_1, u_2 . The chosen degree- h_+ monomial gives $b_+ \leq h_+ \mu_z$, whence

$$1 \leq s_+ = \deg_z(g_+)_{b_+} \leq h_+.$$

No change of the original eigenparameters is made.

Step 4. In this step, we prepare the one-variable equation and descend its full root multiset. Give $\mathbb{C}(u_1, u_2)$ the monomial valuation $v_0(u_i) = \mu_{u_i}$, $i = 1, 2$, and let K be its completion. Write $\delta_0 \mathbb{Z}$ for its value group. Evaluation embeds $\mathbb{C}[[u_1, u_2]]$ into K : positive weights give convergence, and a nonzero finite leading polynomial cannot vanish in the monomial graded ring. The residue field is $\mathbb{C}(\tau)$; η acts isometrically and trivially on this field. In a finite auxiliary extension K'/K , choose ξ of value μ_z , by adjoining a suitable root of a base uniformizer. The extension is complete by Theorem 12.1. Substitute $z = \xi X$ in g_+ and divide by a coefficient of smallest value. The result is restricted, since the coefficients of X^j have values tending to infinity. It has norm one and reduction of degree exactly s_+ . There is no cancellation of the leading coefficient after evaluation in the monomial residue field.

Theorem 12.1, applied over K' , factors this restricted series into a unit and a monic polynomial of degree s_+ and norm one. All roots of that polynomial have norm at most one, and the unit has no zero on the closed unit disc. Thus the evaluated g_+ has exactly s_+ roots, with multiplicity, in the disc $v_0(z) \geq \mu_z$. They are algebraic over K , since K'/K is finite. The uniqueness assertion of the same theorem makes their multiset stable under the absolute Galois group of K . Its monic root polynomial is therefore in $K[z]$ and has degree s_+ . In particular every selected root ζ satisfies

$$[K(\zeta) : K] \leq s_+.$$

The auxiliary rescaling degree $[K' : K]$ does not enter this bound.

Evaluation defines an injection $R_+ \rightarrow K(\zeta)$. To prove injectivity, suppose its prime kernel were nonzero. The quotient would be a complete local domain of dimension at most one. Dimension zero contradicts the nonzero image of u_1 . In dimension one, u_1 is a parameter, so completeness makes the quotient finite over $\mathbb{C}[[u_1]]$. Its element u_2 would satisfy a monic equation over $\mathbb{C}[[u_1]]$, contrary to the injection of $\mathbb{C}[[u_1, u_2]]$ just proved. Tensoring the finite separable extension $\text{Frac } A / \text{Frac } R_+$ with $K(\zeta)$ and choosing a field factor gives an embedding of $\text{Frac } A$ into a finite complete field L , with

$$[L : K] \leq q_+ s_+ \leq E_4.$$

Integrality gives nonnegative values on A , and $\mathfrak{n}^{T_4} \subset I$ makes every nonzero element of $\mathfrak{n}$ have positive value. Varying the root and the field embedding yields at most $q_+ s_+$ valuations. The semi-invariance of g_+ , the invariant disc, and the action on A make this set η -stable.

Step 5. In this step, we return to the algebraic surface field and choose a power fixing the valuation and residue field, with the prescribed conormal phase. In either sign case, restrict a chosen valuation to the function field F_S of the selected algebraic normalization component. The functions u_1, u_2 are algebraically independent, and $F_S / \mathbb{C}(u_1, u_2)$ is finite. The restricted valuation v_S is discrete, centred at $P^\#$, and has residue field finite over $\mathbb{C}(\tau)$. It is an algebraic divisorial

valuation: normalize a model of the monomial divisor in the finite extension $F_S/\mathbb{C}(u_1, u_2)$. Its orbit size, ramification index, and residue degree are each at most $K_\star$. This bounds local valuation data, with no bound on the degree of that algebraic field extension.

Set $C_\star = K_\star!$. The element $\eta^{C_\star}$ fixes v_S , and its residue automorphism has order at most $K_\star$, because it fixes $\mathbb{C}(\tau)$. Thus $\eta^{C_\star^2}$ fixes the full residue field. Let $e_S \leq K_\star$ be the ramification index of v_S over v_0 , and take a uniformizer ϖ of value δ_0/e_S . A base Laurent monomial of value δ_0 is ϖ^{e_S} times a unit. If α is the conormal phase of $\eta^{C_\star^2}$, comparison of its leading term gives

$$e_S \alpha \equiv C_\star^2 \delta_0 \pmod{\mathbb{Z}}.$$

As $e_S \mid C_\star$, the conormal phase of $\sigma = \eta^{P_\star}$ is $P_\star \delta_0/e_S$. It follows that on the graded field for $P_\star v_S$, σ acts on every homogeneous piece by the phase of its value.

Step 6. In this step, we construct the original ambient prime and compute its score with both finite differentials retained. Let S_1 be the selected algebraic normalization component, restricted near $P^\#$. Form

$$M' = M_0 \times_{\mathbb{A}_{u_1, u_2}^2} S_1.$$

Near the section induced by $S_1 \rightarrow M_0$, the relevant component is normal, smooth of relative dimension two over S_1 , and generically finite over M_0 . Denote the remaining two ambient coordinates by x, y , with phases μ_x, μ_y . Subtract their section values and call the resulting relative parameters X_1, X_2 . They keep their characters and generate the section ideal. Both full equations f, Z_E belong to (X_1, X_2) .

On a model carrying the divisor of v_S , take the algebraic Gauss valuation w over $P_\star v_S$ with

$$w(X_1) = 1 + P_\star \mu_x, \quad w(X_2) = 1 + P_\star \mu_y.$$

The weights are positive. After a common scaling they are integral weights on ϖ, X_1, X_2 ; the associated monomial prime has residue transcendence degree three. Relative etale coordinates give its algebraic realization on M' . Its restriction to $\mathbb{C}(M_0)$ is λord_F for a primitive algebraic ambient prime. The centre is the original point, since the section values of x, y , as well as u_1, u_2, X_1, X_2 , have positive value.

Take the original top form $\Omega = du_1 \wedge du_2 \wedge dx \wedge dy$. Separability of $F_S/\mathbb{C}(u_1, u_2)$ gives

$$\Omega = du_1 \wedge du_2 \wedge dX_1 \wedge dX_2$$

after pullback. At the surface DVR, write $du_1 \wedge du_2 = \varpi^{b_S} u_S d\varpi \wedge d\tau_S$, where u_S is a unit and τ_S is a separating residue parameter. The finite-DVR differential calculation in Lemma 3.8 gives

$$(b_S + 1)v_S(\varpi) = \mu_{u_1} + \mu_{u_2}.$$

The term $e_S - 1$ of the tame different is included here. Applying the same differential calculation at the finite restriction from the Gauss prime to F gives

$$(12.2) \quad \lambda A_{M_0}(F) = P_\star(\mu_{u_1} + \mu_{u_2}) + w(X_1) + w(X_2).$$

Thus this is the discrepancy on the original ambient germ. Since $f, Z_E \in (X_1, X_2)$ and U is regular,

$$w(f), w(Z_E) > 1 - P_\star \epsilon, \quad w(U) \geq 0.$$

Substitution into (12.2) yields

$$\lambda A_{M_0, \mathcal{D}_0}(F) < 2 + 4P_\star \epsilon - (2 - a)(1 - P_\star \epsilon) = a + (6 - a)P_\star \epsilon < a + 6P_\star \epsilon.$$

On the Gauss graded field, the action of σ is the phase of the value, both on the base and on X_1, X_2 . This remains true after restriction to $\mathbb{C}(M_0)$. Lemma 12.3 supplies the lower bound, the full original inertia, the conormal phase, and the character congruence. The completed parameter map supplies the valuation bounds in the two-positive case, and the descended multiset of roots supplies them in the three-positive case. The same ambient construction proves the assertion in both cases. $\square$

Lemma 12.6. *In Setup 12.2, suppose W is smooth. Assume also that the equation of the selected formal branch of $E^\#$ corresponding to $P^\#$ has character value at η with a signed phase of absolute value less than ϵ . Put*

$$B_3 = \lceil 3/c_0 \rceil, \quad L_3 = B_3(m+1), \quad K_3 = \binom{B_3 + B_3(3m+1)}{B_3+1}.$$

If $(2L_3 + m)\epsilon < 1$ and $(K_3 + 1)\epsilon < 1$, there is a positive monomial valuation on the original smooth canonical cover, a rational multiple of a primitive algebraic divisorial valuation, whose full boundary discrepancy Λ satisfies

$$a \leq \Lambda < a + 4\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma) \pmod{\mathbb{Z}}.$$

Its primitive divisor restricts through the actual finite cover with the ramification and full inertia of Lemma 3.8.

The extra branch character controls the possible nonpositive ambient phase; the score still uses the full equations.

Proof. We bound the branch data and then construct the positive-weight valuation on the original smooth cover.

Step 1. In this step, we bound the branch multiplicity and use its character to control the nonpositive coordinate phases. The ordinary blowup of the smooth threefold point has log discrepancy three. The lc pair $(W, c_0 E^\#)$ therefore gives $\text{mult}_o(E^\#) \leq 3/c_0$. In a regular local ring this is the multiplicity of its Cartier surface, so the selected reduced branch has multiplicity at most B_3 . Lemma 12.4 applies with $d = 3$ and $B = B_3$.

Write z_{br} for the selected formal branch equation and $\beta \in (-\epsilon, \epsilon)$ for its phase. Write x_i for the ambient eigenparameters and μ_i for their signed phases at η . At least two of these phases are positive. If exactly two are positive, denote their coordinates by u_1, u_2 , and the remaining coordinate by y . The branch ring $R_{\text{br}} = \mathbb{C}[[u_1, u_2, y]]/(z_{\text{br}})$ and its normalization are Cohen–Macaulay of dimension two. The parameter-ideal lengths agree by finite birational multiplicity additivity. Hence

$$k_y := \text{ord}_y z_{\text{br}}(0, 0, y) = \text{length}(A/(u_1, u_2)) \leq K_3.$$

The character of the first nonzero term on this axis gives $\beta \equiv k_y \mu_y \pmod{\mathbb{Z}}$. Since $|\beta - k_y \mu_y| < (K_3 + 1)\epsilon < 1$, the two representatives are equal. In particular $\beta \leq 0$. If N_- is the number of nonpositive ambient phases and j_β is one for $\beta \leq 0$ and zero otherwise, we have $N_- \leq j_\beta \leq 1$.

Step 2. In this step, we construct the original positive-weight valuation and compute its full boundary score and character. Give the original eigenparameters the positive weights

$$w(x_i) = \begin{cases} \mu_i, & \mu_i > 0, \\ 1 + \mu_i, & \mu_i \leq 0. \end{cases}$$

They are rational and define an algebraic monomial prime after primitive normalization, for example by the weighted blowup in these regular parameters. Every monomial of z_{br} has positive weight congruent to β , so

$$w(Z_E) \geq w(z_{\text{br}}) \geq \beta + j_\beta, \quad w(U) \geq 0.$$

The first inequality uses factorization only in the completed regular local ring to estimate the full regular equation. Consequently

$$\begin{aligned} \Lambda &= \sum_i w(x_i) - (1-a)w(Z_E) - w(U)/m \\ &\leq N_- - (1-a)j_\beta + \sum_i \mu_i - (1-a)\beta \\ &< a + 4\epsilon. \end{aligned}$$

The η -action on the monomial graded field is the phase of the value. Lemma 12.3 gives the lower bound and character congruence. The prime is over the original smooth canonical cover, so its primitive restriction and full inertia are precisely those of Lemma 3.8. $\square$

Lemma 12.7. *In the curve case of Theorem 7.16, let D be a reduced component of a fibre of $S \rightarrow C$, and let $J \leq H$ fix D pointwise. If $J \neq 1$, its character ψ_D on the normal line of D is faithful. For the full fibre multiplicity n_D , $\psi_D^{n_D}$ is the character on a base parameter of C . Put $B_2 = \lceil 2/c_0 \rceil$, where $c_0 = c_{\text{ct}}(3, a)$. On the actual smooth transverse surface there is a coordinate root of degree $e \leq B_2$ and an equivariant etale chart in which the selected branch is $v = 0$, the full E -equation is $vF_{\text{other}}(t, v)$, and the full mR -equation is $U(t, v)$. Suppose a nonidentity $\gamma \in J$ has t -character $\psi_D(\gamma)$ and v -character with representatives δ and β , respectively, where $0 < \delta < \epsilon < 1/2$ and $|\beta| < \epsilon$. For the weights $w(t) = \delta$, $w(v) = 1 + \beta$, set*

$$\Lambda = e\delta + 1 + \beta - (1 - a)w(vF_{\text{other}}) - w(U)/m.$$

Then

$$a \leq \Lambda < a + (B_2 + 1)\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma) \pmod{\mathbb{Z}}.$$

Here the coordinate root has bounded degree, while the orders of the other branches and of the remaining boundary may be arbitrary.

Proof. We construct the transverse chart, retain its finite Jacobian, and restrict the resulting valuation to the original field.

Step 1. In this step, we construct the actual transverse branch and bound its coordinate order. Put $\Gamma = \nu(D)$. The terminal threefold Y is smooth at the generic point of this curve by Theorem 7.6 [CKM88, Proposition 6.10 and Corollary 6.13]; Lemma 3.11 gives the same assertion over the ground field. Its transverse local ring is regular of dimension two. The contact pair of Theorem 7.16 has coefficient $b_E = 1 - a + c_0\ell \geq c_0$ along E . The ordinary blowup of the generic curve has bare log discrepancy two, so

$$b_E \text{mult}_\Gamma(E) \leq 2.$$

The normalization DVR of the selected plane branch is the DVR of S along D . In transverse eigenparameters x, y , the smaller finite branch order is at most the multiplicity of the full plane curve. Choose x with that order and denote it by e . Thus $1 \leq e \leq 2/c_0 \leq B_2$.

For an algebraic realization of the transverse surface, choose a nonconstant rational parameter on Γ , lift and average it over J , and take the generic fibre over this invariant parameter. Its differential along Γ is nonzero. Finite separable residue extensions split the selected branch; these are etale constant extensions with trivial J -action. All subsequent maps are algebraic over this transverse field.

The group J acts trivially on the residue field at D . Its action on $k(S) = k(E)$ is faithful by Theorem 7.16. The faithful tame conormal character in Lemma 3.8 therefore identifies its action on a branch parameter with the faithful character ψ_D . A parameter of C has order n_D along D and invariant nonzero leading coefficient in $k(D)$, proving the fibre-character identity.

Step 2. In this step, we construct the coordinate root and branch graph while retaining the full equations and Jacobian. The character of x is ψ_D^e . Adjoin the actual coordinate root $x = t^e$ with character ψ_D . On the normalization of the selected branch the ratio of t to an eigenuniformizer has nonzero invariant residue. Choose one such point; the lifted J -action preserves it. The map from the normalized branch to the t -line is etale there. Base change the root surface by this etale map and subtract from y its value on the branch. This gives an eigenparameter v for which the branch is $v = 0$. The full pulled equation of E is $vF_{\text{other}}(t, v)$. Its selected factor has multiplicity one, since the coordinate root is etale at the generic point of that branch. All remaining factors stay in F_{other} . The full equation U of mR is regular. The pulled transverse differential is

$$dx \wedge dy = et^{e-1}dt \wedge dv.$$

Thus the finite crepant boundary on this chart includes $-(e-1)(t=0)$, and its discrepancy at the indicated monomial valuation is exactly the displayed Λ .

Step 3. In this step, we compare the original discrepancy and character and prove the score bounds. Before the etale map the root extension is Galois, with deck group μ_e , and its full action is $\mu_e \times J$. The two actions commute and intersect trivially, since the chosen lift of J acts faithfully on the original field. For a primitive prime T' in the root chart, let T be its primitive restriction and $e_{T'}$ its ramification index. The finite discrepancy and inertia calculation of Lemma 3.8, with the displayed Jacobian retained, gives

$$A_{\text{chart}}(T') = e_{T'} A_{Y, B_Y}(T), \quad |I_{\mu_e \times J}(T')| = e_{T'} |I_J(T)|.$$

Passing to the lifted J decreases the inertia denominator. The etale chart has ramification one and inertia injects into the preceding inertia. Hence the original all-prime comparison gives $A_{\text{chart}}(T')/|I_J(T')| \geq a$. These primes spread over Γ ; after primitive restriction their residue transcendence degree is two, so they are prime divisors over the original threefold with centre in E .

Let $n = \text{ord}(\gamma)$. Both weights have denominators dividing n . If b_w is the greatest common divisor of their integral numerators, then $w = (b_w/n) \text{ord}_{T'}$. The action of γ is the scalar determined by each weight on the monomial graded field, so $\langle \gamma \rangle \subset I_J(T')$. It follows that

$$\Lambda = \frac{b_w}{n} A_{\text{chart}}(T') \geq ab_w \geq a.$$

The full pulled natural tensor is, up to an invariant nonzero constant,

$$\frac{t^{m(e-1)}(dt \wedge dv)^m}{(vF_{\text{other}})^{m(1-a)}U}.$$

Division by the invariant longitudinal differential retains the original character $\chi|_J$. Each full semi-invariant equation has monomial value congruent to its character. The tensor therefore gives $m\Lambda \equiv \text{phase } \chi(\gamma) \pmod{\mathbb{Z}}$. Finally, effectivity gives

$$w(vF_{\text{other}}) \geq 1 + \beta, \quad w(U) \geq 0,$$

and hence

$$\Lambda \leq e\delta + a(1 + \beta) < a + (B_2 + 1)\epsilon.$$

The constructions and finite-DVR comparisons are algebraic in characteristic zero, so this proof also applies over the ground field of Theorem 7.16. $\square$

13. EQUIVARIANT PENCILS AND COMPLETE FIBRES

We retain the original curve exit of Theorem 7.16. The geometry of its complete fibres and an equivariant pencil on its Stein curve provide the points and characters needed to study the original stabilizer. All fibre schemes below include their multiplicities.

Theorem 13.1 (Equivariant threefold MMP). *Let G be finite and (X, B) a G - $\mathbb{Q}$ -factorial dlt threefold pair projective over a G -variety S , with B invariant. Every $(K_X + B)$ -negative extremal face of $\overline{\text{NE}}(X/S)^G$ has its equivariant contraction. For a small extremal contraction the flip exists; divisorial steps and flips preserve dlt and G - $\mathbb{Q}$ -factoriality. A sequence of these threefold flips terminates. A klt G -pair admits a small projective crepant G - $\mathbb{Q}$ -factorialization. A normal G -variety is G - $\mathbb{Q}$ -factorial if and only if its quotient is $\mathbb{Q}$ -factorial [Pro21, Lemma 1.1.3, Theorems 3.3.1 and 3.4.3, Proposition 4.2.2, Corollary 4.4.5].*

Theorem 13.2 (Normal ruled surfaces). *Let S be a normal integral separated algebraic space of dimension two proper over the ground field, and let $\pi : S \rightarrow C$ be a surjective morphism to a smooth projective curve whose general fibre is $\mathbb{P}^1$. Then S is projective and has rational singularities, $R^i \pi_* \mathcal{O}_S = 0$ for all $i > 0$, and every reduced irreducible curve in a fibre is isomorphic to $\mathbb{P}^1$. If a full scheme fibre is irreducible and reduced, then π is smooth along that fibre [Nak15, Proposition 2.33(1)–(4)].*

Theorem 13.3 (Embedding a Cohen–Macaulay curve). *Let Q be a projective Cohen–Macaulay curve over an algebraically closed field, and let L be an invertible sheaf on Q . The curve may be reducible or nonreduced. If*

$$\deg(L|_B) \geq 2p_a(B) + 1$$

for every nonempty generically Gorenstein subcurve $B \subset Q$, then L is very ample. More generally, for a zero-dimensional subscheme $Z \subset Q$, the restriction

$$H^0(Q, L) \longrightarrow H^0(Z, L|_Z)$$

is surjective if

$$\deg(L|_B) > 2p_a(B) - 2 + \text{length}(Z \cap B)$$

for every such subcurve B [CFHR99, Theorem 1.1].

Lemma 13.4 (Automatic adjunction on a curve). *Let Q be a projective Cohen–Macaulay curve over an algebraically closed field, with dualizing sheaf ω_Q . For a nonzero morphism $\varphi : \mathcal{F} \rightarrow \omega_Q$ from a coherent sheaf, let $B \subset Q$ be defined by $\text{Ann}(\text{im } \varphi)$. Then B is Cohen–Macaulay and φ factors canonically as*

$$\mathcal{F} \rightarrow \mathcal{F}|_B \longrightarrow \omega_B = \text{Hom}_Q(\mathcal{O}_B, \omega_Q) \hookrightarrow \omega_Q,$$

where the middle map is generically surjective. If an invertible sheaf L and a zero-dimensional $Z \subset Q$ admit an injection $\mathcal{I}_Z L \hookrightarrow \omega_Q$, then Q is generically Gorenstein [CFHR99, Lemmas 2.3–2.4].

Lemma 13.5 (Components over a curve in a dlt boundary). *Fix $s \in [0, 1)$. Let Y be a normal quasi-projective threefold, and let B be an effective real boundary such that (Y, B) is dlt and Y is $\mathbb{Q}$ -factorial. Let E be a prime divisor over Y with curve centre $T \subset [B]$ and $A_{Y,B}(E) \leq s$. There is a projective birational modification $Y' \rightarrow Y$ with Y' $\mathbb{Q}$ -factorial, relative Picard number one, and E its only exceptional divisor. The number of irreducible components of a general fibre of $E \rightarrow T$ is bounded in terms of s alone. If $s < 1/2$, that general fibre is irreducible [BL21, Lemma 5.3].*

Theorem 13.6 (Rational chains in log Fano fibres). *Let Y, Z be normal varieties, let $B \geq 0$ be an effective rational divisor on Y with $K_Y + B$ $\mathbb{Q}$ -Cartier, and assume that (Y, B) is log canonical. Let $f : Y \rightarrow Z$ be a projective morphism. If $-(K_Y + B)$ is f -ample, then each connected component of each nonempty reduced fibre over a closed point of Z is rationally chain connected [HM07, Corollary 1.3(2)].*

Theorem 13.7 (The bicanonical system of a del Pezzo surface). *Let R be a smooth projective del Pezzo surface. Then $1 \leq (-K_R)^2 \leq 9$, and the complete linear system $| -2K_R |$ is base-point-free and defines a finite morphism [DI09, degree-by-degree discussion following Theorem 3.8, p. 458].*

Set-up 13.8. In the curve case of Theorem 7.16, use the same normalization and Stein factorization $S \rightarrow C \rightarrow h(E)$. Set

$$M = \text{lcm}(1, \dots, m), \quad \mathcal{L} = \nu^*(\mathcal{O}_Y(-ME)|_E), \quad d_0 = \lfloor 2M/a \rfloor, \quad c_0 = c_{\text{ct}}(3, a).$$

All actions and linearizations are induced by the original group H .

Write $\pi : S \rightarrow C$ for the first map in this factorization.

Proposition 13.9. *In Setup 13.8, ME is Cartier near E , the surface S has rational singularities, and $\mathcal{L}$ is relatively very ample over C . There is an integer $1 \leq d \leq d_0$ such that every full scheme fibre Q satisfies, for all $n \geq 0$,*

$$h^0(Q, \mathcal{L}^n) = dn + 1, \quad H^1(Q, \mathcal{L}^n) = 0.$$

The sheaves $\pi_* \mathcal{L}^n$ are locally free and commute with base change. The complete section multiplication maps are surjective, and the relative ideal is generated by

$$\ker(\text{Sym}^2 \pi_* \mathcal{L} \longrightarrow \pi_* \mathcal{L}^2),$$

a vector bundle of rank $d(d-1)/2$. Every reduced fibre component D_i is a smooth rational curve. Writing the full fibre cycle as $\sum n_i D_i$, one has

$$\sum n_i \deg(\mathcal{L}|_{D_i}) = d, \quad \sum n_i \leq d.$$

These structures retain their natural H -actions.

The complete section spaces retain the nilpotent directions of a nonreduced fibre. The quadratic presentation will be used with these spaces, rather than with the sections of its reduction.

Proof. We first obtain an ample line bundle and acyclic full fibres. We then apply the curve embedding criterion and regularity to their complete section rings. By Lemma 3.11, we may establish the geometric assertions over $\mathbb{C}$: the marked diagram and its full fibres have a common realization, and the cohomology, invertibility and surjectivity assertions descend and ascend by faithful flatness. A resolution can be included in that diagram, so the same applies to rational singularities.

Step 1. In this step, we construct the ample line bundle and determine the cohomology and degrees of the full fibres. The prime E is $\mathbb{Q}$ -Cartier by equivariant $\mathbb{Q}$ -factoriality. Theorem 7.16 gives $r_P = \text{ind}_P(K_Y) \leq m$ at every closed point of E . Lemma 8.3 makes $r_P E$ Cartier there, and $r_P \mid M$. Thus ME is Cartier on a neighbourhood of E . The line $\mathcal{L}$ is π -ample because $-E$ is h -ample and normalization is finite.

The surface S is projective, and its geometric generic fibre over the smooth projective C is $\mathbb{P}^1$. Theorem 13.2 therefore applies. In particular, S has rational singularities and $R^1\pi_*\mathcal{O}_S = 0$. A normal surface is Cohen–Macaulay. A base parameter is a nonzero element in the local domains of S , so π is flat and each full fibre Q is a Cohen–Macaulay curve. Top-degree cohomology and base change give $H^1(Q, \mathcal{O}_Q) = 0$. The Euler characteristic is constant in this flat projective family and equals that of its general $\mathbb{P}^1$; hence

$$h^0(Q, \mathcal{O}_Q) = 1, \quad \chi(Q, \mathcal{O}_Q) = 1.$$

For any nonempty Cohen–Macaulay subcurve $B_1 \subset Q$, the quotient sequence from $\mathcal{O}_Q$ to $\mathcal{O}_{B_1}$ gives $H^1(B_1, \mathcal{O}_{B_1}) = 0$, since its kernel has no second cohomology. Consequently

$$p_a(B_1) = 1 - h^0(B_1, \mathcal{O}_{B_1}) \leq 0.$$

In particular each reduced irreducible component has arithmetic genus zero. Its normalization sequence shows that it is a smooth rational curve.

The degree d of $\mathcal{L}$ on a fibre is a positive integer independent of the fibre. On a general fibre F_C , the normalized adjunction of Theorem 7.16 and $\mathcal{N}^{[M]} = \mathcal{L}$ give

$$-2 + \Delta \cdot F_C = -\frac{ad}{M}.$$

The general fibre avoids the singular points and vertical boundary, so $\Delta \cdot F_C \geq 0$. Thus $d \leq 2M/a$. The degree formula on the full curve, or Lemma 10.3, gives

$$\chi(Q, \mathcal{L}^n) = dn + 1, \quad d = \sum_i n_i \deg(\mathcal{L}|_{D_i}).$$

Each component degree is a positive integer; hence $\sum_i n_i \leq d$.

Step 2. In this step, we prove fibrewise vanishing and construct the relative complete embedding. If $n \geq 1$ and $H^1(Q, \mathcal{L}^n) \neq 0$, duality gives a nonzero map $\mathcal{L}^n \rightarrow \omega_Q$. Lemma 13.4 factors it over a nonempty Cohen–Macaulay subcurve B_1 as $\mathcal{L}^n|_{B_1} \rightarrow \omega_{B_1}$, generically surjectively. This map is generically an isomorphism: the dualizing module of an Artinian local ring has the same length as the ring. Its kernel is then zero because an invertible sheaf on a Cohen–Macaulay curve has no finite-length subsheaf. Comparison of Euler characteristics yields

$$n \deg(\mathcal{L}|_{B_1}) \leq 2p_a(B_1) - 2 \leq -2,$$

contrary to ampleness. Together with the case $n = 0$, this proves the asserted vanishing and dimension formula. Moreover

$$\deg(\mathcal{L}|_{B_1}) \geq 1 \geq 2p_a(B_1) + 1$$

for every such subcurve. Theorem 13.3 makes $\mathcal{L}|_Q$ very ample, including when Q is nonreduced.

Cohomology and base change now make every $\pi_*\mathcal{L}^n$ locally free of rank $dn + 1$ and compatible with base change. Evaluation is surjective on every fibre, hence surjective. With $V_1 = \pi_*\mathcal{L}$, it defines a proper morphism $j : S \rightarrow \mathbb{P}_C(V_1)$ whose fibre maps are closed embeddings. It is quasi-finite, hence finite. The cokernel of $\mathcal{O}_{\mathbb{P}_C(V_1)} \rightarrow j_*\mathcal{O}_S$ vanishes on every fibre and therefore vanishes by Nakayama's lemma. Thus j is a closed embedding, with $j^*\mathcal{O}(1) = \mathcal{L}$.

Step 3. In this step, we prove quadratic generation and retain the natural group action on the relative presentation. Embed a fibre by its complete system in $\mathbb{P}^d$, with ideal sheaf $\mathcal{I}_Q$. If $d = 1$, a pure curve of degree one in $\mathbb{P}^1$ is $\mathbb{P}^1$, and its ideal is zero. Otherwise completeness gives $H^1(\mathcal{I}_Q(1)) = 0$, and $H^1(\mathcal{O}_Q) = 0$ gives $H^2(\mathcal{I}_Q) = 0$. The ideal sequence and the cohomology of projective space give $H^i(\mathcal{I}_Q(2 - i)) = 0$ for $i \geq 3$. Thus $\mathcal{I}_Q$ is 2-regular. It follows from Theorem 10.2 that its homogeneous ideal is generated by quadrics and that $H^1(\mathcal{I}_Q(n)) = 0$ for $n \geq 1$. The latter vanishing identifies its homogeneous coordinate ring with its complete section ring.

The maps $\text{Sym}^n V_1 \rightarrow \pi_*\mathcal{L}^n$ are therefore surjective on every fibre, hence surjective. Their kernels K_n are locally free and commute with base change. In degree two,

$$\text{rank } K_2 = \binom{d+2}{2} - (2d+1) = \frac{d(d-1)}{2}.$$

For $n \geq 2$, the maps $K_2 \otimes \text{Sym}^{n-2} V_1 \rightarrow K_n$ are fibrewise surjective, hence surjective. This proves the relative presentation. Normalization, evaluation, multiplication and the kernel maps all respect the natural H -linearizations, so the presentation retains the original action. $\square$

Proposition 13.10. *In Setup 13.8, let $K_C = \ker(H \rightarrow \text{Aut } C)$ and $J_T = \ker(H \rightarrow \text{Aut } h(E))$. Then*

$$|K_C| \leq (4m+3)^4, \quad [J_T : K_C] \leq 2.$$

We bound the kernel using the two fixed points of its action on a geometric generic fibre, with the original log character at both points.

Proof. Put $K = K_C$. We construct a smooth transverse surface in an unramified extension of the original field, produce a discrepancy score at each pole, and apply character packing. The case $K = 1$ requires only the final field-degree argument.

Step 1. In this step, we construct an actual transverse surface and retain the discrepancy comparison at its two poles. Choose a nonconstant $t_0 \in k(h(E))$. Lift it to $\mathcal{O}_{Z, \eta_{h(E)}}$ and average over K . Since K fixes $h(E)$ pointwise, the result is an actual K -invariant rational function t with residue t_0 . After shrinking along the generic point of $h(E)$, take the generic fibre of this map to $\mathbb{A}^1$. Theorem 7.6 and generic smoothness show that its source is a smooth surface near the exceptional curves. The field extension and the same conclusion over the standing field are justified by Lemma 3.11.

The exceptional reduced components on this smooth surface are smooth rational curves. Here is the embedded assertion needed below. Theorem 7.8, applied to $\mathcal{O}_Y$, gives $R^1 h_* \mathcal{O}_Y = 0$: the pair is klt and $-(K_Y + B_Y)$ is relatively trivial, hence nef and big for the birational morphism. After taking the transverse surface and a finite separable extension of its coefficient field, this vanishing persists. Near the generic centre the fibres have dimension at most one. Top-degree cohomology and base change therefore give $H^1(\mathcal{O}) = 0$ for the full exceptional fibre. Each reduced irreducible component is a quotient of that fibre, so also has $H^1(\mathcal{O}) = 0$. Its arithmetic genus is zero, and its normalization sequence makes it a smooth $\mathbb{P}^1$.

Take a finite separable extension $L/k(t)$ splitting the chosen geometric component and the two fixed points of its K -action. The original K acts identically on L . On the selected smooth surface Σ , denote this component by D_1 . It is K -stable because K fixes the Stein field. Its action is faithful: an element trivial on the geometric generic fibre and on $k(C)$ is trivial on $k(S)$, and the original H -action there is faithful. Thus $K \neq 1$ has two poles Q_+, Q_- on D_1 , with inverse faithful parameter characters.

If P' is an algebraic prime over Σ centred at a pole, its restriction P to $k(Y)$ is primitive and divisorial. Indeed a model of the restricted valuation, base-changed by $L/k(t)$, has unramified divisorial local rings: this field extension is finite étale. The residue field has transcendence degree two over k , and the valuation has ramification index one. Discrepancy is unchanged under this extension and localization; the invariant differential dt contributes no order. Inertia on P' injects into inertia on P . Theorem 7.16 consequently gives

$$A_{\Sigma, B_{\Sigma}}(P') = A_{Y, B_Y}(P) \geq a|I_K(P)| \geq a|I_K(P')|.$$

Step 2. In this step, we identify the natural log tensor and full boundary equation at a pole. At either pole Q , write $B_{\Sigma} = (1-a)D_1 + R_Q$, retaining all other components in $R_Q \geq 0$. Choose actual regular eigenparameters (u_Q, z_Q) , with $D_1 = (z_Q = 0)$, by projecting generators of its ideal and of its curve cotangent line. Divide the original natural tensor by $(dt)^m$ and pull it to Σ ; call the result τ_{Σ} . Its character remains $\chi|_K$. The function

$$h_Q = \frac{(du_Q \wedge dz_Q)^m}{\tau_{\Sigma} z_Q^{m(1-a)}}$$

is a regular eigenfunction with divisor mR_Q . In additive character notation,

$$\text{wt}(h_Q) = m \text{wt}(u_Q) + ma \text{wt}(z_Q) - \chi|_K.$$

This identity includes the other branch of E whenever it passes through the chosen pole.

Step 3. In this step, we construct positive-weight valuations with the original lower discrepancy bound and character. Let $\gamma \in K$ have order $e > 1$, with parameter phases $0 < \delta < \epsilon$ and $|\beta| < \epsilon$ at one pole, where $\epsilon < 1$. Give u_Q, z_Q weights $\delta, 1 + \beta$, and put

$$\Lambda = \delta + a(1 + \beta) - \frac{w(h_Q)}{m}.$$

If b is the greatest common divisor of the two positive integers $e\delta, e(1 + \beta)$, the primitive weighted blowup has orders $(e/b)(\delta, 1 + \beta)$. The element γ acts on its associated graded ring by weighted scalars, hence trivially on its residue field. Its inertia therefore has order at least e . The preceding lower bound yields

$$\frac{e}{b}\Lambda \geq ae, \quad \Lambda \geq ab \geq a.$$

Regularity of h_Q and positivity of the weights give

$$a \leq \Lambda < a + (1 + a)\epsilon \leq a + 2\epsilon.$$

Every monomial of h_Q has its eigencharacter, so its weighted order is congruent to that character modulo $\mathbb{Z}$. The tensor identity gives

$$m\Lambda \equiv \text{phase } \chi(\gamma) \pmod{\mathbb{Z}}.$$

These are scores of actual primitive primes before rescaling; no equality of an auxiliary closed-point minimum with a has been assumed.

Step 4. In this step, we use the two pole characters to bound the curve-action kernel. Set $Q_0 = 4m + 3$ and $\epsilon = 1/Q_0$. If $|K| > Q_0^4$, Lemma 8.9, with trivial excluded subgroup, supplies a nonidentity γ whose four parameter characters at Q_+, Q_- have signed phases in $(-\epsilon, \epsilon)$. The two curve phases sum to zero and are nonzero, since their characters are inverse and faithful. After

replacing γ by its inverse, apply the score construction to γ at Q_+ and to γ^{-1} at Q_- . Write the resulting scores as $a + \eta_+$ and $a + \eta_-$. Then

$$0 \leq \eta_{\pm} < 2\epsilon, \quad m(\eta_+ + \eta_-) \in \mathbb{Z}, \quad 0 \leq m(\eta_+ + \eta_-) < \frac{4m}{4m+3} < 1.$$

Both $\eta_{\pm}$ vanish. Since $ma \in \mathbb{Z}$, the character congruence forces $\chi(\gamma) = 1$, contrary to faithfulness of the original character. Hence $|K_C| \leq (4m+3)^4$.

Finally $K_C \subset J_T$, and J_T/K_C acts faithfully on the field extension $k(C)/k(h(E))$. Its degree is at most two by Theorem 7.16, which gives $[J_T : K_C] \leq 2$. $\square$

We record two preparations for constructing the pencil on the original Stein curve.

Lemma 13.11. *Let (Z_1, Δ_1) be a quasi-projective dlt threefold pair with effective rational boundary. Suppose a finite group J preserves the pair and Z_1 is J - $\mathbb{Q}$ -factorial. Then there is an ordinary projective small $\mathbb{Q}$ -factorialization $q : Z_1^q \rightarrow Z_1$, crepant for the strict transform of (Z_1, Δ_1) . It is an isomorphism at the generic point of every curve in $[\Delta_1]$. No J -action on Z_1^q is asserted.*

Proof. The reduced divisor $[\Delta_1]$ is invariant, hence $\mathbb{Q}$ -Cartier. For sufficiently small rational $\varepsilon > 0$, the pair

$$(Z_1, \Delta_1 - \varepsilon[\Delta_1])$$

is klt. This follows on a dlt log resolution: the strict coefficients are below one, and the finitely many exceptional discrepancies remain positive. The total log divisor is $\mathbb{Q}$ -Cartier because the subtracted divisor is. Theorem 4.8, with empty extraction set, supplies q . Smallness makes the strict transform of $K_{Z_1} + \Delta_1$ its pullback, since they agree in codimension one.

At the generic point of a curve in the floor, the transverse dlt surface has one floor component and is plt, or has two floor components and is simple normal crossing. In the first case Theorem 6.5 gives a cyclic quotient of a smooth surface; in the second it is smooth. Both are $\mathbb{Q}$ -factorial. A projective small modification of a $\mathbb{Q}$ -factorial germ is an isomorphism: push down a relatively ample Cartier divisor and pull back its $\mathbb{Q}$ -Cartier pushdown. The difference is exceptional and has no prime component, so it is zero. Every contracted curve would then have both positive and zero degree against this divisor. There are no such curves, and the remaining finite birational morphism to a normal germ is an isomorphism. $\square$

Lemma 13.12. *Let S_1, S_2 be normal projective surfaces over an algebraically closed field of characteristic zero with identified function fields. Let $\pi_i : S_i \rightarrow C_i$ be contractions to smooth projective curves with geometric generic fibre $\mathbb{P}^1$. If $g(C_1) > 0$, then*

$$k(C_1) = k(C_2) \subset k(S_1) = k(S_2).$$

If a finite group preserves these subfields, their identification and the induced isomorphism of smooth projective curves are equivariant.

Proof. On a common resolved projective model, the rational map to C_1 is constant on a general rational fibre over C_2 , because no nonconstant map from $\mathbb{P}^1$ to a positive-genus curve exists. Consequently the functions from $k(C_1)$ belong to the algebraic closure of $k(C_2)$ in the surface field. That algebraic closure is $k(C_2)$ by the Stein property. Thus $k(C_1) \subset k(C_2)$, and C_2 also has positive genus. Interchanging the rulings proves the reverse inclusion. The equality takes place in the original surface field, so it respects every automorphism preserving the subfields. $\square$

We shall use rational chains only to recognize individual fibre curves.

Lemma 13.13. *Let $f : X_1 \rightarrow Z_1$ be a projective morphism of normal varieties, and let $B_1 \geq 0$ be a rational divisor such that (X_1, B_1) is lc and $K_{X_1} + B_1$ is $\mathbb{Q}$ -Cartier. Assume that $-(K_{X_1} + B_1)$ is f -ample. Then every irreducible curve component of a closed fibre of dimension at most one has normalization $\mathbb{P}^1$.*

Proof. Theorem 13.6 makes each connected component of the reduced fibre rationally chain connected. A rational chain through a general point of one of its curve components must contain a rational curve dominating that component: the point lies on no other component. Its normalization is therefore $\mathbb{P}^1$. This argument makes no assertion that the whole fibre is connected. $\square$

Absolute log Fano positivity also gives the rational surfaces used below.

Lemma 13.14. *Let (R_1, Λ_1) be a projective dlt surface pair with effective rational boundary and ample $-(K_{R_1} + \Lambda_1)$. Then R_1 is rational.*

Proof. The reduced floor is $\mathbb{Q}$ -Cartier. Along one floor component this follows from Theorem 6.5, and where two meet the dlt surface is simple normal crossing. Off the floor the assertion is vacuous. Decreasing its coefficient by a sufficiently small positive rational number makes the pair klt and preserves ampleness of the negative adjoint. If the floor is empty, no decrease is needed. Theorem 4.13 therefore makes R_1 rationally connected. Rational curves through general points lift to its smooth resolution. The resolution is a rationally connected smooth surface, so its minimal model is rational or ruled; the base of a ruled minimal model has genus zero, since a rational curve cannot dominate a positive-genus curve. Thus R_1 is rational. $\square$

We use adjunction to a floor of a dlt threefold with its full different. The floor is normal, and Theorem 4.5 gives an lc adjunction pair. It is dlt: on a dlt log resolution, restrict the crepant boundary to the strict floor. Every exceptional curve of the restricted resolution comes from an exceptional divisor of coefficient less than one; simple normal crossings exclude an additional strict floor along its generic point. This gives a dlt log resolution of the adjunction pair. Its boundary is effective and rational, as the codimension-two different formula and the simple normal crossing intersections show.

The next lemma measures a map between the original residue fields. Its auxiliary ordinary modifications need carry no group action.

Lemma 13.15. *Let a finite group G_1 preserve a quasi-projective dlt threefold pair (Z_1, Δ_1) with effective rational boundary, and assume Z_1 is G_1 - $\mathbb{Q}$ -factorial. Let F_1 be an exceptional prime over Z_1 , with complete curve centre T_1 contained in a floor prime R_1 . Fix a rational number $0 < u_1 < 1$, and assume*

$$0 \leq A_{Z_1, \Delta_1}(F_1) < 1 - u_1.$$

Suppose a normal projective model of $\kappa(F_1)$ has a contraction to a smooth projective curve C_1 of positive genus, with geometric generic fibre $\mathbb{P}^1$. Let $J_1 \leq G_1$ preserve F_1, R_1, T_1 and the subfield $\mathbb{C}(C_1) \subset \kappa(F_1)$. Then the inclusion $\mathbb{C}(T_1) \subset \kappa(F_1)$ induces a J_1 -equivariant finite morphism

$$C_1 \longrightarrow T_1^\nu, \quad \deg(C_1 \rightarrow T_1^\nu) \leq \lceil 2/u_1 \rceil.$$

In the adjunction $K_{R_1} + \Delta_{R_1} = (K_{Z_1} + \Delta_1)|_{R_1}$, the coefficient of T_1 in Δ_{R_1} is at least u_1 .

Proof. We bound the number of geometric components of an extracted fibre, identify its Stein field, and compute the adjunction coefficient.

Step 1. In this step, we extract the chosen prime and bound its geometric generic fibre components. Apply Lemma 13.11 with the group G_1 . The resulting ordinary small $\mathbb{Q}$ -factorialization is an isomorphism at the generic point of T_1 . On it keep coefficient one on the transform of R_1 and slightly reduce the other floor coefficients. On a dlt log resolution this makes the pair plt. Choose the reduction rational and sufficiently small that the discrepancy δ of F_1 still satisfies

$$0 < \delta < 1 - u_1 < s, \quad s = 1 - u_1/2.$$

Write (Z_1^q, Δ') for this plt pair. All divisors used in the reduction are $\mathbb{Q}$ -Cartier.

Lemma 13.5 extracts exactly F_1 by a projective morphism $p : Z'_1 \rightarrow Z_1^q$ of ordinary relative Picard number one. Denote its exceptional prime by F'_1 , and write

$$K_{Z'_1} + \Delta'' = p^*(K_{Z_1^q} + \Delta'), \quad \text{coeff}_{F'_1} \Delta'' = 1 - \delta > 1 - s.$$

This boundary is effective and rational, and the pair is plt. Negativity and relative Picard number one make $-F'_1$ p -ample. The exceptional locus is exactly F'_1 : an effective F'_1 has nonnegative degree on every curve outside its support. For small rational $\eta > 0$, $(Z'_1, \Delta'' + \eta F'_1)$ is still plt and its negative adjoint is p -ample. Lemma 13.13 therefore makes every geometric generic exceptional fibre component rational. One can apply the lemma on a general closed transverse slice, or spread the geometric generic fibre to such slices.

Let μ be the number of these geometric components. The strict floor R'_1 is relatively ample: if $p^*R_1^q = R'_1 + b_1 F'_1$, then $b_1 > 0$, so $R'_1 \equiv_p -b_1 F'_1$. Both R'_1 and R_1^q are normal. Their birational morphism is an isomorphism over the generic point of T_1 , because a proper birational morphism of normal surfaces is an isomorphism at codimension-one points of the target. Thus every geometric component of a general exceptional fibre meets R'_1 at the same point.

Slice transversely to T_1 at a general point. In the crepant surface boundary each exceptional fibre component has coefficient greater than $1 - s$. The index correction in adjunction has the form $1 - 1/e + b/e \geq b$, so this remains true if the slice meets a quotient singularity. The coefficient-one strict floor passes through all these components. Adjunction to that floor has coefficient at most one at their common point. By the different formula in Theorem 6.4, this implies that the sum of their coefficients, counted with their positive integral local multiplicities, is at most one. Consequently

$$\mu(1 - s) < 1, \quad \mu < 2/u_1.$$

Step 2. In this step, we identify the original Stein field and its equivariant finite morphism. Normalize F'_1 and take its Stein factor over $T_1^\vee$. The intermediate normal projective curve C'_1 has function field equal to the algebraic closure of $\mathbb{C}(T_1)$ in the same $\kappa(F_1)$. Its degree over $T_1^\vee$ is μ : the generic extension is separable, and its embeddings index the geometric components of the normalized generic fibre. That fibre over C'_1 is geometrically rational. Lemma 13.12 identifies $\mathbb{C}(C'_1) = \mathbb{C}(C_1)$ inside $\kappa(F_1)$. Since J_1 preserves $\mathbb{C}(T_1)$, it preserves this algebraic closure. The induced finite map on the smooth projective curves is therefore equivariant for the original actions, even though the ordinary modifications used to measure its degree were not.

Step 3. In this step, we bound the adjunction coefficient on the original floor. There is a component of $R'_1 \cap F'_1$ mapping birationally to T_1 . Its adjunction coefficient has the form

$$1 - \frac{1}{r} + \frac{(1 - \delta)n_1 + v_1}{r}, \quad r \geq 1, \quad n_1 \geq 1, \quad v_1 \geq 0.$$

It is at least u_1 , since $1 - \delta > u_1$ and $u_1 < 1$. Crepant adjunction pushes this coefficient to the original generic point of T_1 . Restoring the other floor coefficients can only increase the different: on the local index cover it adds the restriction of an effective divisor. This proves the required bound for Δ_{R_1} . $\square$

The surface calculation below turns this finite map into an equivariant pencil with an explicit degree.

Lemma 13.16. *Let a finite cyclic group J_1 preserve a normal projective rational lc surface pair (R_1, Λ_1) , with $\Lambda_1 \geq 0$ rational and $K_{R_1} + \Lambda_1 \sim_{\mathbb{Q}} 0$. Let $D_1 \subset R_1$ be an invariant prime whose coefficient is at least a real number $u_1 > 0$. Let C_1 be a smooth projective positive-genus curve with a J_1 -equivariant finite morphism $C_1 \rightarrow D_1^\vee$ of degree n_1 . Then C_1 admits a J_1 -equivariant nonconstant morphism to $\mathbb{P}^1$ of degree at most $18n_1/u_1$.*

Proof. If $D_1^\vee \simeq \mathbb{P}^1$, the given map suffices, since $u_1 \leq 1$. Otherwise we pass to a smooth minimal group surface and use its anticanonical system or its conic bundle.

Step 1. In this step, we obtain an effective crepant boundary on a smooth minimal group surface. Take the minimal smooth resolution of R_1 , with its lifted J_1 -action. Its crepant boundary is effective. Indeed, if E_i are the exceptional curves and Λ_{str} is the strict boundary, its exceptional coefficients b_i satisfy

$$\sum_i b_i(E_i \cdot E_j) = -(K + \Lambda_{\text{str}}) \cdot E_j \leq 0.$$

Here $K \cdot E_j \geq 0$ by adjunction and minimality: the only negative case would be an exceptional smooth rational (-1) -curve. Also $\Lambda_{\text{str}} \cdot E_j \geq 0$. The exceptional intersection matrix is negative definite with nonnegative off-diagonal entries; its inverse has nonpositive entries. Hence $b_i \geq 0$, and lc gives $b_i \leq 1$.

Contract invariant disjoint collections of (-1) -curves until the smooth rational group surface is minimal. The pushed boundary remains effective and its adjoint remains $\mathbb{Q}$ -linearly trivial. The image D_2 of D_1 survives, since its normalization has positive genus. By Theorem 6.1 the resulting smooth surface R_2 is del Pezzo or has an equivariant conic bundle $R_2 \rightarrow \mathbb{P}^1$. An action kernel can be removed before this operation and then restored.

Step 2. In this step, we bound the equivariant pencil degree in the del Pezzo and conic-bundle alternatives. In the del Pezzo case put $A_2 = -K_{R_2}$, and let Λ_2 be the pushed boundary. Theorem 13.7 gives $A_2^2 \leq 9$ and a finite morphism defined by $|2A_2|$. Since A_2 is ample,

$$0 < u_1(A_2 \cdot D_2) \leq A_2 \cdot \Lambda_2 = A_2^2 \leq 9.$$

The pullback of $2A_2$ to C_1 has degree at most $18n_1/u_1$. The natural J_1 -linearization on the bicanonical system induces a representation on its restricted section space. This space has dimension at least two because the associated morphism is finite on D_2 . Choose two independent eigensections. Their ratio is nonconstant and transforms by a character. Removing the common zero divisor gives the asserted morphism with the stated degree bound.

In the conic-bundle case D_2 is horizontal, since every irreducible fibre component is rational. For a general fibre L_2 ,

$$0 < u_1(D_2 \cdot L_2) \leq \Lambda_2 \cdot L_2 = 2.$$

The ruling composed with $C_1 \rightarrow D_2'$ is an equivariant map of degree at most $2n_1/u_1$, which also suffices. $\square$

For the point-centre case we need connected generic fibres on each individual exceptional floor, even when the contraction is equivariant.

Lemma 13.17. *Let a finite group G_1 act on a quasi-projective dlt threefold pair (X_1, Γ_1) with rational boundary, and let $g : X_1 \rightarrow X_2$ be an equivariant projective birational contraction to a normal variety X_2 . Assume X_1 is G_1 - $\mathbb{Q}$ -factorial, $\dim N^1(X_1/X_2)^{G_1} = 1$, and $-(K_{X_1} + \Gamma_1)$ is g -ample. Suppose g contracts a prime divisor and every contracted prime is a floor component. Then its exceptional locus is a single G_1 -orbit of floor primes. If one such prime R_1 maps to a curve B_1 , the induced morphism $R_1 \rightarrow B_1'$ is a contraction with geometric generic fibre $\mathbb{P}^1$.*

Proof. We first locate the entire exceptional locus and then examine a transverse fibre.

Step 1. In this step, we identify the entire exceptional locus as one orbit of floor primes. Let D_1 be the sum of the translates of a contracted prime. It is an invariant effective exceptional $\mathbb{Q}$ -Cartier divisor. It is not g -nef by Lemma 4.9. The invariant relative divisor space has dimension one, so $-D_1$ is numerically a positive multiple of the relatively ample $-(K_{X_1} + \Gamma_1)$; hence it is g -ample. Every contracted curve lies in D_1 , because an effective $\mathbb{Q}$ -Cartier divisor has nonnegative degree on a curve not contained in it. The fibres are connected. An isolated point outside D_1 cannot accompany a positive-dimensional fibre, and a finite birational morphism to a normal variety is an isomorphism. Thus the entire exceptional locus is $\text{Supp } D_1$.

Step 2. In this step, we prove that an individual floor has connected geometric generic fibre equal to a projective line. Over a general geometric point of B_1 the fibre has dimension one and

all its components lie in exceptional floor primes. A general transverse slice gives a normal dlt surface pair. Its adjoint is negative on every fibre component; the slicing divisor is pulled back from the target and has degree zero there. Adjunction to a component L_1 therefore gives

$$\deg(K_{L_1} + \text{Diff}_{L_1}) < 0.$$

The component is smooth, the different is effective, and hence $L_1 \simeq \mathbb{P}^1$. Every incidence with another floor curve contributes one to this different. At such an incidence the dlt surface is simple normal crossing, and three floor curves cannot meet. Thus each vertex of the fibre's incidence graph has degree at most one, counting separate intersection points separately. The graph is connected, so it has at most two vertices.

The floor R_1 is normal. Its generic fibre over B_1' is normal by localization, and geometrically normal in characteristic zero. Distinct geometric components of this fibre are therefore disjoint. Two such components would require a third component from another floor to connect the whole fibre, contrary to the bound just proved. Its geometric generic fibre is consequently one $\mathbb{P}^1$. The Stein factor of $R_1 \rightarrow B_1'$ is finite and birational and hence is B_1' . This proves the contraction assertion. $\square$

The following observation extends the equivariance of a pencil while controlling its degree.

Lemma 13.18. *Let C_1 be a smooth projective integral curve over an algebraically closed field of characteristic zero. Let a finite cyclic group J act on C_1 , and let $J_1 \leq J$ have index j . Suppose there is a J_1 -equivariant nonconstant morphism $\varphi_1 : C_1 \rightarrow \mathbb{P}^1$ of degree at most b , for a positive integer b and some action of J_1 on $\mathbb{P}^1$. Then there is a J -equivariant nonconstant morphism $\varphi : C_1 \rightarrow \mathbb{P}^1$ of degree at most jb , for some action of J on $\mathbb{P}^1$.*

Proof. Let σ generate J . A finite-order projective linear transformation of $\mathbb{P}^1$ is diagonalizable in characteristic zero. We can therefore choose a rational coordinate on the target whose pullback $f \in k(C_1)$ satisfies $\sigma^{j*}f = \lambda f$ for a root of unity λ . This also applies when the target action is trivial. The space

$$V_1 = \text{span}_k\{f, \sigma^*f, \dots, \sigma^{(j-1)*}f\}$$

is J -stable. It has an eigenbasis and contains a nonconstant eigenfunction v , since it contains the nonconstant function f . Its polar divisor satisfies

$$\text{div}_\infty(v) \leq \sum_{i=0}^{j-1} \text{div}_\infty(\sigma^{i*}f), \quad \deg \text{div}_\infty(v) \leq jb.$$

The function v defines a nonconstant morphism on the smooth projective curve C_1 . Its eigencharacter defines the required multiplicative action on the target. $\square$

Proposition 13.19. *In Setup 13.8, put*

$$c = \min(c_0, a, 1/2), \quad u = (1 - a + c/2)/2, \quad N_{\text{pen}} = \lceil 2/u \rceil, \quad P_{\text{pen}} = \lceil 18N_{\text{pen}}/u \rceil.$$

The same curve C admits a full- H -equivariant nonconstant morphism

$$\phi : C \longrightarrow \mathbb{P}^1, \quad \deg \phi \leq D_\star = 2P_{\text{pen}}.$$

For genus zero the degree can be one. The action on the target is allowed to be trivial.

The curve and the action in this proposition are those of the original exit. Auxiliary boundaries will provide a pencil without replacing them.

Proof. For genus zero, transport the H -action through an isomorphism $C \simeq \mathbb{P}^1$. Suppose henceforth that $g(C) > 0$. We first construct a rational plt floor. A curve centre on that floor gives the pencil directly. For a point centre, we follow the same prime through an equivariant dlt MMP and use its last divisorial step. We work over $\mathbb{C}$ first.

Step 1. In this step, we construct a rational plt floor over the original closed point. Keep the original cover $(V \ni o, B_V)$ and the original prime F . By Theorem 5.9, choose an original invariant contact function g , and write $\ell = \text{ord}_F(g)$. The choice of c gives

$$\ell \geq 1, \quad c\ell \leq a, \quad B_V^* = B_V + \frac{c}{2} \text{div}_V(g).$$

The pair (V, B_V^*) is klt: its discrepancies are the averages of the positive discrepancies of (V, B_V) and the nonnegative ones for the full contact addition. Since $2u = 1 - a + c/2$,

$$0 < \frac{a}{2} \leq A_{V, B_V^*}(F) = a - \frac{c\ell}{2} \leq a - \frac{c}{2} = 1 - 2u.$$

In particular $0 < u < 1/2$, and all these coefficients are rational.

Lemma 6.3 gives an actual H -equivariant extraction $f : W_0 \rightarrow V$ of one reduced prime R_0 , with $-R_0$ relatively ample and $(W_0, R_0 + B_0)$ plt, where B_0 is the strict transform of B_V^* . The reduced fibre over o is precisely R_0 . Indeed a contracted curve outside R_0 would have nonnegative degree against R_0 , contradicting ampleness of $-R_0$; connectedness of birational fibres excludes isolated extra points. For $\alpha = A_{V, B_V^*}(R_0) > 0$,

$$K_{W_0} + R_0 + B_0 = f^*(K_V + B_V^*) + \alpha R_0.$$

Plt adjunction gives a projective klt log Fano surface on R_0 , which is rational by Lemma 13.14. Here R_0 is the reduced floor supplied by the extraction lemma; no finite Weil pullback multiplicity is assigned to it.

Shrink V to an invariant affine neighbourhood. For a sufficiently large divisible integer j , choose a general member

$$A_j \in |-j(K_{W_0} + R_0 + B_0)|.$$

This system is generated after increasing j , by relative ampleness. On a log resolution, a general member meets the boundary transversally and avoids its strata; with coefficient $1/j$ it preserves plt. Average A_j/j over H , obtaining L_0 . Discrepancies of the average are averages of positive discrepancies away from the fixed floor, so

$$\Omega_0 = B_0 + L_0, \quad (W_0, R_0 + \Omega_0) \text{ plt}, \quad K_{W_0} + R_0 + \Omega_0 \sim_{\mathbb{Q}, V} 0.$$

These divisors are effective and rational.

The field $\kappa(F)$ is not rational. Otherwise a rational surface would dominate the positive-genus curve C , whereas restriction to general rational curves makes every such map constant. Thus $F \neq R_0$, and plt and the effective addition $\alpha R_0 + L_0$ give

$$0 < A_{W_0, R_0 + \Omega_0}(F) \leq 1 - 2u.$$

Step 2. In this step, we obtain the pencil when the original prime has a curve centre on the plt model. Decrease the coefficient of the $\mathbb{Q}$ -Cartier R_0 slightly to obtain a klt pair. Theorem 13.1 supplies a small projective H - $\mathbb{Q}$ -factorialization $q : W \rightarrow W_0$. Write R_W, Ω_W for the strict transforms. Smallness gives

$$K_W + R_W + \Omega_W = q^*(K_{W_0} + R_0 + \Omega_0), \quad R_W = q^*R_0.$$

The pair remains plt. The floor R_W is a normal projective rational surface with an invariant klt adjunction boundary Θ_W and $K_{R_W} + \Theta_W \sim_{\mathbb{Q}} 0$. The same discrepancy bound holds for F .

If its centre on W is a curve T , apply Lemma 13.15 with $G_1 = J_1 = H$ and $u_1 = u$. It gives an actual equivariant map

$$C \longrightarrow T^\nu, \quad n = \deg(C \rightarrow T^\nu) \leq N_{\text{pen}}, \quad \text{coeff}_T \Theta_W \geq u.$$

The centre is complete because it lies over o . Lemma 13.16 gives an H -equivariant pencil of degree at most $18n/u \leq P_{\text{pen}}$. We may therefore suppose that the centre on W is a point P .

Step 3. In this step, we construct an equivariant dlt MMP sequence ending at W . Choose a small positive rational ε such that

$$B_\varepsilon = (1 - \varepsilon)R_W + \Omega_W \quad \text{is klt}, \quad A_{W, B_\varepsilon}(F) < 1 - u.$$

Take an equivariant projective log resolution $\psi : Z_1 \rightarrow W$ carrying F . Set

$$\Gamma_1 = (B_\varepsilon)_{\text{str}} + \sum_i E_i, \quad K_{Z_1} + \Gamma_1 = \psi^*(K_W + B_\varepsilon) + \sum_i a_i E_i, \quad a_i = A_{W, B_\varepsilon}(E_i) > 0,$$

where the sum runs over all exceptional primes. This is a log-smooth dlt pair with rational boundary.

Run the equivariant $(K + \Gamma)$ -MMP over W on extremal rays of the invariant cone, using the equivariant MMP theorem already invoked. Its steps preserve ordinary dlt singularities and H - $\mathbb{Q}$ -factoriality. All steps are birational, since the generic fibre over W is a point. Every contracted prime is exceptional over W and has coefficient one in Γ . Divisorial steps decrease the relative Picard number, and the threefold flips terminate. On the final nef model the effective exceptional difference $\sum a_i E_i$ must be zero by negativity. Thus its morphism to W is small.

This last morphism is an isomorphism. Average a relatively ample Cartier divisor over H , push it to W , and pull back its $\mathbb{Q}$ -Cartier pushdown. The two divisors agree in codimension one and hence agree everywhere as Weil divisors; their difference has no exceptional prime. Relative ampleness then excludes any contracted curve. The remaining finite birational morphism to normal W is an isomorphism. We have obtained

$$Z_1 \dashrightarrow Z_2 \dashrightarrow \cdots \dashrightarrow Z_N = W.$$

Let T_i be the centre of the same F on Z_i . The pushed discrepancy identity has an effective exceptional difference at every stage. Therefore

$$0 \leq A_{Z_i, \Gamma_i}(F) \leq A_{W, B_\varepsilon}(F) < 1 - u.$$

Every T_i is H -invariant and projective, since it lies in the fibre over P . Initially it is the floor divisor F ; finally it is P .

Step 4. In this step, we use the last relevant divisorial contraction to obtain a finite map from the original Stein curve. Choose the last divisorial step $\varphi_\ell : Z_\ell \rightarrow Z_{\ell+1}$ for which T_ℓ has positive dimension and is contained in the exceptional locus. Such a step exists because the original divisor F has been contracted. Lemma 13.17 applies to every divisorial step of this sequence.

The image $T_{\ell+1}$ is a point or a rational curve. Indeed, if it is a curve, its generic point must eventually enter a non-isomorphism, since the final centre is P . Until the first such step the centre curves are birationally identified. This first step cannot be divisorial by the choice of ℓ , since its entire exceptional locus consists of contracted primes. It is therefore a flip. The centre curve is a component of a fibre of the small contraction defining that flip. Its source pair is dlt with effective rational boundary, its adjoint is $\mathbb{Q}$ -Cartier, and its negative adjoint is ample for the contraction. A small contraction of a threefold has fibres of dimension at most one. Lemma 13.13 makes this component rational. Hence $T_{\ell+1}^\nu \simeq \mathbb{P}^1$.

If $\dim T_\ell = 2$, then T_ℓ is the H -invariant model of F . The exceptional-orbit conclusion makes it the sole exceptional prime of φ_ℓ . If its image is a point, dlt adjunction makes it a projective dlt log Fano surface, which is rational by Lemma 13.14. If its image is a curve, the connected generic-fibre conclusion gives a geometrically rational ruling over $T_{\ell+1}^\nu \simeq \mathbb{P}^1$. Theorem 6.7 gives a point on the smooth genus-zero generic curve over $\mathbb{C}(\mathbb{P}^1)$; it is therefore $\mathbb{P}^1$ over that field. Its surface field is rational in this case as well. Both cases contradict the positive-genus ruling of $\kappa(F)$. Thus $\dim T_\ell = 1$.

Choose an exceptional floor prime R containing T_ℓ . At the generic point of a curve in a dlt threefold at most two floor primes meet: an intersection of two is generically simple normal crossing, and three floors meet only in codimension three. Every H -translate of R contains the

invariant T_ℓ . Consequently

$$H_0 = \text{Stab}_H(R), \quad [H : H_0] \leq 2.$$

Dlt adjunction gives an H_0 -invariant dlt pair

$$(R, \Gamma_R), \quad K_R + \Gamma_R = (K_{Z_\ell} + \Gamma_\ell)|_R.$$

Apply the curve-trace lemma with ambient group $G_1 = H$ and subgroup $J_1 = H_0$. This use requires only the established H - $\mathbb{Q}$ -factoriality of Z_ℓ . It gives

$$v : C \longrightarrow T'_\ell, \quad n = \deg v \leq N_{\text{pen}}, \quad \text{coeff}_{T_\ell} \Gamma_R \geq u.$$

The field of the source is the original $k(C)$ inside $\kappa(F)$.

Step 5. In this step, we construct the pencil in each possible image configuration and return full equivariance over the original field. If $\varphi_\ell(R)$ is a point, the surface R is projective and rational, and $-(K_R + \Gamma_R)$ is ample. Take a general member of a sufficiently divisible multiple of this ample divisor, divide by that multiple, and average over H_0 . Adding the result gives an effective invariant rational boundary $\Lambda \geq \Gamma_R$ with

$$(R, \Lambda) \text{ dlt}, \quad K_R + \Lambda \sim_{\mathbb{Q}} 0.$$

To check dlt, choose the general member on a dlt log resolution transverse to the boundary and avoiding every lc stratum. Its coefficient is less than one. Averaging preserves the positive exceptional discrepancies and the same floor. The rational-surface pencil lemma applied to v gives degree at most $18n/u$.

Suppose next that $\varphi_\ell(R) = B_1$ is a curve but $\varphi_\ell(T_\ell)$ is a point. The divisorial-fibre lemma gives a contraction $R \rightarrow B'_1$ with geometric generic fibre $\mathbb{P}^1$. The pair (R, Γ_R) has effective rational boundary and negative adjoint ample over B'_1 . Its fibres have dimension one. The rational-fibre-curve lemma shows that $T'_\ell \simeq \mathbb{P}^1$, so v itself is an H_0 -equivariant pencil of degree n .

Finally suppose R maps to a curve B_1 and T_ℓ dominates it. Then $B_1 = T_{\ell+1}$ and $B'_1 \simeq \mathbb{P}^1$. For a general fibre $L_1 \simeq \mathbb{P}^1$ of $R \rightarrow B'_1$, relative log Fano positivity gives

$$0 < u(T_\ell \cdot L_1) \leq \Gamma_R \cdot L_1 < 2.$$

The intersection $T_\ell \cdot L_1$ is the degree of $T'_\ell \rightarrow B'_1$. Composing it with v gives an equivariant pencil of degree at most $2n/u$. All centre curves and their images here are complete, since they lie over P .

The curve-centre case already gave the pencil for all of H . In the point-centre case each of the three images gives an H_0 -equivariant pencil of degree at most P_{pen} . Lemma 13.18 extends it to all of H with degree at most $2P_{\text{pen}} = D_\star$.

Finally apply Lemma 3.11 to realize the original marked diagram and its Stein curve over a common algebraically closed field embedded in k and $\mathbb{C}$. The complex argument gives an equivariant pencil of bounded degree. Existence of such a pencil descends to this field of definition: for each degree at most $D_\star$ and each character of the cyclic group, graphs of morphisms to $\mathbb{P}^1$ with that target action form a scheme of finite type, and equivariance is a closed condition in the morphism locus. A nonempty such locus over $\mathbb{C}$ has a point over the algebraically closed field of definition. Base extension then gives the pencil over k . The original Stein field and its action are retained throughout. $\square$

For a diagonal torus action in projective coordinates, the *least-weight locus* of an invariant projective scheme means the intersection of its reduction with the projective subspace of least-weight coordinates. Here the least weight is taken among the coordinates whose restrictions to the reduction are nonzero.

Lemma 13.20. *Let $\pi : S \rightarrow C$ be a projective flat morphism from a normal integral surface to a smooth curve, with relatively very ample line bundle $\mathcal{L}$. Fix a full Cartier fibre Q of degree $d \geq 1$. Suppose $h^0(Q, \mathcal{L}) = d + 1$, its complete section multiplication maps are surjective, its ideal is*

generated by quadrics, and $\pi_*\mathcal{L}$ is locally free and commutes with base change near $c = \pi(Q)$. Assume each reduced component has positive integral $\mathcal{L}$ -degree. Let a finite cyclic group preserve the data and linearize $\mathcal{L}$. Choose eigenbases of $H^0(Q, \mathcal{L})$ and of the quadratic ideal space of its complete embedding. Suppose their character values at an element γ have signed phases of absolute value less than ϵ , its base parameter phase has a representative $0 < \tau < \epsilon$, and

$$(2d^2 + d + 4)\epsilon < 1.$$

These weights define an auxiliary torus action on the same full fibre. Its least-weight locus on Q_{red} is nonempty. Either it contains a component fixed pointwise by γ , or it is finite and every character of the actual cotangent space of S at each of its points has a representative in $(0, 2\epsilon)$.

The conclusion concerns the cotangent space of the surface itself. We use normality of that surface to control nilpotent sections of the full fibre; the auxiliary torus need act only on the fibre.

Proof. We construct the auxiliary action, exclude sections of excessively small weight by division by the base parameter, and then compute the cotangent characters at the least-weight points. We restrict throughout to the cyclic group generated by γ .

Step 1. In this step, we construct the auxiliary torus on the full fibre and lift its eigensections to the surface. Write $x_0, \dots, x_d$ for the coordinate eigensections and $\lambda_0, \dots, \lambda_d$ for their chosen signed phases. If the monomial $x_i x_j$ occurs in a quadratic eigenrelation of phase ρ , then

$$\lambda_i + \lambda_j - \rho \in \mathbb{Z}, \quad |\lambda_i + \lambda_j - \rho| < 3\epsilon < 1.$$

This integer is zero. Every quadratic generator is therefore homogeneous for the rational weights λ_i . After clearing denominators these weights define a torus action on the homogeneous coordinate ring of the same Q . A finite element of this torus induces γ . The multiplication hypothesis identifies this ring with $\bigoplus_{n \geq 0} H^0(Q, \mathcal{L}^n)$.

Choose an actual eigenparameter t at c of phase τ by projecting a local parameter to its cotangent character. On an invariant neighbourhood of c , local freeness and base change allow us to lift each x_i to a section of $\mathcal{L}$. Character projection makes the lifts eigensections with the same restrictions. After shrinking, they form a frame of $\pi_*\mathcal{L}$ and give the relative complete embedding. Every homogeneous polynomial in the x_i consequently has an actual eigenlift of its indicated degree and character near Q .

Step 2. In this step, we exclude the smallest nilpotent weights by division by the base parameter. Put $b_* = \min_i \lambda_i$. We claim that, for $1 \leq j \leq d$, no nonzero section $s \in H^0(Q, \mathcal{L}^j)$ of auxiliary weight $j b_*$, with an actual eigenlift $\tilde{s}$, can vanish on Q_{red} . Write the fibre cycle as $\sum_D n_D D$. The degree formula and the positive integral component degrees give $n_D \leq d$. Vanishing on the reduction gives $\text{ord}_D(\tilde{s}) \geq 1$ for every D . Thus $\tilde{s}^d/t$ has no divisorial pole near Q ; normality of S makes it a regular section of $\mathcal{L}^{jd}$.

Suppose $v_k = \tilde{s}^d/t^k$ is regular, with $1 \leq k \leq d$. Its character has phase $jdb_* - k\tau$. If its restriction to Q were nonzero, that character would occur in $H^0(Q, \mathcal{L}^{jd})$. By complete multiplication there would be a degree- jd monomial weight W of this character, with

$$jdb_* \leq W < j d \epsilon.$$

It follows that

$$0 < W - (jdb_* - k\tau) < (2jd + k)\epsilon < 1, \quad W - (jdb_* - k\tau) \in \mathbb{Z},$$

a contradiction. Hence v_k vanishes on the full fibre. Its ideal is (t) , so v_{k+1} is regular. Induction shows that $\tilde{s}^d/t^{d+1}$ is regular.

Since $s \neq 0$, the section $\tilde{s}/t$ is not regular near all of Q . Normality then gives a component D with $\text{ord}_D(\tilde{s}) < n_D$. At this component

$$\text{ord}_D(\tilde{s}^d/t^{d+1}) = d \text{ord}_D(\tilde{s}) - (d+1)n_D < 0,$$

contradicting regularity. This proves the claim.

Let b be the least coordinate weight whose restriction to Q_{red} is nonzero. If $b_* < b$, a basis section of weight b_* violates the claim with $j = 1$. Thus $b_* = b$. In any chart with a nonzero weight- b coordinate, all affine coordinate weights are nonnegative.

Step 3. In this step, we analyze the least-weight locus and, when it is finite, eliminate its nonconstant zero-weight local coordinates. Choose a point of Q_{red} at which some weight- b coordinate is nonzero. Its limit under the torus as the parameter tends to zero belongs to the least-weight locus, so this locus is nonempty. If it contains a curve component, the torus fixes that component pointwise, and so does γ .

Otherwise write its points as $P_1, \dots, P_r$. A general hyperplane containing the least-weight projective subspace contains no component of Q_{red} . It meets that reduced curve in degree at most d , while containing the r distinct points. Consequently $r \leq d$.

Fix P_i , and choose a weight- b coordinate x_0 nonzero there. For another weight- b coordinate x_j , set $c_j = x_j(P_i)/x_0(P_i)$. Choose a homogeneous polynomial A_i of degree $r - 1$ in the weight- b coordinates, nonzero at P_i and zero at every other P_h : a product of one separating linear form for each $h \neq i$ does this. For $r = 1$ take $A_i = 1$. The section

$$F_i = A_i(x_j - c_j x_0)$$

has degree r and auxiliary weight rb .

This section vanishes on every reduced component. On a component where all weight- b coordinates vanish, this follows from its expression. On any other component, projection to the weight- b coordinate space is invariant under the torus. The component is not fixed pointwise, since that would put it in the finite least-weight locus. Its torus action therefore has a dense orbit, and the projection is constant, with value one of the P_h . The expression for F_i vanishes at that value.

The section F_i has an actual eigenlift. Since $r \leq d$, the divisibility claim forces it to be zero on the full Q . Near P_i , both A_i and x_0 are units after trivializing their line bundles. Hence

$$x_j/x_0 = c_j$$

on the full local fibre. All its zero-weight affine coordinates are constants; this conclusion includes its nilpotent directions.

Step 4. In this step, we identify positive representatives for the actual surface cotangent characters. The relative embedding shows that $\mathfrak{m}_{S, P_i}/\mathfrak{m}_{S, P_i}^2$ is generated by the class of t and the affine coordinate differences. For $\lambda_j > b$, the coordinate x_j/x_0 vanishes at P_i and has character represented by

$$0 < \lambda_j - b < 2\epsilon.$$

For $\lambda_j = b$, the actual lifted difference $x_j/x_0 - c_j$ restricts to zero on Q , so it equals tv for a regular function v . The difference is invariant and t has nontrivial character τ ; thus v has character $-\tau$ and vanishes at the fixed point P_i . The product tv contributes zero to the cotangent space. Finally t has phase $\tau \in (0, \epsilon)$. The surviving generators all have phases in $(0, 2\epsilon)$, which proves the assertion. $\square$

We isolate the finite-curve calculation used for a normalized branch.

Lemma 13.21. *Let D_1 be a smooth integral curve over an algebraically closed field of characteristic zero, and let $D_1^\sharp$ be a finite disjoint union of smooth integral curves. Suppose a cyclic group Δ of order r acts generically freely on $D_1^\sharp$, with finite quotient $q : D_1^\sharp \rightarrow D_1$. Let a finite abelian group $\tilde{J}$ act on this diagram, with exact sequence*

$$1 \longrightarrow \Delta \longrightarrow \tilde{J} \longrightarrow J \longrightarrow 1.$$

Assume J fixes $P \in D_1$, and fix $\xi \in q^{-1}(P)$. Put $L_1 = \text{Stab}_{\tilde{J}}(\xi)$, and let ψ and θ be the cotangent characters at ξ and P , respectively, and let χ_1 be a faithful character of J . Then L_1 fits into an

exact sequence

$$1 \longrightarrow \mu_f \longrightarrow L_1 \longrightarrow J \longrightarrow 1, \quad f \leq r,$$

with ψ^f equal to the inflation of θ . The inflation of χ_1 to L_1 has kernel μ_f .

Proof. A geometric fibre of a finite group quotient is a single group orbit. Every lift in $\tilde{J}$ of an element of J therefore carries ξ to a Δ -translate of ξ . Correcting by that translate gives an element of L_1 , so $L_1 \rightarrow J$ is surjective. Its kernel is Δ_ξ .

The action of Δ_ξ on the completed local ring at ξ is faithful, by generic freeness. In characteristic zero, projection of a uniformizer to its cotangent character gives an eigenuniformizer. An element with trivial cotangent character fixes this uniformizer and the completed local ring, hence is the identity. Thus Δ_ξ is identified by its cotangent character with μ_f , where $f = |\Delta_\xi| \leq r$. If z is an L_1 -eigenuniformizer, the completed quotient ring is

$$k[[z]]^{\Delta_\xi} = k[[z^f]].$$

The pullback of a parameter at P consequently has leading term a nonzero scalar times z^f . Comparing the L_1 -actions on leading terms proves $\psi^f = \theta$ after inflation. Faithfulness of χ_1 gives the last kernel assertion. $\square$

Lemma 13.22. *In Setup 13.8, let D be a reduced fibre component, $P \in D$, and $J \leq H$ preserve D and fix P . On the canonical cover at $\nu(P)$, normalize the full inverse image of E , then the reduced inverse image of D . For any chosen point above P , its stabilizer has an exact sequence*

$$1 \longrightarrow \mu_f \longrightarrow L_J \longrightarrow J \longrightarrow 1, \quad f \leq r \leq m,$$

where r is the local canonical degree. If ψ is its character on the lifted curve parameter and θ the original curve parameter character at P , then $\psi^f = \theta$. The natural m -log character is inflated from the original $\chi|_J$ and has kernel μ_f .

Here the equality of parameter characters is read after inflation along $L_J \rightarrow J$. No faithfulness of ψ on all of L_J is asserted.

Proof. We retain the actual canonical cover through normalization of the surface and curve, then compute the branch stabilizer.

Step 1. In this step, we construct the actual canonical cover and its full reduced Cartier inverse divisor. Put $P_0 = \nu(P)$. The original all-prime comparison in Theorem 7.16 gives a lower bound a for the closed-point minimum of the finite crepant quotient by J at the image of P_0 . This follows from Lemma 3.11, since $I_J \subset I_H$. Denote that exact minimum by b ; then $b \geq a > 0$. Apply Proposition 8.8 at this minimum. It supplies the canonical cover $W \rightarrow Y$ and the abelian extension $\tilde{J}$, with deck group $\Delta_0 = \mu_r$. The bound $r \leq m$ is supplied by Theorem 7.16.

The full Weil inverse image $E^\sharp$ is Cartier by Lemma 8.3, because W is Gorenstein terminal. It is generically reduced, since the cover is quasi-étale. The smooth or isolated hypersurface description in Proposition 8.8 makes W Cohen–Macaulay. Thus this Cartier divisor is Cohen–Macaulay with no embedded components, and is reduced. Exactness of finite-group invariants gives $(\mathcal{O}_{E^\sharp})^{\Delta_0} = \mathcal{O}_E$: the invariant part of the Cartier ideal is precisely the ideal of functions vanishing on the reduced prime E .

Step 2. In this step, we identify the normalized surface quotient and its generic deck torsor over D . Let $S^\sharp$ be the normalization of $E^\sharp$, allowing disconnected components. Taking integral closures commutes here with invariants, since an invariant integral element lies in the invariant total fraction ring. Hence $S^\sharp/\Delta_0 = S$. The canonical cover is étale over the smooth locus of Y , and normalization commutes with étale base change. By Theorem 7.6, the image of the curve D has smooth generic point in Y . The map $S^\sharp \rightarrow S$ is consequently a deck torsor over the generic point of D .

Step 3. In this step, we compute the normalized curve-branch stabilizer and its parameter and log characters. Normalize the reduced inverse image of D in $S^\sharp$, obtaining a finite normal

curve scheme $D^\sharp \rightarrow D$ near P . The curve D is smooth by Proposition 13.9; the same invariant normalization argument gives $D^\sharp/\Delta_0 = D$. Its deck action is generically free. The entire $\tilde{J}$ preserves this diagram, since J preserves D . Lemma 13.21 applies and yields the claimed exact sequence and $\psi^f = \theta$. The natural log character is the inflation of the faithful $\chi|_J$ by Proposition 8.8, so its kernel on L_J is exactly μ_f . The field realizations, normalizations and natural characters are preserved by Lemma 3.11; this proves the assertion over the standing field as well. $\square$

14. CURVE EXITS

We retain the notation of Setup 13.8, and write $\pi : S \rightarrow C$ for the Stein ruling. We bound the full original stabilizer by applying the local estimates of Section 12 above the two poles of an equivariant pencil. A single finite abelian group will carry all the characters needed at these poles.

Proposition 14.1. *Fix $m \geq 1$, a rational $a \in (0, 1]$ with ma integral, and an integer $D \geq 1$. There is an integer $C_{\text{pen}}(m, a, D)$ such that, in Setup 13.8, every specified full- H -equivariant nonconstant morphism $C \rightarrow \mathbb{P}^1$ of degree at most D gives*

$$|H| \leq C_{\text{pen}}(m, a, D).$$

The assertion holds over every algebraically closed field of characteristic zero.

The bound controls the kernel of the pencil action as well as its image. In particular, the pencil action may be trivial.

Proof. We first work over $\mathbb{C}$. We bound the kernel of the pencil action, pack the characters on all its pole fibres, and use the local estimates to obtain two discrepancy scores with inverse original characters.

Step 1. In this step, we choose the uniform constants used in the character packing and local estimates. Use $L_4, E_4, P_\star$ from Proposition 12.5, L_3, K_3 from Lemma 12.6, and B_2 from Lemma 12.7. They depend only on m, a . Put

$$\begin{aligned} A_\star &= \max\{6, B_2 + 1\}, & K_{\text{base}} &= 2(4m + 3)^4, \\ F_D &= (2Dd_0)!, & N_D &= D(d_0^2 + 21d_0 + 4) + 2. \end{aligned}$$

Choose

$$\begin{aligned} R_D &= \max\{4/a, E_4, 4P_\star, 2L_4 + m, 2d_0^2 + d_0 + 4, D + 1, d_0 + 1, \\ &\quad P_\star(2L_3 + m), P_\star(K_3 + 1), 4mA_\star P_\star\}, \end{aligned}$$

and set $Q_D = \lfloor R_D \rfloor + 1$ and $\epsilon = 1/Q_D$. We prove the assertion with

$$(14.1) \quad C_{\text{pen}}(m, a, D) = F_D D K_{\text{base}} P_\star Q_D^{N_D}.$$

Step 2. In this step, we bound the pencil kernel and preserve all components above its two poles. Let $\phi : C \rightarrow \mathbb{P}^1$ be the specified morphism, and let $\beta : H \rightarrow \text{Aut}(\mathbb{P}^1)$ be its action on the target. By Proposition 13.10, the kernel of H on C has order at most K_{base} . The image of $\ker \beta$ on C acts by automorphisms of $\mathbb{C}(C)/\mathbb{C}(\mathbb{P}^1)$, so its order is at most $\deg \phi$. Thus

$$(14.2) \quad |\ker \beta| \leq D K_{\text{base}}.$$

This proves the bound when the target action is trivial.

Otherwise the nontrivial cyclic image is conjugate to an action $z \mapsto \zeta z$. Denote its two fixed points by b_+, b_- . There are at most $2D$ points of C over these poles. Proposition 13.9 gives a common fibre degree $1 \leq d \leq d_0$, and at most d_0 reduced components in each full fibre. Hence there are at most $2Dd_0$ such components. The subgroup

$$H_0 = H^{F_D} = \{g^{F_D} : g \in H\}$$

preserves every component and every base point over the two poles, and $[H : H_0] \leq F_D$.

Step 3. In this step, we collect all required characters on one finite abelian group. For each $c \in \phi^{-1}(\{b_+, b_-\})$, let Q_c be the full fibre of π . Choose H_0 -eigenbases of $H^0(Q_c, \mathcal{L})$ and of the quadratic ideal space in its complete embedding. Record their characters and the character on a local parameter of C at c . The dimensions supplied by Proposition 13.9 show that this records

$$(d+1) + \frac{d(d-1)}{2} + 1 = \frac{d^2 + d + 4}{2}$$

characters per fibre. Record also the two target parameter characters at b_+, b_- . The total so far is at most $D(d_0^2 + d_0 + 4) + 2$.

For every reduced component D' of these fibres, set $J_{D'} = \ker(H_0 \rightarrow \text{Aut}(D'))$. If $J_{D'} \neq 1$, record the two characters in the transverse chart of Lemma 12.7: the normal character $\psi_{D'}$ on S , and the character on the other parameter of that chart. Extend each to H_0 . Such an extension exists because H_0 is cyclic: a root of the prescribed value on a subgroup generator gives the value on a generator of H_0 . These extensions add at most $4Dd_0$ characters; they will only be used on elements of $J_{D'}$.

Each component on which H_0 acts nontrivially is a smooth rational curve, so its cyclic image has exactly two fixed points. List these points, with repetitions when a point lies on more than one component, as P_i . There are at most $4Dd_0$ entries. At $\nu(P_i)$, take the actual canonical cover and its abelian extension of H_0 from Proposition 8.8. Lemma 13.22 supplies a stabilizer L_i , preserving a selected lifted curve point and the corresponding normalization point of the full inverse image of E , with

$$1 \longrightarrow \mu_{f_i} \longrightarrow L_i \longrightarrow H_0 \longrightarrow 1, \quad f_i \leq r_i \leq m.$$

All these are finite abelian groups. Their natural log characters are inflated from the same original $\chi|_{H_0}$.

If the canonical cover is smooth, record its three ambient eigenparameter characters, chosen for the full abelian extension, and the character of an equation of the selected formal branch of the inverse image of E . The branch is preserved by L_i ; in the smooth completed local ring its principal ideal has a semi-invariant generator, obtained by projecting a generator to its character modulo the maximal ideal times that ideal. If the cover is singular, record the four eigenparameter characters of its smooth ambient germ. These are precisely the local coordinate data of Setup 12.2 and Lemma 12.6. Each entry contributes four characters, for a total of at most $16Dd_0$.

Form the finite abelian fibre product

$$A_{\text{aux}} = \prod_{H_0} L_i, \quad p : A_{\text{aux}} \twoheadrightarrow H_0.$$

When the list is empty, take $A_{\text{aux}} = H_0$ and p the identity. Pull the recorded characters back from their factors or from H_0 . Their number is at most

$$D(d_0^2 + d_0 + 4) + 2 + 4Dd_0 + 16Dd_0 = N_D.$$

The surjection p retains every local deck kernel.

Step 4. In this step, we choose short characters outside the kernel of the powered pencil action. Define

$$B_{\text{bad}} = \{g \in H_0 : \beta(g)^{P_\star} = 1\}.$$

Lemma 8.9 and (14.2) give $|B_{\text{bad}}| \leq DK_{\text{base}} P_\star$. Suppose, towards a contradiction, that

$$(14.3) \quad |H_0| > DK_{\text{base}} P_\star Q_D^{N_D}.$$

Apply Lemma 8.9 to p and the recorded characters. We obtain $\eta \in A_{\text{aux}}$ whose image $\gamma = p(\eta)$ lies outside B_{bad} , with every recorded character having a signed phase in $(-\epsilon, \epsilon)$. In particular, $\beta(\gamma^{P_\star}) \neq 1$.

The two target parameter characters at γ are inverse and nontrivial. Relabel the poles so that the signed phase at b_+ is positive. Use (η, γ) above b_+ , and (η^{-1}, γ^{-1}) above b_- . All recorded phases remain in $(-\epsilon, \epsilon)$, and the target parameter phase is positive at the pole under consideration.

Step 5. In this step, we obtain positive base phases and treat the component and finite-locus alternatives on the full pole fibres. Write η_0, γ_0 for the elements chosen at either pole, and $\tau \in (0, \epsilon)$ for its target phase. Choose any c above that pole. If $e_c \leq D$ is the ramification index of ϕ at c , and δ_c is the recorded signed phase of a base parameter there, equivariance gives

$$e_c \delta_c \equiv \tau \pmod{\mathbb{Z}}.$$

The absolute value of their difference is less than $(D+1)\epsilon < 1$; hence $e_c \delta_c = \tau$ and $0 < \delta_c < \epsilon$.

The ruling π is flat: its local rings are torsion-free over the discrete valuation rings of the smooth base. The remaining hypotheses of Lemma 13.20 follow from Proposition 13.9. Since $(2d^2 + d + 4)\epsilon < 1$, that lemma applies to the full fibre Q_c and the element γ_0 .

Suppose first that its least-weight locus contains a component D' . Then $\gamma_0 \in J_{D'}$, so the recorded extensions give its actual transverse characters. Write ρ for the signed phase of $\psi_{D'}(\gamma_0)$, and $n_{D'}$ for the full fibre multiplicity. Lemma 12.7 gives $n_{D'}\rho \equiv \delta_c \pmod{\mathbb{Z}}$. As $n_{D'} \leq d_0$ and $(d_0 + 1)\epsilon < 1$, this is an equality of rational representatives. In particular, $0 < \rho < \epsilon$.

Suppose instead that the least-weight locus is finite, and choose a point P in it and a reduced component D' through P . We check that P occurs in the list P_i . If γ_0 acted trivially on D' , every coordinate nonzero on D' would have the same projective character. Their signed weights differ by less than $2\epsilon < 1$, so these weights would be equal as rational numbers. The auxiliary torus would then fix D' pointwise. Since P has least weight, the whole component would belong to the least-weight locus, contradicting its finiteness. Thus γ_0 acts nontrivially on D' . Its fixed points on this rational curve are the two fixed points of the cyclic H_0 -image, which were listed. Consequently $P = P_i$ for some entry, and Lemma 13.20 gives representatives in $(0, 2\epsilon)$ for every character of the actual cotangent space of S at P .

Step 6. In this step, we apply all local alternatives to the prescribed powered original character. In the component alternative apply Lemma 12.7 to $\gamma_0^{P_\star}$, with shortness parameter $P_\star\epsilon$. This element is nonidentity because its pencil action is nontrivial. Its normal phase is $P_\star\rho > 0$, and its other recorded phase has absolute value less than $P_\star\epsilon$. The inequality $4P_\star\epsilon < 1$ keeps these representatives in the required ranges. The lemma supplies a score Λ with

$$a \leq \Lambda < a + (B_2 + 1)P_\star\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma_0^{P_\star}) \pmod{\mathbb{Z}}.$$

In the finite alternative, project η_0 to the branch stabilizer L_i attached to P , and denote the image by η_i . If its canonical cover is smooth, use $\eta_i^{P_\star}$ and $\gamma_0^{P_\star}$ in Lemma 12.6. The three ambient phases and the formal branch phase have absolute value less than $P_\star\epsilon$. Every coarse cotangent phase has a representative in $(0, 2P_\star\epsilon)$. Our choice of Q_D ensures

$$(2L_3 + m)P_\star\epsilon < 1, \quad (K_3 + 1)P_\star\epsilon < 1.$$

The resulting score satisfies

$$a \leq \Lambda < a + 4P_\star\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma_0^{P_\star}) \pmod{\mathbb{Z}}.$$

If the canonical cover is singular, apply Proposition 12.5 directly to η_i , with shortness parameter ϵ . Its four ambient phases were recorded, and coarse cotangent positivity was established above. The inequalities

$$(2L_4 + m)\epsilon < 1, \quad 4\epsilon < a, \quad E_4\epsilon < 1, \quad 4P_\star\epsilon < 1$$

hold by construction. This supplies

$$a \leq \Lambda < a + 6P_\star\epsilon, \quad m\Lambda \equiv \text{phase } \chi(\gamma_0^{P_\star}) \pmod{\mathbb{Z}}.$$

The component lemma covers a positive-dimensional least-weight locus; the smooth and singular cover results cover every point in a finite least-weight locus. Each uses the full boundary and the original natural character.

Step 7. In this step, we combine the opposite pole scores and conclude the bound over the original field. We have obtained scores Λ_+, Λ_- satisfying

$$a \leq \Lambda_{\pm} < a + A_{\star} P_{\star} \epsilon, \quad m\Lambda_{\pm} \equiv \pm \text{phase } \chi(\gamma^{P_{\star}}) \pmod{\mathbb{Z}}.$$

Since $ma \in \mathbb{Z}$, the number

$$m((\Lambda_+ - a) + (\Lambda_- - a))$$

is a nonnegative integer. It is less than $2mA_{\star}P_{\star}\epsilon < 1$, and hence is zero. Both scores equal a , so the displayed congruences force $\chi(\gamma^{P_{\star}}) = 1$. The original character is faithful, which contradicts $\beta(\gamma^{P_{\star}}) \neq 1$. Thus (14.3) is impossible. Multiplication by $[H : H_0] \leq F_D$ proves (14.1) over $\mathbb{C}$.

For an arbitrary algebraically closed field of characteristic zero, apply Lemma 3.11 to the original cover and marked exit, together with their normalization, Stein factorization, natural actions and specified pencil. Its common complex realization retains these data, the all-prime comparisons, m, a , the pencil degree and $|H|$. The complex bound therefore gives the stated bound over the original field. $\square$

Theorem 14.2. *Fix $m \geq 1$ and a rational $a \in (0, 1]$ with ma integral. There is an integer $C_{\text{cur}}(m, a) > 0$ such that, for every original cover datum and every curve contraction supplied by Theorem 7.16, the full original stabilizer satisfies*

$$|H| \leq C_{\text{cur}}(m, a).$$

The assertion holds over every algebraically closed field of characteristic zero, without any further condition on the genus of the Stein curve, normality of the exceptional divisor, conductor orders, or smoothness and Gorensteinness of the terminal source.

The equivariant pencil on the original Stein curve makes the preceding bound uniform over all curve exits.

Proof. Proposition 13.19 gives a full- H -equivariant nonconstant morphism $C \rightarrow \mathbb{P}^1$ of degree at most $D_{\star}$, with $D_{\star}$ depending only on a and $c_0 = c_{\text{ct}}(3, a)$. Apply Proposition 14.1 to this morphism and set

$$C_{\text{cur}}(m, a) = C_{\text{pen}}(m, a, D_{\star}).$$

Both propositions hold over every algebraically closed field of characteristic zero and retain the original group H . $\square$

15. COMPUTING CURVES ABOVE ONE

We retain the original cover data of Definition 3.10. We bound the stabilizer of a prescribed computing divisor when its centre on the terminal model is a curve. The estimate uses the two fixed points of the action on that curve and keeps the natural log character at both points.

Theorem 15.1. *Let original cover data have exact minimum $a = p/q > 1$, with mB integral and $q \mid m$. Suppose the centre of the prescribed computing lift F on an H -equivariant effective terminal pair model is a complete curve C . Then $C \simeq \mathbb{P}^1$, H acts faithfully on C , F is its ordinary generic blowup valuation, and*

$$|H| \leq Q(m) = [4m(m + 5) + 3]^8.$$

The assertion holds over every algebraically closed field of characteristic zero, including singular and non-Gorenstein poles.

The bound concerns the full stabilizer of the chosen lift. In particular, it does not require a choice of a new computing divisor.

Proof. We first use the generic computing valuation to determine the boundary multiplicity. We then construct a short-weight valuation at each fixed point, keeping the ramification of its canonical and coordinate-root covers. Character packing at the two points completes the estimate.

Step 1. In this step, we determine the generic boundary multiplicity and retain the original all-prime lower bound. Write $g : (Y, B_Y) \rightarrow (V, B_V)$ for the given terminal pair model. Lemma 7.12 gives $C \simeq \mathbb{P}^1$, the faithful H -action on C , and the identification of F with its ordinary generic blowup valuation. The discrepancy of this codimension-two blowup is therefore

$$(15.1) \quad j = m \operatorname{mult}_C B_Y \in \{0, \dots, m-1\}, \quad a = 2 - \frac{j}{m}, \quad ma = 2m - j, \quad a \geq 1 + \frac{1}{m}.$$

Here the multiplicity is the sum of the generic transverse multiplicities of the boundary components, weighted by their coefficients. Its integrality after multiplication by m follows from the crepant identity and the integral Cartier divisor $m(K_Y + B_Y)$.

For a primitive prime T over Y centred at a point of C , let T_X be its primitive restriction to $k(X)$. Its centre is x . Lemma 3.8 and crepancy give

$$(15.2) \quad \frac{A_{Y,B_Y}(T)}{|I_H(T)|} = \frac{|I_G(T)|}{|I_H(T)|} A_{X,B}(T_X) \geq a.$$

The equality $I_H(T) = H \cap I_G(T)$ is an equality in the original function field; it does not require G to act on Y . The natural m -log line on Y has the original faithful character $\chi|_H$, by Proposition 7.5. Lemma 3.11 allows us to prove the bound over $\mathbb{C}$, retaining this diagram, character, and all-prime inequality.

Set

$$R = 4m(m+5) + 3, \quad \epsilon = R^{-1}.$$

There is nothing to prove if $H = 1$. Otherwise the faithful cyclic action on C has two fixed points P_+, P_- , with inverse tangent characters. Indeed a generator in $\operatorname{Aut}(\mathbb{P}^1)$ is diagonalizable and has distinct eigenlines; in coordinates at these eigenlines its multipliers are inverse. We first work at one of these points, denoted P .

Step 2. In this step, we construct the canonical cover and the stabilizer of a normalized curve branch. Let a_P be the exact closed-point minimum of the finite crepant quotient $(Y, B_Y)/H$ at the image of P . By (15.2), $a_P \geq a$. Apply Proposition 8.8 at this minimum. It supplies the canonical cover $(W \ni z) \rightarrow (Y \ni P)$, an abelian extension

$$1 \longrightarrow \Delta = \mu_\ell \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1, \quad \ell = \operatorname{ind}_P(K_Y) \leq m,$$

and a smooth ambient germ M_0 . If W is singular, its equation in M_0 is a semi-invariant f , and $\dim M_0 = 4$. Otherwise $M_0 = W$ has dimension three. Let h be the full regular semi-invariant equation of mB_W , lifted to M_0 , and write the pulled-back log generator as $\tau_W = \theta_W^m/h$, where θ_W is a canonical generator, obtained by residue in the hypersurface case. Its character is the inflation of $\chi|_H$, with kernel exactly Δ . For every primitive ambient prime T centred at z , the same proposition gives

$$(15.3) \quad \frac{A_{M_0}(T) - T(f) - T(h)/m}{|I_{\tilde{H}}(T)|} \geq a_P \geq a,$$

with $T(f)$ omitted in the smooth case. The intrinsic inequality on (W, B_W) follows from Lemma 3.8.

Normalize the reduced inverse image of the curve germ $C \ni P$ in W , and call the resulting normal curve S . Over the generic point of C , the canonical cover is a Δ -torsor: Y is smooth there, and its canonical cover is etale. Moreover the quotient S/Δ is the curve germ $C \ni P$. To see the latter equality, invariants of the reduced inverse image give $\mathcal{O}_{C,P}$; an invariant in its normalization is integral over $\mathcal{O}_{C,P}$ and belongs to $k(C)$, so normality of C puts it in $\mathcal{O}_{C,P}$. Consequently Δ acts transitively on the points of S above P .

Choose one such point ξ and let $L = \text{Stab}_{\tilde{H}}(\xi)$. Transitivity gives a surjection $\pi_L : L \rightarrow H$. Put $b = |\Delta \cap L|$, so $b \leq \ell \leq m$, and denote the inflated log character by $\chi_L = (\chi|_H) \circ \pi_L$. The action of L on the smooth curve germ $S \ni \xi$ is faithful. An element in its kernel acts trivially on C , hence maps to the identity in H ; it then fixes a geometric generic point of the Δ -torsor and is itself the identity. A finite-order automorphism of a smooth curve germ with derivative one is the identity: its first nonzero higher term would be multiplied by its order under iteration. Thus L is cyclic and its tangent character ψ is faithful. If θ denotes the H -character of a parameter of C at P , tame ramification on the normalized curve gives

$$(15.4) \quad 1 \longrightarrow \mu_b \longrightarrow L \longrightarrow H \longrightarrow 1, \quad \psi^b = \theta \circ \pi_L, \quad b \leq m.$$

Indeed the pulled-back parameter has order b at ξ , so its leading term has character ψ^b .

Step 3. In this step, we construct an algebraic coordinate-root chart straightening the selected branch. Choose ambient eigenparameters at z . One of them, say v , has finite order e on $S \ni \xi$, with $1 \leq e \leq b$. The pullback of a parameter of C has order b and lies in the ideal generated by these parameters, so their least finite order is at most b . Its leading term also shows that the character of v on L is ψ^e . Take the coordinate-root cover

$$M_1 \longrightarrow M_0, \quad v = t^e.$$

It is smooth at the point over z , with coordinates t and the remaining ambient parameters. The action $\gamma^*t = \psi(\gamma)t$ lifts L . Together with the root deck group it gives $\Gamma = \mu_e \times L$, with quotient $M_1/\Gamma = M_0/L$. The ramification divisor is exactly $(e-1)(t=0)$.

In the normalization of the base change of S , there are e points over ξ , unramified over S , and t is a parameter at each. Each is fixed by the chosen lift of L : if u is an eigenparameter on S of character ψ , the nonzero residues of t/u distinguish these points and are fixed by L . Choose one point and an invariant smooth curve neighbourhood U on which $t : U \rightarrow \mathbb{A}^1$ is etale. The fibre product

$$M_2 = M_1 \times_{\mathbb{A}_t^1} U$$

is an equivariant etale neighbourhood of M_1 at the selected point. The graph of $U \rightarrow M_1$ is a smooth embedded curve D in M_2 . Subtracting from each remaining coordinate its restriction to U gives regular eigenparameters t, x_1, x_2, x_3 with $D = (x_1 = x_2 = x_3 = 0)$ in the hypersurface case. In the smooth case there are just t, x_2, x_3 .

Continue to write f, h for the full pulled-back equations. The component of $f = 0$ through the generic point of D is smooth there and has multiplicity one. On this component the generic normal order of h along D is the integer j in (15.1), since both finite covers are etale at the generic point of this branch. The retraction $M_2 \rightarrow U$ lets us write the linear normal part of f over the discrete valuation ring of U . Divide its coefficients by their smallest t -order N , and use a unit coefficient to replace one normal parameter. This is an invertible equivariant change of parameters and gives

$$(15.5) \quad f = t^N x_1 + G_2, \quad G_2 \in (x_1, x_2, x_3)^2, \quad N \geq 0.$$

All coefficient quotients are regular functions on U . In particular, writing characters additively here, $\text{char}(f) = N\psi + \text{char}(x_1)$.

Step 4. In this step, we retain ramification and primitive normalization in the ambient and intrinsic discrepancy bounds. On M_1 , the crepant pullback of the ambient divisor is

$$\text{div}(f) + \frac{1}{m} \text{div}(h) - (e-1) \text{div}(t),$$

again omitting f in dimension three. For a primitive prime T_1 over M_1 , its primitive restriction T_0 to M_0 , and their ramification index r_T , finite discrepancy comparison and the tower of inertia groups give

$$\begin{aligned} A_{M_1}(T_1) - T_1(f) - T_1(h)/m + (e-1)T_1(t) &= r_T(A_{M_0}(T_0) - T_0(f) - T_0(h)/m), \\ |I_\Gamma(T_1)| &= r_T |I_L(T_0)|. \end{aligned}$$

These are the finite discrepancy identities of Lemma 3.8, with the displayed different retained in the pullback. The numerator is positive by (15.3). Dividing by the inertia of the selected subgroup $L \leq \Gamma$ therefore preserves the lower bound a . Etale restriction has ramification one and injects the new inertia group into the old one. Consequently, on M_2 , every primitive ambient prime T at its selected point satisfies

$$(15.6) \quad \frac{A_{M_2}(T) - T(f) - T(h)/m + (e-1)T(t)}{|I_L(T)|} \geq a.$$

We also use the intrinsic version on the normalization of the selected component of $f = 0$. To obtain it, normalize the full finite inverse image of W in M_1 . The stabilizer in Γ of a component has invariant field $k(W)^L$ and contains the chosen lift of L on the component through our branch. Its action is faithful: a kernel element is trivial on $k(W)$, hence is a root deck element, and fixing the nonzero function t makes it the identity. Finite discrepancy comparison for this full stabilizer, followed by restriction to L , gives the same lower bound. Normalization commutes with the etale base change used above. Thus, on any subsequent birational chart, if $A_*(T)$ denotes discrepancy for the possibly signed crepant pullback of (W, B_W) , then

$$(15.7) \quad \frac{A_*(T)}{|I_L(T)|} \geq a$$

for every primitive prime centred over the selected point. All restrictions here are to the original finite function fields; the only intervening etale restriction has ramification one.

We now explain the normalization for the rational weights we will use. Let $\gamma \in L$ have order $d > 1$, and suppose the phase of $\psi(\gamma)$ is δ , with $0 < \delta < \epsilon$. In a smooth eigenparameter chart, take positive rational weights congruent to the coordinate characters at γ , with weight δ on t . Since ψ is faithful, δ has denominator exactly d . Let c be the greatest common divisor of the integer weights obtained by multiplying by d . Then $\gcd(c, d) = 1$, and the primitive monomial prime has weights d/c times the chosen weights. It is an algebraic prime on the normalized weighted blowup in these regular parameters. The action of γ on the associated graded ring is weighted scalar, so it fixes the degree-zero fraction field and belongs to the inertia group of this prime. If Λ denotes its discrepancy in the rational normalization, either (15.6) or (15.7) therefore gives

$$(15.8) \quad \frac{d}{c}\Lambda \geq a|I_L(T)| \geq ad, \quad \Lambda \geq ca \geq a.$$

Step 5. In this step, we construct a short-weight valuation at a pole with the original natural character. Suppose that the phases of γ on the other parameters have representatives α_i with $|\alpha_i| < \epsilon$. In the hypersurface case put

$$w_0 = (\delta, w_1, w_2, w_3), \quad w_i = 1 + \alpha_i, \quad \beta = \min(w_2, w_3), \quad u_0 = N\delta.$$

(15.6) and (15.8) give

$$\Lambda_0 = e\delta + w_1 + w_2 + w_3 - w_0(f) - w_0(h)/m \geq a.$$

If $u_0 \geq \beta$, (15.5) implies

$$w_0(f) \geq \min\{u_0 + w_1, 2 \min(w_1, w_2, w_3)\} \geq 2 \min(w_1, w_2, w_3).$$

Since h is regular, this would give

$$\Lambda_0 < 1 + (e+5)\epsilon \leq 1 + \frac{m+5}{R} < 1 + \frac{1}{m} \leq a,$$

a contradiction. Hence $u_0 < \beta$.

Take the t^N -chart of the blowup of (t^N, x_1, x_2, x_3) , interpreted as the identity when $N = 0$. Write $x_i = t^N y_i$. The full transformed equation, after removing its t^{2N} factor, is

$$f_N = t^{-2N} f = y_1 + G_N, \quad G_N \in (y_1, y_2, y_3)^2.$$

The selected strict transform is therefore smooth at the fixed origin, with regular eigenparameters t, y_2, y_3 . It is the component containing the strict transform of D . The pullback of the original canonical generator is

$$(15.9) \quad \theta_W = (\text{invariant unit}) t^{N+e-1} dt \wedge dy_2 \wedge dy_3.$$

Indeed the root map contributes t^{e-1} , and the ambient substitution contributes t^{3N} to the top form and t^{2N} to f . Residue adjunction supplies the remaining factor t^N . The units from the equivariant parameter changes are semi-invariant units at fixed points, so their characters are trivial. This calculation uses the full pulled-back equation, including any other components meeting the selected component.

Give t, y_2, y_3 the positive weights $\delta, w_2 - u_0, w_3 - u_0$, respectively, and let w be the resulting monomial valuation. These weights are congruent to their characters at γ . By (15.9), their crepant discrepancy is

$$\Lambda = e\delta + w_2 + w_3 - u_0 - w(h)/m \geq a,$$

where the lower bound follows from (15.7) and (15.8). If $j \geq 1$, the full equation h vanishes on D , so $h \in (x_1, x_2, x_3)$ before the blowup and $h = t^N h_1$ with h_1 regular afterward. The generic normal order of h_1 along the strict curve is j . On this smooth chart the powers of the curve ideal (y_2, y_3) are saturated with respect to nonzero functions on the curve: their successive normal graded pieces are free over its regular local ring. Thus $h_1 \in (y_2, y_3)^j$, and

$$w(h) \geq u_0 + j \min(w_2 - u_0, w_3 - u_0) = j\beta - (j-1)u_0.$$

It follows that

$$\begin{aligned} \Lambda &\leq e\delta + w_2 + w_3 - \frac{j}{m}\beta - \left(1 - \frac{j-1}{m}\right)u_0 \\ &< a + (e+2+j/m)\epsilon < a + (e+3)\epsilon \leq a + (m+3)\epsilon. \end{aligned}$$

Here $j < m$ makes the coefficient of u_0 positive. When $j = 0$, regularity of h instead gives $\Lambda < a + (e+2)\epsilon$.

If W is smooth, no hypersurface equation or birational chart is needed. In the straightened chart t, x_2, x_3 , saturation of the smooth curve ideal gives $h \in (x_2, x_3)^j$. Let w be the monomial valuation with weights $\delta, 1 + \alpha_2, 1 + \alpha_3$. Its discrepancy satisfies

$$\Lambda = e\delta + w_2 + w_3 - w(h)/m, \quad a \leq \Lambda < a + (e+3)\epsilon.$$

Thus both cases give

$$(15.10) \quad a \leq \Lambda < a + (m+3)\epsilon.$$

The character of this discrepancy is still the original one. In the hypersurface case the tensor in (15.9), divided by h after taking its m -th power, gives

$$\chi_L = m((e-N)\psi + \text{char}(x_2) + \text{char}(x_3)) - \text{char}(h)$$

in additive character notation. In the smooth case the same formula has $N = 0$. Every term of a semi-invariant has its character, so a monomial valuation whose weights have the coordinate phases takes that phase modulo $\mathbb{Z}$. Applying this to the displayed tensor yields

$$(15.11) \quad m\Lambda \equiv \text{phase}(\chi(\bar{\gamma})) \pmod{\mathbb{Z}},$$

where $\bar{\gamma} = \pi_L(\gamma)$ is the image in H .

Step 6. In this step, we pack the characters at both fixed points and bound the full original stabilizer. Perform these constructions at P_+ and P_- , obtaining L_+, L_- , tangent characters ψ_+, ψ_- , and kernel orders $b_+, b_- \leq m$. Form the finite abelian fibre product

$$A = L_+ \times_H L_-.$$

List on A the coordinate characters of the two straightened charts, after the linear normal changes leading to (15.5). There are at most eight characters. If $|H| > R^8$, Lemma 8.9, with the excluded

subgroup equal to $\{1\}$, supplies $\eta \in A$ whose image $h_0 \in H$ is nonidentity and all of whose listed signed phases have absolute value less than ϵ .

Let v_+, v_- be its two tangent phases. The inverse tangent characters on C and (15.4) give

$$b_+v_+ + b_-v_- \in \mathbb{Z}, \quad |b_+v_+ + b_-v_-| < 2m/R < 1.$$

This integer is zero. Neither phase is zero, since each projection of η has nonidentity image h_0 and its tangent character is faithful. The phases therefore have opposite signs. Replace η by its inverse if necessary so that $v_+ > 0$. Use its projection at P_+ and the inverse projection at P_- . Both tangent phases are positive, so (15.10) gives numbers Λ_+, Λ_- with

$$0 \leq \Lambda_{\pm} - a < (m+3)/R.$$

Their characters in (15.11) are $\chi(h_0)$ and its inverse. Since ma is integral,

$$m((\Lambda_+ - a) + (\Lambda_- - a)) \in \mathbb{Z}, \quad 0 \leq m((\Lambda_+ - a) + (\Lambda_- - a)) < \frac{2m(m+3)}{R} < 1.$$

Both nonnegative differences vanish. (15.11) then gives $\chi(h_0) = 1$, contrary to the faithfulness of the original character on H . Hence $|H| \leq R^8 = Q(m)$. The field comparison already invoked preserves the prescribed valuation, its stabilizer and the model, so the bound holds over the original algebraically closed field as well. $\square$

16. EXACT MINIMA GREATER THAN ONE

We retain the original cover and natural character of Section 3. We write $P(m)$ for the bound in Proposition 9.8 and $Q(m)$ for the bound in Theorem 15.1. We first separate the selection of a computing divisor from the bound on its stabilizer.

The results over $\mathbb{C}$ used below are terminal smoothness in codimension two, the isolated cDV description, the ordinary point-blowup formula, and relative vanishing. We use them in the precise forms of Theorems 7.6, 7.1, 7.7, and 7.8. For an arbitrary algebraically closed characteristic-zero field, apply Lemma 3.11 to the original cover, its natural line and a chosen equivariant terminal model. The lemma retains the full fibre and all primitive-prime comparisons on a common realization which also embeds in $\mathbb{C}$. Smoothness, dimension of the singular locus, and hypersurface multiplicity are preserved by these extensions; cohomology on the proper fibre and the degrees and lengths used below are preserved by flat base change. Thus each application of these results is made after this realization. The final index bounds return by the same lemma, and smoothness descends under field extension. All covers and valuations in the arguments remain those of the realized original diagram.

Lemma 16.1. *Let original cover data have exact minimum $a = p/q > 1$, with mB integral and $q \mid m$. Let $t \geq 1$ be an integer. If the prescribed computing lift F satisfies $[G : \text{Stab}_G(F)] \leq t$, then*

$$|G| \leq t \max\{P(m), Q(m)\}, \quad \text{ind}_x(K_X + B) \leq mt \max\{P(m), Q(m)\}.$$

Thus the stabilizer estimates apply after the computing divisor has been selected on the original cover.

Proof. Put $H = \text{Stab}_G(F)$. Choose an H -equivariant terminal pair model $g : (Y, B_Y) \rightarrow (V, B_V)$ by Proposition 7.5. Its natural m -log character is faithful, and its local canonical indices are at most m . By Theorem 3.9,

$$A_{Y, B_Y}(F) = a, \quad I_H(F) = 1.$$

For every primitive prime T with centre in $g^{-1}(o)$, Lemma 3.8 and crepancy give

$$(16.1) \quad \frac{A_{Y, B_Y}(T)}{|I_H(T)|} \geq \frac{A_{Y, B_Y}(T)}{|I_G(T)|} \geq a.$$

The valuation action of G in the middle expression is its action on $k(V)$; it does not require a G -action on this H -model.

The centre of F is a point or a complete curve by Lemma 7.12. In the point case it is H -fixed, and Proposition 9.8 gives $|H| \leq P(m)$. In the curve case, Theorem 15.1 gives $|H| \leq Q(m)$. The identities $|G| = [G : H]|H|$ and $\text{ind}_x(K_X + B) \leq m|G|$ prove the claims. $\square$

Lemma 16.2. *Let original cover data have exact minimum $a > 1$, and let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a full G -equivariant terminal pair model from Proposition 7.5. Set*

$$Z = g^{-1}(o)_{\text{red}}, \quad E_m = \binom{m+4}{4} - 1.$$

Then $\text{embdim}_P Y \leq E_m$ for every closed point $P \in Z$. If Z is one-dimensional, the tangent lines of its components at any junction are linearly independent in $T_P Y$, and at most E_m components meet there.

The branch bound concerns the reduced fibre on the given model; it does not require a replacement by a normal crossings curve.

Proof. We bound the ambient cotangent dimension and then identify the local ring of the reduced fibre.

Step 1. In this step, we bound the ambient cotangent dimension using the local canonical cover. Fix $P \in Z$, and put $\ell = \text{ind}_P(K_Y) \leq m$. Apply Proposition 8.8 with the trivial group and with the exact minimum of $(Y \ni P, B_Y)$. This minimum is positive and attained by Theorem 3.9. The canonical cover has an equivariant smooth ambient germ of dimension $d \leq 4$. In completed eigenparameters $u_1, \dots, u_d$, the completed local ring of Y is a quotient of

$$k[[u_1, \dots, u_d]]^{\mu_\ell}.$$

Indeed, taking invariants is exact in characteristic zero, so it commutes with passing to the semi-invariant hypersurface quotient when that quotient is present.

Every invariant monomial of degree greater than ℓ has a proper nonconstant invariant factor of degree at most ℓ . To see this, list ℓ of its variable characters in $\mathbb{Z}/\ell\mathbb{Z}$. Among their $\ell + 1$ partial sums two coincide. Their difference gives the required factor. Induction on the degree shows that invariant monomials of degrees one through ℓ topologically generate the invariant ring. There are at most

$$\binom{\ell + d}{d} - 1 \leq E_m$$

such monomials. Passing to a quotient and then to its cotangent space cannot increase this number. Completion preserves the cotangent dimension, proving the first assertion.

Step 2. In this step, we identify the reduced fibre's local ring and prove independence of its tangent lines. Suppose that Z has c components $C_1, \dots, C_c$. They are smooth rational curves by Lemma 7.12. For the normalization $\nu : \coprod_i C_i \rightarrow Z$, put $\mathcal{Q} = \nu_* \mathcal{O}_{\coprod_i C_i} / \mathcal{O}_Z$. The same lemma gives $H^0(Z, \mathcal{O}_Z) = k$ and $H^1(Z, \mathcal{O}_Z) = 0$, whence

$$\sum_{P \in Z} \text{length } \mathcal{Q}_P = c - 1.$$

If r_P branches meet at P , taking their residues gives a surjection $\mathcal{Q}_P \rightarrow k^{r_P}/k$, with diagonal constants in the denominator. Thus its length is at least $r_P - 1$. The incidence tree gives $\sum_P (r_P - 1) = c - 1$, so equality holds at every junction. Consequently

$$\hat{\mathcal{O}}_{Z,P} = \{(h_1, \dots, h_{r_P}) \in \prod_i k[[t_i]] : h_1(0) = \dots = h_{r_P}(0)\}.$$

Its maximal ideal is the direct sum of the branch maximal ideals, and products from different summands vanish. Its cotangent dimension is therefore r_P . The closed immersion $Z \subset Y$ proves both tangent independence and $r_P \leq \text{embdim}_P Y \leq E_m$. $\square$

Lemma 16.3. *Let original cover data have exact minimum $a > 1$, and let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a full equivariant terminal pair model. Put $Z = g^{-1}(o)_{\text{red}}$. Every closed subscheme $T \subset Y$ supported in Z satisfies*

$$H^1(T, \mathcal{O}_T) = 0.$$

This includes the symbolic second thickening of a component and its union with the reduced fibre.

Proof. By the field reduction above we may use Theorem 7.8 [Fuj09b, Theorem 2.48 and Corollary 2.49, p. 55] over $\mathbb{C}$. Shrink V to an affine neighbourhood of o over which all fibres have dimension at most one, as permitted by Lemma 7.12 and upper semicontinuity. The divisor $-(K_Y + B_Y)$ is pulled back from V , hence is g -nef. It is g -big because g is birational; its generic fibre is a point. The pair is klt, so there are no lc centres to test. Applying the cited vanishing theorem to the integral Cartier divisor zero gives $R^i g_* \mathcal{O}_Y = 0$ for $i > 0$, and therefore $H^1(Y, \mathcal{O}_Y) = 0$.

We also have $H^2(Y, \mathcal{I}_T) = 0$. Here is a direct argument in this relative dimension. Choose a general relative hyperplane divisor D_0 containing no component of the fibre. After shrinking V again, its proper morphism to V is quasi-finite and hence finite. The same is true of nD_0 . For n sufficiently large, relative Serre vanishing and affineness of V give $H^2(Y, \mathcal{I}_T(nD_0)) = 0$. The quotient in

$$0 \longrightarrow \mathcal{I}_T \longrightarrow \mathcal{I}_T(nD_0) \longrightarrow \mathcal{Q}_n \longrightarrow 0$$

is supported on the finite V -scheme nD_0 , so $H^1(Y, \mathcal{Q}_n) = 0$. This proves the asserted H^2 -vanishing. Finally the cohomology sequence of $0 \rightarrow \mathcal{I}_T \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_T \rightarrow 0$ gives $H^1(T, \mathcal{O}_T) = 0$. $\square$

Definition 16.4. Let Y be a terminal threefold and $C \subset Y$ a smooth complete curve whose generic point lies in the smooth locus of Y . Write $\mathcal{I}_C^{(j)}$ for the symbolic power, obtained by saturating $\mathcal{I}_C^j$ at the generic point of C . Saturation of $\mathcal{I}_C \omega_Y$ in ω_Y likewise means the subsheaf of sections whose image in $\omega_Y / \mathcal{I}_C \omega_Y$ vanishes generically along C . Set

$$\mathcal{F}_C = \mathcal{I}_C / \mathcal{I}_C^{(2)}, \quad \mathcal{W}_C = \omega_Y / \text{sat}_{\omega_Y}(\mathcal{I}_C \omega_Y).$$

These are torsion-free sheaves on C , of ranks two and one. Let

$$\alpha_C : \det \mathcal{F}_C \otimes \omega_C \longrightarrow \mathcal{W}_C$$

be the map induced by top differentials. For any positive integer r with rK_Y Cartier near $P \in C$, let

$$\mu_{C,r} : \mathcal{W}_C^{\otimes r} \longrightarrow \mathcal{O}_Y(rK_Y)|_C$$

be the map induced by multiplication of canonical forms. The *conormal correction* at P is

$$\delta_P(C) = i_P(C) + w_P(C), \quad i_P(C) = \text{length}_P \text{coker } \alpha_C, \quad w_P(C) = \frac{1}{r} \text{length}_P \text{coker } \mu_{C,r}.$$

Lemma 16.5. *Let original cover data have exact minimum $a \in (1, 2)$, and let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a full equivariant terminal pair model with one-dimensional reduced fibre Z . Set*

$$R = \left\lfloor \frac{1}{a-1} \right\rfloor, \quad D = \left\lceil 1 + \frac{3}{a-1} \right\rceil.$$

For a component C of Z , put $\beta_C = \text{mult}_C B_Y$, and let S_C consist of its junctions, the points of $C \cap \text{Sing } Y$, and its smooth ambient points with $\text{mult}_P B_Y > 1$. Then:

- (i) The corrections of Definition 16.4 are well defined and nonnegative. Their support is exactly $C \cap \text{Sing } Y$, and

$$\begin{aligned} 0 \leq \beta_C \leq 2 - a < 1, \quad -4 \leq K_Y \cdot C \leq \frac{3\beta_C}{1 - \beta_C}, \\ \deg \mathcal{F}_C = K_Y \cdot C + 2 - \sum_{P \in C} \delta_P(C), \\ \delta_P(C) \geq \frac{1}{m} \quad (P \in C \cap \text{Sing } Y), \quad \sum_{P \in C} \delta_P(C) \leq 1 + \frac{3}{1 - \beta_C} \leq D. \end{aligned}$$

- (ii) There are at most mD singular ambient points on C , at most D smooth ambient junctions, and at most D smooth ambient points with boundary multiplicity greater than one. The set S_C has at most $(m + 2)D$ points and contains every point of C which is the centre of an original computing lift.
- (iii) Every subgroup of G preserving C has pointwise kernel on C of order at most R . Every original computing lift whose centre on Y is C is its ordinary generic blowup valuation.

The estimates apply to every component of the full fibre. In particular, the marked set is controlled before a computing component is selected.

Proof. We construct the intrinsic corrections, control the boundary's leading normal form, and then bound the marked points and the component-action kernel.

Step 1. In this step, we define the conormal correction maps and compute their degree identities. Theorem 7.6, with the field reduction above, makes Y smooth at the generic point of C . The ordinary transverse blowup and (16.1) give $2 - \beta_C \geq a$. In particular $0 \leq \beta_C < 1$. Put

$$M = m \text{ lcm}(1, \dots, m), \quad k_C = M\beta_C.$$

The local canonical index bound in Proposition 7.5 makes both MK_Y and MB_Y Cartier near the fibre. The sum $M(K_Y + B_Y)$ is pulled back from a trivial line on the germ. Thus k_C is an integer with $0 \leq k_C < M$, and

$$\deg \mathcal{O}_Y(MB_Y)|_C = -MK_Y \cdot C.$$

On the smooth curve C , the torsion-free sheaves $\mathcal{F}_C$ and $\mathcal{W}_C$ are locally free. For local lifts x, y of conormal sections and a curve parameter s , the map α_C sends $(x \wedge y) \otimes ds$ to $dx \wedge dy \wedge ds$. The map from top differentials to the reflexive canonical sheaf is regular: it is regular on the smooth locus and extends to the reflexive sheaf across codimension two. Altering a lift by a symbolic square gives a section vanishing at the generic point of C , hence zero in $\mathcal{W}_C$. This proves that α_C is a regular, generically invertible map. The same saturation argument defines the regular, generically invertible map $\mu_{C,r}$. Replacing r by a multiple multiplies its local vanishing order by that multiple, so $w_P(C)$ is independent of r .

Write $i_P = i_P(C)$, $w_P = w_P(C)$, and $\delta_P = \delta_P(C)$. Taking degrees, using $C \simeq \mathbb{P}^1$, gives

$$(16.2) \quad \deg \mathcal{W}_C = K_Y \cdot C - \sum_P w_P, \quad \deg \mathcal{F}_C = K_Y \cdot C + 2 - \sum_P \delta_P.$$

All corrections are nonnegative, finitely supported, and zero at smooth ambient points. The exact sequence of the symbolic second thickening of C , together with Lemma 16.3, gives $H^1(C, \mathcal{F}_C) = 0$: constants lift to the thickening. Consequently

$$\mathcal{F}_C \simeq \mathcal{O}_C(d_1) \oplus \mathcal{O}_C(d_2), \quad d_1, d_2 \geq -1.$$

In particular

$$(16.3) \quad \deg \mathcal{F}_C \geq -2, \quad \max(d_1, d_2) \leq \deg \mathcal{F}_C + 1, \quad \sum_P \delta_P \leq K_Y \cdot C + 4.$$

Step 2. In this step, we compute the correction determinant on an actual normalized branch. Fix $P \in C$. Apply Proposition 8.8 with the trivial group and its own positive attained local

minimum. It supplies the actual canonical cover $\pi : W \rightarrow Y$, of degree $\ell = \text{ind}_P(K_Y) \leq m$, and a smooth ambient chart. Normalize the reduced inverse image of C , and choose a point ξ above P . The deck group $\Delta = \mu_\ell$ is generically a torsor on this preimage, since Y is smooth generically along C . Its invariant normalization is C : the invariant ring is integral over $\mathcal{O}_C$, lies in $k(C)$, and C is normal. The deck group therefore acts transitively on the points above P . The ramification index on the chosen normalized branch is $f_P = |\text{Stab}_\Delta(\xi)| \leq \ell$.

Some regular ambient parameter v has finite positive order $e_P \leq f_P$ on this branch. Indeed, a lifted parameter of C has order f_P and belongs to the ideal generated by the ambient parameters. Take the actual coordinate-root cover $v = t^{e_P}$. The normalization of the branch base change is unramified at a point above ξ , and t is its uniformizer: in the complete branch DVR, v is a unit times an e_P -th power of a uniformizer, and the unit has an e_P -th root in an étale neighbourhood. On an open normalized branch U , the map $t : U \rightarrow \mathbb{A}^1$ is étale. Base change the ambient chart along this map and subtract the restrictions of its other coordinates to U . The graph is now a smooth embedded curve D_P , the t -axis. These are algebraic coordinate-root and étale maps; no formal branch is substituted for an original field.

In the hypersurface case let Φ be the full pulled equation, including any other components. Its linear normal coefficients generate an ideal (t^{N_P}) in the curve DVR. Row operations on the normal parameters put it in the form

$$\Phi = t^{N_P}u + \text{terms in } (u, v_1, v_2)^2.$$

It is generically smooth along D_P , so N_P is finite. Its torsion-free conormal $\mathcal{F}_{D_P}$ is free on v_1, v_2 . In the smooth-cover case use the ordinary conormal and put $N_P = 0$. The map induced by functions,

$$\mathcal{F}_C \otimes \mathcal{O}_{D_P} \longrightarrow \mathcal{F}_{D_P},$$

is injective and generically invertible. Symbolic squares map to generic normal order at least two and hence to zero in the torsion-free target. Let $L_P \geq 0$ be the cokernel length.

Let η be a local canonical section inducing a generator of $\mathcal{W}_C$, and write $\pi^*\eta = g_P\theta_W$, where θ_W generates ω_W . The coefficient g_P is regular, by normality and regularity away from the isolated cover point. The definition of $\mu_{C,r}$ gives $\text{ord}_t(g_P|_{D_P}) = f_P w_P$. Residue adjunction and the coordinate-root differential give, at the generic point of D_P ,

$$\theta_W = \text{unit} \cdot t^{e_P-1-N_P} dt \wedge dv_1 \wedge dv_2.$$

The derivative of a parameter of C has order $f_P - 1$, and the determinant of the conormal map has order L_P . Comparing the pullback of α_C with this expression therefore gives

$$(16.4) \quad N_P + L_P = f_P \delta_P - f_P + e_P \leq f_P \delta_P.$$

At a non-Gorenstein point the canonical deck character is faithful and nontrivial, by Theorem 3.9 for the bare canonical cover. The invariant section $\pi^*\eta$ makes g_P have its inverse character, so g_P vanishes at the cover point. Its nonzero branch restriction gives $w_P \geq 1/f_P \geq 1/m$. At a singular Gorenstein point, the cover is trivial and the embedded smooth curve can be straightened without a root cover. Then $f_P = e_P = 1$, $L_P = 0$, and singularity of the ambient hypersurface gives $N_P \geq 1$. Equation (16.4) gives $\delta_P = N_P \geq 1$. This proves the asserted support and the lower bounds for the corrections.

Step 3. In this step, we bound the poles of the leading boundary form and deduce the component degree estimates. Put $\mathcal{L}_B = \mathcal{O}_Y(MB_Y)|_C$. The degree k_C leading normal form of the boundary section is a nonzero rational section of $\text{Sym}^{k_C} \mathcal{F}_C \otimes \mathcal{L}_B$. We claim that it extends to a regular section of

$$(16.5) \quad \text{Sym}^{k_C} \mathcal{F}_C \otimes \mathcal{L}_B \otimes \mathcal{O}_C \left(\sum_P [k_C \delta_P] P \right).$$

Use the actual local chart just constructed, and lift a regular equation of MB_Y to its ambient chart. Over $k((t))$, solve $\Phi = 0$ for u . The solution starts in normal degree two. Its degree j

coefficients have poles of order at most $(j-1)N_P$: substitution into a monomial with b other normal factors and c occurrences of u costs at most $(j-b-c+1)N_P$, including the single division by t^{N_P} , and $b+c \geq 2$. This proves the bound inductively in j .

The leading degree k_C form of the boundary equation therefore has poles at most $k_C N_P$ in the conormal basis on D_P . The inverse conormal matrix has poles at most L_P , by its adjugate and determinant. Transforming the degree k_C tensor back costs at most $k_C L_P$. Lower normal degrees of the boundary vanish identically, so only this linear matrix affects the leading form. The resulting pole order on D_P is at most

$$k_C(N_P + L_P) \leq k_C f_P \delta_P.$$

A pole of order j on C pulls back with order $f_P j$, which proves (16.5). The argument also applies when $k_C = 0$ or when the cover is smooth.

Using (16.2) and (16.3), each line summand of (16.5) has degree at most

$$(16.6) \quad k_C(\deg \mathcal{F}_C + 1) - M K_Y \cdot C + \sum_P [k_C \delta_P] \leq 3k_C - (M - k_C) K_Y \cdot C.$$

One summand has a nonzero section, so the right side is nonnegative. This gives $K_Y \cdot C \leq 3\beta_C/(1-\beta_C)$. Equations (16.2) and (16.3) now give

$$K_Y \cdot C \geq -4, \quad \sum_P \delta_P \leq 4 + \frac{3\beta_C}{1-\beta_C} = 1 + \frac{3}{1-\beta_C} \leq D.$$

This proves (i), and bounds the number of singular ambient points by mD .

Step 4. In this step, we bound the junctions, singular points, and smooth points of large boundary multiplicity. Let $\mathcal{E}_C$ be the image of $\mathcal{I}_Z$ in $\mathcal{F}_C$. The exact sequence for the union of Z with the symbolic second thickening of C has kernel $\mathcal{E}_C$ and quotient $\mathcal{O}_Z$. Constants lift, so Lemma 16.3 gives $H^1(C, \mathcal{E}_C) = 0$. It is a rank-two vector bundle and has degree at least -2 . At a smooth ambient junction with r_P branches, the independent tangent lines from Lemma 16.2 impose at least $r_P - 1$ independent conditions on its image in $\mathcal{F}_C|_P$. Thus

$$\sum_{P \text{ smooth junction}} (r_P - 1) \leq \text{length}(\mathcal{F}_C/\mathcal{E}_C) \leq \deg \mathcal{F}_C + 2 = K_Y \cdot C + 4 - \sum_P \delta_P \leq D.$$

At a smooth ambient point with $\text{mult}_P B_Y > 1$, a local equation of MB_Y has ordinary order at least $M+1$. Its degree k_C normal form vanishes along C there to order at least $M+1-k_C$. The correction divisor in (16.5) is zero at this point. Choose a nonzero scalar section after splitting that bundle. Its degree, by (16.6) and $K_Y \cdot C \geq -4$, is at most $4M - k_C$. The number of these points is consequently at most

$$\left\lfloor \frac{4M - k_C}{M + 1 - k_C} \right\rfloor \leq \frac{4 - \beta_C}{1 - \beta_C} = 1 + \frac{3}{1 - \beta_C} \leq D.$$

At every other smooth ambient point, Theorem 7.7 [Kaw17, Proposition 6], applied to the local Cartier boundary as an effective rational ideal, gives $\text{mld}(Y \ni P, B_Y) = 3 - \text{mult}_P B_Y \geq 2$. An original computing lift has discrepancy $a < 2$, so its point centre is in S_C . Counting singular points, smooth junctions, and the remaining marked smooth points proves (ii).

Step 5. In this step, we bound the component-action kernel and identify the original curve-centred computing valuations. Let $J \leq G$ fix C pointwise. At its smooth generic point choose two eigen normal parameters. A lifted rational parameter on C can be averaged to be J -invariant. These parameters identify the completed transverse action with its faithful two-dimensional linear representation. If an element of order n has both normal eigenvalues nontrivial, its positive fractional weights and those of its inverse have coordinate sums adding to two. One sum is at most one. The primitive integral weights on that ray give an actual transverse weighted blowup prime; the element is in its inertia. Dividing the smooth bare discrepancy by this inertia gives at most one, and subtracting the effective boundary only decreases it. Its centre on Y is C , contrary to (16.1) and $a > 1$. Thus J is the union of the two normal-character kernels. A group cannot

be the union of two proper subgroups: if neither contains the other, multiplying an element in each of their set differences gives an element in neither. One character is therefore trivial on J , and faithfulness makes the other faithful. If $J \neq 1$, choose a generator whose nontrivial phase is $1/|J|$. The weights $(1, 1/|J|)$ give in the same way $a \leq 1 + 1/|J|$. Hence $|J| \leq R$; for $J = 1$ this holds because $a < 2$.

Finally, the transverse smooth surface has boundary multiplicity $\beta_C < 1$. Its ordinary point blowup uniquely computes $2 - \beta_C$. One may apply the strict clause of Theorem 7.7, or check it over $k(C)$ by successive point blowups: strict boundary multiplicities never exceed β_C . A subsequent blowup at a point on one exceptional curve of discrepancy b_1 has discrepancy $b_1 + 1 - \beta'$, while at an intersection of two such curves it has discrepancy $b_1 + b_2 - \beta'$, with $\beta' \leq \beta_C < 1$. Starting with $2 - \beta_C > 1$, both formulas are strictly larger than each exceptional discrepancy at that centre. Geometric constant extension is unramified and leaves these discrepancies unchanged. For an original computing lift with centre C , the lower comparison and its discrepancy a force $2 - \beta_C = a$. It is therefore this unique ordinary prime, which is defined over $k(C)$. This proves (iii). $\square$

Lemma 16.6. *Let original cover data have exact minimum $a \in (1, 2)$, and let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a full equivariant terminal pair model. Put $Z = g^{-1}(o)_{\text{red}}$ and*

$$R = \left\lfloor \frac{1}{a-1} \right\rfloor, \quad E_m = \binom{m+4}{4} - 1, \quad K_0 = 2m + 10 + \left\lceil \frac{8}{a-1} \right\rceil, \\ J(m, a) = 3Rm^2 E_m! K_0^7.$$

For every junction $P \in Z$ with at least three branches,

$$|\text{Stab}_G(P)| \leq J(m, a).$$

The argument allows a non-Gorenstein point and singular embedded lifts of all three branches.

Proof. We retain three normalized branches in the full canonical-cover extension, choose an element with short characters, and use their independent tangent directions to exclude every possible phase pattern.

Step 1. In this step, we retain three normalized branches in the full local canonical-cover extension. Put $H = \text{Stab}_G(P)$, and let H_1 be the kernel of its permutation action on all r_P branches. By Lemma 16.2,

$$[H : H_1] \leq r_P! \leq E_m!.$$

Choose three components C_1, C_2, C_3 through P . They are smooth, so H_1 preserves each entire component. The actual quotient pair $(Y, B_Y)/H_1$ at the image of P has a positive attained minimum $b_P \geq a$, by (16.1) and Theorem 3.9. Apply Proposition 8.8 with this b_P . We obtain the actual canonical cover $W \rightarrow Y$ and an abelian extension

$$1 \longrightarrow \Delta = \mu_\ell \longrightarrow A \longrightarrow H_1 \longrightarrow 1, \quad \ell \leq m,$$

acting on a smooth ambient germ M_0 of dimension d . If W is smooth then $d = 3$ and $M_0 = W$. Otherwise $d = 4$ and $W = (f = 0)$. In the latter case Theorem 7.1 [Rei83, Theorem (1.1), pp. 138–139], used over $\mathbb{C}$ as specified above, makes W an isolated cDV singularity. Its equation has a nonzero quadratic part: a general Cartier hyperplane section is a Du Val surface of multiplicity two, so the hypersurface equation has order two.

Choose actual regular eigenparameters $x_1, \dots, x_d$ for the ambient A -action. Dropping the effective boundary term from the canonical-chart comparison gives, for every primitive ambient prime T centred at the marked point,

$$(16.7) \quad \frac{A_{M_0}(T) - T(f)}{|I_A(T)|} \geq a,$$

with the f -term omitted if $d = 3$.

For each $i \in \{1, 2, 3\}$, normalize the reduced preimage of C_i and choose a point ξ_i above P . As in the normalized-branch construction in Lemma 16.5, its Δ -quotient is C_i , and Δ acts transitively on the points above P . Thus $A_i = \text{Stab}_A(\xi_i)$ surjects onto H_1 : a deck element corrects the image of any lift. Moreover $[A : A_i] \leq \ell$. Put $A_* = A_1 \cap A_2 \cap A_3$. Then

$$(16.8) \quad |A_*| \geq \frac{|A|}{\ell^3} = \frac{|H_1|}{\ell^2} \geq \frac{|H|}{E_m!m^2}.$$

Choose an eigenuniformizer u_i of the smooth normalized branch at ξ_i , and let ψ_i be its A_* -character. A finite-order automorphism of a smooth curve germ with derivative one is the identity: iterating its first nonzero higher term multiplies that term by the order. Consequently $\ker \psi_i$ acts trivially on the branch. Its image in H_1 fixes C_i pointwise and has order at most R , by Lemma 16.5. Its intersection with Δ is trivial, since the canonical cover is a deck torsor over the generic point of C_i . Hence

$$(16.9) \quad |\ker \psi_i| \leq R.$$

Write $f_i \leq \ell$ for the ramification index of the normalized branch over C_i . A curve parameter pulls back with order f_i , and every function from Y restricts with order a multiple of f_i .

Step 2. In this step, we choose short characters and derive the positive-weight discrepancy bound. Suppose that $|H| > J(m, a)$, and put $\epsilon = 1/K_0$. Equation (16.8) gives $|A_*| > 3RK_0^7$. Use the box argument of Lemma 8.9 for the d coordinate characters and the three ψ_i , excluding the union of their three kernels. This union has at most $3R$ elements by (16.9). There are at most $K_0^{d+3} \leq K_0^7$ boxes. One contains more than $3R$ elements, whose differences from a fixed element are distinct; one difference avoids all three kernels. We obtain $\gamma \in A_*$ with signed phases α_j on the coordinates and σ_i on the normalized branches satisfying

$$(16.10) \quad |\alpha_j| < \epsilon, \quad 0 < |\sigma_i| < \epsilon.$$

Inverting γ negates all these phases. The ambient action is faithful, so at least one $\alpha_j \neq 0$. Our choice of K_0 ensures

$$(16.11) \quad (m+1)\epsilon < 1, \quad 6\epsilon < a-1, \quad 6\epsilon < 1.$$

Set $w_j = \alpha_j$ if $\alpha_j > 0$, and $w_j = 1 + \alpha_j$ otherwise. In particular a zero phase has positive weight one. If n is the order of γ , let c be the greatest common divisor of the positive integers nw_j . The weights nw_j/c are primitive and define an algebraic weighted blowup prime in the actual etale parameter chart. The element γ acts by a weighted scalar and belongs to its inertia. Indeed $\gcd(c, n) = 1$, since a common divisor would make a nonidentity power of γ fix all parameters. Its primitive bare discrepancy is $(n/c)(\sum_j w_j - w(f))$, and its inertia has order at least n . Thus (16.7) implies

$$(16.12) \quad \Lambda := \sum_j w_j - w(f) \geq ca \geq a,$$

again omitting $w(f)$ in dimension three.

In dimension four, let δ be the signed weight of a monomial in the nonzero quadratic part of f . All its quadratic monomials have the same real signed weight: their characters agree and two such weights differ in absolute value by less than $4\epsilon < 1$. We have $|\delta| < 2\epsilon$. Every monomial of f has positive w -weight congruent to δ modulo $\mathbb{Z}$. Consequently

$$(16.13) \quad \delta \leq 0 \implies w(f) \geq 1 + \delta.$$

This includes $\delta = 0$, because f has no constant term. If $\delta > 0$, either $w(f) \geq 1 + \delta$, or its weight- δ part is a nonzero polynomial P_δ involving only the positive-phase coordinates. The other coordinates have weight at least $1 - \epsilon > \delta$.

Step 3. In this step, we relate the branch phases to the three independent actual tangent directions. On the i -th normalized branch write $x_j(u_i) = \sum_{s \geq 1} c_{js} u_i^s$. A nonzero coefficient satisfies

$$s\sigma_i - \alpha_j \in \mathbb{Z}.$$

For $1 \leq s \leq f_i \leq m$, (16.11) forces this integer to be zero. Thus through order f_i , only coordinates whose phases have the same sign as σ_i occur.

The invariant-monomial generators constructed in Lemma 16.2 also generate the completed local ring of Y here. Taking the coefficient of $u_i^{f_i}$ in these generators reads the tangent direction of C_i : every product of two elements of the maximal ideal has order at least $2f_i$. If only one ambient coordinate x has the required sign, only its invariant powers can contribute. If its Δ -character has order h , these powers are powers of x^h . The latter is a function on Y , so its nonzero order on the branch is at least f_i . Only x^h can contribute to order f_i ; its higher powers cannot. All such tangent directions therefore lie on one fixed line, independently of the branch. If no coordinate has the required sign, the tangent direction would be zero.

The three tangent lines are independent by Lemma 16.2. It follows that at most one positive and at most one negative phase cannot accommodate these three branches. If there are two positive phases and just one negative phase, at least two branches have positive σ_i , with independent tangent directions.

We need one refinement. Suppose $d = 4$, the phases are positive on x, y , negative on a third coordinate, and zero on a fourth, and suppose

$$\delta = \alpha_x > 0, \quad P_\delta = c_0 y^b, \quad c_0 \neq 0, \quad b \geq 2.$$

For a branch with $\sigma_i > 0$, assume y has a first nonzero term of order $s \leq f_i$. Then $s\sigma_i = \alpha_y$. Put $N_i = bs$, so $N_i\sigma_i = \delta$. For every parameter term of order $t \leq N_i$, the congruence $t\sigma_i - \alpha_j \in \mathbb{Z}$ is again an equality, since its absolute value is less than $3\epsilon < 1$. Modulo $u_i^{N_i+1}$, the negative and zero coordinates vanish, $y = c_y u_i^s$ with $c_y \neq 0$, and x is a multiple of $u_i^{N_i}$. Since f has no linear term, its coefficient of $u_i^{N_i}$ is exactly $c_0 c_y^b \neq 0$, contradicting the inclusion of this actual branch in W . Thus y contributes nothing to the tangent direction of any positive branch. All positive branch tangents then lie on the single line from x . The argument uses $N_i\sigma_i = \delta < 2\epsilon$; it imposes no bound on N_i or on the branch's contact order.

Step 4. In this step, we exclude every phase pattern and conclude the junction-stabilizer bound. If $d = 3$ and all phases are nonzero, invert γ so that at most one is negative. Then $\Lambda < 1 + 3\epsilon < a$, contrary to (16.12). If exactly two phases are nonzero and have the same sign, orient them positively. The zero phase contributes one, giving $\Lambda < 1 + 2\epsilon < a$. If these two phases have opposite signs, or if there is at most one nonzero phase, the tangent-direction argument excludes three independent branches.

Now suppose $d = 4$. If all four phases are nonzero and one orientation has at most one negative phase, then $\Lambda \leq \sum_j w_j < 1 + 4\epsilon < a$. Otherwise each orientation has two negative phases. Choose the one with $\delta \leq 0$. Equation (16.13) gives

$$\Lambda < 2 + 4\epsilon - (1 - 2\epsilon) = 1 + 6\epsilon < a.$$

Suppose exactly three phases are nonzero. If they have the same sign, orient them positively; their weight sum is less than $1 + 3\epsilon < a$. Otherwise orient them with two positive and one negative, and let z be the zero-phase coordinate. The weight sum is less than $2 + 3\epsilon$. If $\delta \leq 0$, or if $w(f) \geq 1 + \delta$, equations (16.13) and (16.11) give $\Lambda < 1 + 5\epsilon < a$. It remains that $\delta > 0$ and $P_\delta \neq 0$. No pure power of z occurs in f , since it has trivial character whereas $\delta \notin \mathbb{Z}$. The z -axis lies in W . Isolatedness of its singularity forces a nonzero partial derivative along this axis, so f contains a monomial $x_j z^b$ with $b \geq 1$. Its character implies $\alpha_j = \delta$, since both are short representatives. Call this positive coordinate x , and the other positive coordinate y . A weight- δ monomial cannot contain x , except as the forbidden linear term. Hence $P_\delta = c_0 y^b$ for some $b \geq 2$.

The refinement just proved places all positive branch tangents on one line; all negative ones also lie on one line. Three independent tangents are impossible.

Suppose exactly two phases are nonzero. Opposite signs are excluded by the tangent-direction argument. Otherwise orient both positively, on coordinates x, y , and let z_1, z_2 have phase zero. The quadratic signed weight δ is nonnegative. If it is zero, then $w(f) \geq 1$ and $\Lambda < 1 + 2\epsilon < a$. If $\delta > 0$, the plane $x = y = 0$ lies in W . Only $\partial_x f$ and $\partial_y f$ can be nonzero on this plane. They can be nonzero only when, respectively, $\alpha_x = \delta$ and $\alpha_y = \delta$. If one vanishes identically, the other vanishes at the origin, and its zero locus on this smooth two-dimensional plane has dimension at least one. This contradicts isolatedness. Thus $\alpha_x = \alpha_y = \delta$. There is no nonlinear monomial in x, y of weight δ , so $P_\delta = 0$. Therefore

$$\Lambda \leq 2 + 2\delta - (1 + \delta) = 1 + \delta < a.$$

Finally, at most one nonzero phase is excluded by tangent independence. These cases exhaust all patterns and contradict (16.12). It follows that $|H| \leq J(m, a)$. $\square$

Lemma 16.7. *Let original cover data have exact minimum $a = p/q > 1$, with $q \mid m$, and let $g : (Y, B_Y) \rightarrow (V, B_V)$ be a full G -equivariant terminal pair model. Put $Z = g^{-1}(o)_{\text{red}}$. Let R, N, J be positive integers. Suppose that, when Z is one-dimensional, the following conditions hold:*

- (i) *Every component C has a set S_C of at most N points, invariant under $\text{Stab}_G(C)$, containing its junctions and every point on C which is the centre of an original computing lift.*
- (ii) *For every subgroup preserving C , the kernel of its action on C has order at most R .*
- (iii) *The stabilizer in G of every junction with at least three branches has order at most J .*
- (iv) *Every original computing lift with curve centre on Y is the ordinary generic blowup valuation of that curve.*

Then

$$|G| \leq 2 \max\{RN, J, P(m), Q(m)\}.$$

One subgroup suffices to propagate through the whole tree. No successive index losses occur at its junctions.

Proof. If Z is a point, the whole group fixes the centre of any original computing lift. Equation (16.1) holds for G on the full model. The inertia of the lift is one, so Proposition 9.8 gives $|G| \leq P(m)$.

Step 1. In this step, we choose one subgroup of index at most two preserving a component at the centre of the tree. Suppose that Z is one-dimensional and that $|G| > 2 \max\{RN, J\}$. A finite tree has a central vertex or a central edge, obtained by repeatedly deleting its leaves. The group preserves this centre. Since the incidence graph has component vertices and junction vertices, the ends of a central edge cannot be interchanged. There is therefore a G -fixed vertex. It cannot be a junction with at least three branches, by (iii). If it is a two-branch junction, pass to the subgroup which fixes both incident components. Otherwise retain G . In either case we obtain a subgroup G_0 with

$$[G : G_0] \leq 2, \quad |G_0| > \max\{RN, J\},$$

preserving a component C .

Step 2. In this step, we propagate the chosen subgroup through all components and marked points without further index loss. The image of G_0 on $C \simeq \mathbb{P}^1$ has order greater than N , by (ii). A nontrivial finite cyclic group on $\mathbb{P}^1$ has two fixed points and every other orbit has its full order: diagonalizing a generator gives the action $z \mapsto \zeta z$. Hence (i) forces every point of S_C to be fixed by G_0 . At a junction on C , condition (iii) excludes three or more branches. The incoming component is preserved, so the only other component is preserved by the same G_0 . Repeating along the finite connected tree shows that G_0 preserves every component and fixes every point in every S_C .

Step 3. In this step, we apply the retained subgroup to an original computing centre and conclude the group bound. Choose an original computing lift F . If its centre is a point, (i) makes that point G_0 -fixed. Its discrepancy is a , its inertia in G_0 is one, and (16.1) holds for all primes centred there. Proposition 9.8 therefore bounds the whole G_0 by $P(m)$, even if it moves F . If the centre is a curve, (iv) and functoriality of the ordinary blowup make F invariant under G_0 . Its original orbit has size at most two, and Lemma 16.1 applies. These two alternatives give the required bound. $\square$

Theorem 16.8. *Fix a rational $a \in (1, 2)$ and an integer $m \geq 1$ divisible by its denominator. There is a positive integer $C(m, a)$ such that every klt threefold germ $(X \ni x, B)$ with mB integral and $\text{mld}(X \ni x, B) = a$ satisfies*

$$\text{ind}_x(K_X + B) \leq mC(m, a).$$

Only $K_X + B$ is assumed $\mathbb{Q}$ -Cartier. The assertion holds over every algebraically closed characteristic-zero field.

The component and junction estimates now give a bound for the whole original group, without choosing a fixed computing prime.

Proof. Take the original integral-log cover of Theorem 3.9 and a full G -equivariant terminal pair model from Proposition 7.5. The integer m clears the boundary on this model: the crepant divisor $m(K_Y + B_Y)$ is integral Cartier and K_Y is integral. The natural line retains the original faithful character. Since the denominator of a divides m , every primitive lift of an original computing divisor has discrepancy a and inertia one.

Let R, D be the integers in Lemma 16.5, and let $J(m, a)$ be the integer in Lemma 16.6. If the reduced fibre is a curve, these two lemmas establish all four hypotheses of Lemma 16.7 with $N = (m + 2)D$. In particular the marked sets are invariant because junctions, ambient singularities, and boundary multiplicity are invariant under the component stabilizers. The same propagation lemma includes the point-fibre case. Thus the positive integer

$$C(m, a) = 2 \max\{R(m + 2)D, J(m, a), P(m), Q(m)\}$$

bounds $|G|$. The original cover relation gives $\text{ind}_x(K_X + B) \leq m|G| \leq mC(m, a)$. The field reduction at the beginning of the section preserves this exact original index and minimum, proving the assertion over the stated fields. $\square$

Lemma 16.9. *Let $(X \ni x, B)$ be a klt threefold germ with effective rational boundary and $a = \text{mld}(X \ni x, B) > 2$. Let $m \geq 1$ satisfy mB integral, and take the original integral-log cover and a full equivariant terminal pair model $g : (Y, B_Y) \rightarrow (V, B_V)$. If $g^{-1}(o)$ is zero-dimensional, then X is smooth near x and $\text{ind}_x(K_X + B)$ divides m .*

The point-fibre case follows from the ambient comparison without a separate classification of the bare minimum two case.

Proof. We identify the terminal model with the cover, exclude a singular canonical cover, and then exclude a nontrivial smooth quotient.

Step 1. In this step, we identify the terminal model with the original cover near the point fibre. By upper semicontinuity of fibre dimension, g is quasi-finite over a neighbourhood of o . It is proper and birational, so it is finite and hence an isomorphism there, since V is normal. We identify (Y, B_Y) with (V, B_V) on this neighbourhood. The full original cyclic group fixes o .

Apply Proposition 8.8 to $(V \ni o, B_V)$ with this group and the exact quotient minimum a . Write A for its abelian extension acting on the canonical cover W , and M_0 for its smooth ambient germ. The composite $W \rightarrow V \rightarrow X$ is quasi-étale: both the local canonical cover and the original integral-log cover have this property by Theorem 3.9, applied to the bare terminal pair for the first cover.

Step 2. In this step, we exclude a singular canonical cover using the ambient discrepancy comparison. If W is singular, its equation f in the smooth fourfold M_0 has order at least two. Let T be the ordinary blowup valuation of the ambient point. Then

$$A_{M_0}(T) - T(f) - T(h_B)/m \leq 4 - 2 = 2.$$

This contradicts the all-prime inequality in Proposition 8.8, since its inertia order is at least one and $a > 2$. Thus $W = M_0$ is a smooth threefold germ.

Step 3. In this step, we exclude a nontrivial smooth quotient and obtain the index divisibility. Choose actual regular eigenparameters u_1, u_2, u_3 for the finite abelian action on W , by averaging lifts of a cotangent eigenbasis. The action on these parameters is faithful. Otherwise an element would act trivially on the complete local ring, hence on the local ring and its function field.

Suppose $\gamma \in A$ has order $n > 1$. It cannot be a pseudoreflection: the corresponding fixed coordinate divisor would give codimension-one ramification in $W \rightarrow X$. Let $\theta_i \in [0, 1)$ be its coordinate phases, and let $z \leq 1$ be the number of zero phases. Form two positive rational weight vectors by taking θ_i and $1 - \theta_i$ for each nonzero phase and taking one in both vectors for a zero phase. Their coordinate sums add to $3 + z \leq 4$. Choose the vector w with sum at most two; it is congruent to the phases of either γ or its inverse.

Let c be the greatest common divisor of the positive integers nw_i . The primitive weights nw_i/c define an algebraic weighted blowup prime T in the actual regular parameters. The chosen element acts by a weighted scalar on the associated graded ring and hence trivially on its residue field. Thus $|I_A(T)| \geq n$. The smooth discrepancy formula gives

$$\frac{A_W(T) - T(h_B)/m}{|I_A(T)|} \leq \frac{A_W(T)}{n} = \frac{\sum_i w_i}{c} \leq 2,$$

contradicting Proposition 8.8. It follows that $A = 1$. Hence $W = V = X$ near the marked points and X is smooth. Its integral Weil divisor mB is Cartier, so $m(K_X + B)$ is Cartier as well. $\square$

Proposition 16.10. *Let $\Gamma \subset \mathbb{Q} \cap [0, 1]$ be finite, put $d = d_\Gamma$, and let $(X \ni x, B)$ be a klt threefold germ with $B \in \Gamma$ and $a = \text{mld}(X \ni x, B) \geq 2$. Take its original integral-log cover for $m = d$ and a full equivariant terminal pair model $g : (Y, B_Y) \rightarrow (V, B_V)$. Assume that Y is smooth at the generic point of every curve component of $Z = g^{-1}(o)_{\text{red}}$. Then*

$$\text{ind}_x(K_X + B) \leq d \left(\binom{d+4}{4} - 1 \right) \max\{P(d), Q(d)\}.$$

If $a > 2$, then X is smooth near x and this index divides d .

This reduction isolates the use of smoothness in codimension two from the original-orbit and local-character arguments.

Proof. For every primitive prime T whose centre on Y is contained in Z , the original finite comparison gives

$$A_{Y, B_Y}(T) \geq a |I_G(T)| \geq a.$$

If $C \subset Z$ is a component, generic smoothness as assumed in the statement permits the ordinary transverse blowup. Its primitive valuation F_C has

$$A_{Y, B_Y}(F_C) = 2 - \text{mult}_C B_Y \leq 2.$$

It is a prime over the original field $k(V)$, obtained on the normalization of the blowup of the closure of C .

Step 1. In this step, we reduce $a > 2$ to the point-fibre case and obtain smoothness and index divisibility. If $a > 2$, these inequalities exclude every curve in Z . Lemma 7.12 therefore makes Z a single point. Lemma 16.9 proves the asserted smoothness and divisibility.

Step 2. In this step, we bound the index at $a = 2$ using a fixed point or a bounded component orbit. Now let $a = 2$. Its denominator is one, so $m = d$ clears both the boundary and the

minimum. If Z is a point, any original computing lift has discrepancy two, inertia one, and centre fixed by G . Proposition 9.8 then gives $|G| \leq P(d)$, which suffices.

If Z is a curve, the displayed inequalities force

$$\text{mult}_C B_Y = 0, \quad A_{Y,B_Y}(F_C) = 2, \quad I_G(F_C) = 1$$

for every component. The primitive restriction of F_C therefore computes the original minimum. Functoriality of the ordinary blowup and uniqueness of the valuation centre give $\text{Stab}_G(F_C) = \text{Stab}_G(C)$.

The incidence tree has a vertex fixed by G , as in the proof of Lemma 16.7. If a component is fixed, choose it. If a junction is fixed, choose any incident component. Its entire orbit consists of components through that junction, and hence has size at most $E_d = \binom{d+4}{4} - 1$ by Lemma 16.2. In either case an original computing lift has orbit at most E_d . Lemma 16.1 now gives the stated index bound. $\square$

Theorem 16.11. *For every finite $\Gamma \subset \mathbb{Q} \cap [0, 1]$, there is a positive integer $R_{\geq 2}(\Gamma)$ such that every klt threefold pair $(X \ni x, B)$ with $B \in \Gamma$ and $\text{mld}(X \ni x, B) \geq 2$ satisfies*

$$\text{ind}_x(K_X + B) \leq R_{\geq 2}(\Gamma).$$

If $\text{mld}(X \ni x, B) > 2$, then X is smooth near x and $\text{ind}_x(K_X + B)$ divides d_Γ . Only the total $K_X + B$ is assumed $\mathbb{Q}$ -Cartier, and the ground field may be any algebraically closed field of characteristic zero.

The bound is uniform on the whole interval $[2, \infty)$; only its endpoint requires a nontrivial stabilizer estimate.

Proof. Set $d = d_\Gamma$. On the full terminal pair model for the original cover with coefficient-clearing integer d , the underlying threefold is terminal by Proposition 7.5. Theorem 7.6 [CKM88, Proposition 6.10 and Corollary 6.13, pp. 41–42] makes it smooth at the generic point of each fibre component. For a ground field other than $\mathbb{C}$, this use is justified by the common realization in Lemma 3.11, as explained at the beginning of the section. Thus all hypotheses of Proposition 16.10 hold. It proves both conclusions, with

$$R_{\geq 2}(\Gamma) = d \left(\binom{d+4}{4} - 1 \right) \max\{P(d), Q(d)\}.$$

Here $P(d), Q(d) \geq 1$ and $\binom{d+4}{4} - 1 \geq 1$, so this bound also includes the divisibility conclusion when the minimum is greater than two. No separate $\mathbb{Q}$ -Cartierness is assumed on the initial K_X or B ; all component computations take place on the terminal pair model supplied by the total log divisor. $\square$

17. THE THREEFOLD INDEX THEOREM

We retain the notation for local indices and original cover data. We combine the orbit and stabilizer bounds with the results for minima greater than one.

Theorem 17.1. *For every finite set $\Gamma \subset \mathbb{Q} \cap [0, 1]$ and every real number $a > 0$, there is a positive integer $C_3(a, \Gamma)$ with the following property. Let (X, B) be a klt threefold pair over an algebraically closed field of characteristic zero, with $B \in \Gamma$, and let $x \in X$ be a closed point. If $K_X + B$ is $\mathbb{Q}$ -Cartier and $\text{mld}(X \ni x, B) = a$, then*

$$\text{ind}_x(K_X + B) \leq C_3(a, \Gamma).$$

The bound is uniform in the germ and the ground field once the coefficient set and the exact minimum are fixed.

Proof. We first exclude irrational minima, then bound the index in the three ranges $0 < a \leq 1$, $1 < a < 2$, and $a \geq 2$.

Step 1. In this step, we exclude irrational minima and choose the common denominator-clearing integer. Since $d_\Gamma B$ is integral, Theorem 3.9, applied with $m = d_\Gamma$, shows that the closed-point minimum is rational and attained. Thus no such pair exists when a is irrational, and we may take $C_3(a, \Gamma) = 1$ in that case. Suppose henceforth that $a = p/q$, with p, q coprime positive integers, and set

$$m = \text{lcm}(d_\Gamma, q).$$

Then mB is integral and ma is an integer. This choice of m depends only on Γ and a .

Step 2. In this step, we bound the index for $0 < a \leq 1$ using the original orbit and stabilizer estimates. Choose a computing divisor and a primitive lift F to the original cover $\rho : (V \ni o, B_V) \rightarrow (X \ni x, B)$ of Theorem 3.9 for this m . Let G be its group and put $H = \text{Stab}_G(F)$. These are original cover data in the sense of Definition 3.10; in particular,

$$I_G(F) = 1, \quad A_{V, B_V}(F) = a, \quad \text{ind}_x(K_X + B) \leq m[G : H]|H|.$$

Theorem 6.16 bounds the orbit factor by $[G : H] \leq C_{\text{orb}}(a)$.

Theorem 7.16 supplies an H -equivariant divisorial contraction $h : Y \rightarrow Z$ of the same original prime E , whose image is a point or a curve. It retains the natural character, contact marking, and discrepancy comparisons required by the stabilizer theorems. This applies also at $a = 1$, when the coefficient $1 - a$ of E in the source boundary is zero. If $h(E)$ is a point, Theorem 11.17 gives $|H| \leq C_{\text{pt}}(m, a)$. If $h(E)$ is a curve, Theorem 14.2 gives $|H| \leq C_{\text{cur}}(m, a)$. In both cases the bound is for the same full stabilizer H . The orbit bound and these two alternatives therefore give

$$\text{ind}_x(K_X + B) \leq mC_{\text{orb}}(a) \max\{C_{\text{pt}}(m, a), C_{\text{cur}}(m, a)\}.$$

We take the right-hand side as $C_3(a, \Gamma)$ in this range.

Step 3. In this step, we bound the index for $1 < a < 2$ by the middle-range theorem. Our choice of m satisfies both hypotheses on denominators in Theorem 16.8. Applying it directly to $(X \ni x, B)$ gives

$$\text{ind}_x(K_X + B) \leq mC(m, a).$$

We take $C_3(a, \Gamma) = mC(m, a)$ in this range.

Step 4. In this step, we bound the index for $a \geq 2$ by the large-range theorem. Theorem 16.11 applies directly and gives

$$\text{ind}_x(K_X + B) \leq R_{\geq 2}(\Gamma).$$

We take $C_3(a, \Gamma) = R_{\geq 2}(\Gamma)$, including at $a = 2$.

These ranges exhaust the positive rational minima. Each chosen bound is a positive integer depending only on Γ and a . All the invoked bounds hold over every algebraically closed field of characteristic zero and assume only that the total divisor $K_X + B$ is $\mathbb{Q}$ -Cartier, as required. $\square$

For the klt pairs in Theorem 17.1, the integer $r = \text{lcm}(1, \dots, C_3(a, \Gamma))$ is a uniform Cartier multiple: each integer $\text{ind}_x(K_X + B) \leq C_3(a, \Gamma)$ divides r , so $r(K_X + B)$ is Cartier near x .

Corollary 17.2. *For every real $a > 0$, there is a positive integer $C_3(a)$ such that every three-dimensional klt germ $X \ni x$ over an algebraically closed field of characteristic zero with $\text{mld}(X \ni x) = a$ satisfies $\text{ind}_x(K_X) \leq C_3(a)$.*

Thus the exact positive minimal log discrepancy bounds the canonical index of a klt threefold germ.

Proof. Apply Theorem 17.1 with $B = 0$ and $\Gamma = \{0\}$, and set $C_3(a) = C_3(a, \{0\})$. $\square$

REFERENCES

- [ADK13] D. Abramovich, J. Denef, and K. Karu, *Weak toroidalization over non-closed fields*, Manuscripta Math. **142** (2013), nos. 1–2, 257–271. DOI: <https://doi.org/10.1007/s00229-013-0610-5>.
- [Amb09] F. Ambro, *On the classification of toric singularities*, Combinatorial aspects of commutative algebra, Contemp. Math. **502** (2009), 1–3, Amer. Math. Soc., Providence, RI.
- [Amb16] F. Ambro, *Variation of log canonical thresholds in linear systems*, Int. Math. Res. Not. IMRN **2016** (2016), no. 14, 4418–4448. DOI: <https://doi.org/10.1093/imrn/rnv284>.
- [Art66] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), no. 1, 129–136. DOI: <https://doi.org/10.2307/2373050>.
- [Asc04] M. Aschenbrenner, *Ideal Membership in Polynomial Rings over the Integers*, J. Amer. Math. Soc. **17** (2004), no. 2, 407–441. DOI: <https://doi.org/10.1090/S0894-0347-04-00451-5>.
- [BM24] F. Bernasconi and G. Martin, *Bounding geometrically integral del Pezzo surfaces*, Forum Math. Sigma **12** (2024), e81, 24 pp. DOI: <https://doi.org/10.1017/fms.2024.80>.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. DOI: <https://doi.org/10.1090/S0894-0347-09-00649-3>.
- [Bir16] C. Birkar, *Singularities of linear systems and boundedness of Fano varieties*, [arXiv:1609.05543](https://arxiv.org/abs/1609.05543).
- [Bir19] C. Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, 345–463. DOI: <https://doi.org/10.4007/annals.2019.190.2.1>.
- [BL21] C. Birkar and K. Loginov, *Bounding non-rationality of divisors on 3-fold Fano fibrations*, J. reine angew. Math. **779** (2021), 167–188. DOI: <https://doi.org/10.1515/crelle-2021-0039>.
- [Bir23a] C. Birkar, *Geometry of polarised varieties*, Publ. Math. IHÉS **137** (2023), 47–105. DOI: <https://doi.org/10.1007/s10240-022-00136-w>.
- [Bir23b] C. Birkar, *Singularities on Fano fibrations and beyond*, [arXiv:2305.18770](https://arxiv.org/abs/2305.18770).
- [Bos14] S. Bosch, *Lectures on Formal and Rigid Geometry*, Lecture Notes in Mathematics **2105**, Springer, 2014. DOI: <https://doi.org/10.1007/978-3-319-04417-0>.
- [CFHR99] F. Catanese, M. Franciosi, K. Hulek, and M. Reid, *Embeddings of curves and surfaces*, Nagoya Mathematical Journal **154** (1999), 185–220. DOI: <https://doi.org/10.1017/S0027763000025381>.
- [Che24] B. Chen, *Boundedness of klt complements on Fano fibrations over surfaces*, [arXiv:2403.01154](https://arxiv.org/abs/2403.01154).
- [CH21] G. Chen and J. Han, *Boundedness of (ϵ, n) -complements for surfaces*, Adv. Math. **383** (2021), Paper No. 107703.
- [CKM88] H. Clemens, J. Kollár, and S. Mori, *Higher dimensional complex geometry*, Astérisque **166**, Société Mathématique de France, 1988, 146 pp.
- [DI09] I. V. Dolgachev and V. A. Iskovskikh, *Finite subgroups of the plane Cremona group*, in: *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin*, Vol. I, Progress in Mathematics **269**, Birkhäuser, 2009, 443–548. DOI: https://doi.org/10.1007/978-0-8176-4745-2_11.
- [EM06] L. Ein and M. Mustață, *Jet schemes and singularities*, [arXiv:math/0612862](https://arxiv.org/abs/math/0612862).
- [FQS16] G. Fichou, R. Quarez, and M. Shiota, *Artin approximation compatible with a change of variables*, Canad. Math. Bull. **59** (2016), no. 4, 760–768. DOI: <https://doi.org/10.4153/CMB-2016-034-1>.
- [Fil20] S. Filipazzi, *On a generalized canonical bundle formula and generalized adjunction*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **21** (2020), 1187–1221. DOI: https://doi.org/10.2422/2036-2145.201810_001.
- [Fuj01] O. Fujino, *The indices of log canonical singularities*, Amer. J. Math. **123** (2001), no. 2, 229–253.
- [Fuj09a] O. Fujino, *Effective base point free theorem for log canonical pairs—Kollár type theorem*, Tohoku Math. J. (2) **61** (2009), no. 4, 475–481. DOI: <https://doi.org/10.2748/tmj/1264084495>. [Published text](#).
- [Fuj09b] O. Fujino, *Introduction to the log minimal model program for log canonical pairs*, [arXiv:0907.1506](https://arxiv.org/abs/0907.1506).
- [Fuj09c] O. Fujino, *Kawamata–Viehweg vanishing theorem*, author note, version 1.04, 22 July 2009. <https://www.math.kyoto-u.ac.jp/~fujino/kawamata--viehweg2.pdf>.
- [FG14] O. Fujino and Y. Gongyo, *On the moduli b -divisors of lc-trivial fibrations*, Ann. Inst. Fourier **64** (2014), no. 4, 1721–1735. DOI: <https://doi.org/10.5802/aif.2894>.
- [Fuj22] O. Fujino, *Cone and contraction theorem for projective morphisms between complex analytic spaces*, [arXiv:2209.06382](https://arxiv.org/abs/2209.06382).
- [GHS03] T. Graber, J. Harris, and J. Starr, *Families of Rationally Connected Varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67. DOI: <https://doi.org/10.1090/S0894-0347-02-00402-2>.
- [HM07] C. D. Hacon and J. McKernan, *On Shokurov’s rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136. DOI: <https://doi.org/10.1215/S0012-7094-07-13813-4>.
- [HL25a] J. Han and J. Liu, *On termination of flips and exceptionally noncanonical singularities*, Geom. Topol. **29** (2025), no. 1, 399–441. DOI: <https://doi.org/10.2140/gt.2025.29.399>.
- [HL25b] J. Han and J. Liu, *Classification of threefold enc cDV quotient singularities*, [arXiv:2501.01024](https://arxiv.org/abs/2501.01024).
- [HLL25] J. Han, J. Liu, and Y. Luo, *ACC for minimal log discrepancies of terminal threefolds*, Adv. Math. **480** (2025), 110457. DOI: <https://doi.org/10.1016/j.aim.2025.110457>.

- [HW81] F. Hidaka and K.-i. Watanabe, *Normal Gorenstein Surfaces with Ample Anti-canonical Divisor*, Tokyo Journal of Mathematics **4** (1981), no. 2, 319–330. <https://doi.org/10.3836/tjm/1270215157>.
- [HX09] A. Hogadi and C. Xu, *Degenerations of Rationally Connected Varieties*, Trans. Amer. Math. Soc. **361** (2009), no. 7, 3931–3949. DOI: <https://doi.org/10.1090/S0002-9947-09-04715-1>.
- [How01] J. A. Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2665–2671. DOI: <https://doi.org/10.1090/S0002-9947-01-02720-9>.
- [Ish00] S. Ishii, *The quotients of log-canonical singularities by finite groups*, Singularities—Sapporo 1998, Adv. Stud. Pure Math., **29** (2000), 135–161, Kinokuniya, Tokyo.
- [Jia19] C. Jiang, *A gap theorem for minimal log discrepancies of non-canonical singularities in dimension three*, arXiv:1904.09642.
- [JM12] M. Jonsson and M. Mustață, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier **62** (2012), no. 6, 2145–2209. DOI: <https://doi.org/10.5802/aif.2746>.
- [JM08] S.-Y. Jow and E. Miller, *Multiplier ideals of sums via cellular resolutions*, Math. Res. Lett. **15** (2008), no. 2, 359–373. DOI: <https://doi.org/10.4310/MRL.2008.v15.n2.a13>.
- [Ju⁺26] H. Ju, G. Gao, J. Jiang, B. Wu, Z. Sun, S. Liu, L. Chen, Y. Wang, Y. Wang, Z. Wang, W. He, P. Wu, L. Xiao, R. Liu, B. Dai, and B. Dong, *Automated Conjecture Resolution with Formal Verification*, arXiv:2604.03789.
- [Kat96] F. Kato, *Log Smooth Deformation Theory*, Tohoku Math. J. (2) **48** (1996), no. 3, 317–354. DOI: <https://doi.org/10.2748/tmj/1178225336>.
- [Kaw05] M. Kawakita, *Three-fold divisorial contractions to singularities of higher indices*, Duke Math. J. **130** (2005), no. 1, 57–126. DOI: <https://doi.org/10.1215/S0012-7094-05-13013-7>.
- [Kaw07] M. Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. **167** (2007), no. 1, 129–133. DOI: <https://doi.org/10.1007/s00222-006-0008-z>.
- [Kaw15] M. Kawakita, *The index of a threefold canonical singularity*, Amer. J. Math. **137** (2015), no. 1, 271–280.
- [Kaw17] M. Kawakita, *Divisors computing the minimal log discrepancy on a smooth surface*, Math. Proc. Cambridge Philos. Soc. **163** (2017), no. 1, 187–192. DOI: <https://doi.org/10.1017/S0305004116001043>.
- [Kaw93] Y. Kawamata, *The minimal discrepancy of a 3-fold terminal singularity*, Appendix to V. V. Shokurov, *3-fold log flips*, Russian Acad. Sci. Izv. Math. **40** (1993), no. 1, 193–195. DOI of the enclosing article: <https://doi.org/10.1070/IM1993v040n01ABEH001862>.
- [Kol04] J. Kollár, *Seifert G_m -bundles*, arXiv:math/0404386.
- [Kol05] J. Kollár, *Resolution of Singularities—Seattle Lecture*, arXiv:math/0508332.
- [Kol21] J. Kollár, *Moishezon morphisms*, arXiv:2102.02614.
- [KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, Cambridge, 1998.
- [Li18] C. Li, *Minimizing normalized volumes of valuations*, Math. Z. **289** (2018), nos. 1–2, 491–513. DOI: <https://doi.org/10.1007/s00209-017-1963-3>.
- [Liu26] J. Liu, *Boundedness of complements for log canonical pairs*, <https://jihaoliu.org/notes/boundedness-of-complements-of-log-canonical-pairs-2026-10-11.pdf>.
- [LX21] J. Liu and L. Xiao, *An optimal gap of minimal log discrepancies of threefold non-canonical singularities*, J. Pure Appl. Algebra **225** (2021), no. 9, 106674. DOI: <https://doi.org/10.1016/j.jpaa.2021.106674>.
- [Liu⁺26] J. Liu, G. Gao, Z. Sun, B. Wu, S. Liu, J. Jiang, H. Ju, L. Chen, R. Cheng, X. Zhang, and B. Dong, *Danus: Orchestrating Mathematical Reasoning Agents with Fact-Graph Memory*, arXiv:2607.06447.
- [LM22] S. Lyu and T. Murayama, *The relative minimal model program for excellent algebraic spaces and analytic spaces in equal characteristic zero*, arXiv:2209.08732.
- [Mor20] J. Moraga, *Extracting non-canonical places*, Adv. Math. **375** (2020), 107415, 12 pp. DOI: <https://doi.org/10.1016/j.aim.2020.107415>.
- [Mor21] J. Moraga, *On a toroidalization for klt singularities*, arXiv:2106.15019.
- [NS25a] Y. Nakamura and K. Shibata, *Inversion of adjunction for quotient singularities II: non-linear actions*, Algebraic Geometry **12** (2025), no. 4, 443–496. DOI: <https://doi.org/10.14231/AG-2025-013>.
- [NS25b] Y. Nakamura and K. Shibata, *Shokurov's index conjecture for quotient singularities*, in *Higher dimensional algebraic geometry—a volume in honor of V. V. Shokurov*, London Math. Soc. Lecture Note Ser. **489**, Cambridge Univ. Press, Cambridge, 2025, 220–230.
- [NS26a] Y. Nakamura and K. Shibata, *Inversion of adjunction for quotient singularities III: semi-invariant case*, J. London Math. Soc. (2) **113** (2026), no. 5, e70545. DOI: <https://doi.org/10.1112/jlms.70545>.
- [NS26b] Y. Nakamura and K. Shibata, *A non-klt counterexample to Shokurov's index conjecture*, arXiv:2608.28038.
- [Nak15] N. Nakayama, *A variant of Shokurov's criterion of toric surface*, author preprint, RIMS-1825, May 2015. <https://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1825.pdf>.
- [Ope26a] OpenAI, *Bounded klt complements for Fano contractions*, preprint, 25 September 2026. <https://github.com/openai/math/blob/fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb/preprints/Bounded-klt-complements-for-Fano-contractions-September-25-2026/Bounded-klt-complements-for-Fano-contractions-September-25-2026.pdf>.

- [Ope26b] OpenAI, *Uniform Cartier sections for Fano type contractions*, preprint, 25 September 2026. <https://github.com/openai/math/blob/fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb/preprints/Uniform-Cartier-sections-for-Fano-type-contractions-September-25-2026/Uniform-Cartier-sections-for-Fano-type-contractions-September-25-2026.pdf>.
- [Pro01a] Yu. G. Prokhorov, *Lectures on complements on log surfaces*, MSJ Memoirs **10**, Mathematical Society of Japan, Tokyo, 2001, viii+130 pp. DOI: <https://doi.org/10.2969/msjmemoirs/010010000>.
- [PS09] Yu. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, J. Algebraic Geom. **18** (2009), no. 1, 151–199. DOI: <https://doi.org/10.1090/S1056-3911-08-00498-0>.
- [Pro21] Yu. G. Prokhorov, *Equivariant minimal model program*, Russian Math. Surveys **76** (2021), no. 3, 461–542. DOI: <https://doi.org/10.1070/RM9990>.
- [Rei83] M. Reid, *Minimal Models of Canonical 3-Folds*, Advanced Studies in Pure Mathematics **1** (1983), 131–180. DOI: <https://doi.org/10.2969/aspm/00110131>.
- [Rei94] M. Reid, *Nonnormal del Pezzo Surfaces*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 5, 695–727. DOI: <https://doi.org/10.2977/PRIMS/1195165581>.
- [Rob76] L. Robbiano, *Some properties of complete intersections in good projective varieties*, Nagoya Math. J. **61** (1976), 103–111. DOI: <https://doi.org/10.1017/S0027763000017323>.
- [Sch24] C. Schnell, *Complex Manifolds, Fall 2024, Class 2: The Weierstrass theorems*, 2024 course notes. <https://www.math.stonybrook.edu/~cschnell/mat545/class2.pdf>.
- [Sch01] S. Schröer, *Normal del Pezzo surfaces containing a nonrational singularity*, Manuscripta Math. **104** (2001), no. 2, 257–274. DOI: <https://doi.org/10.1007/s002290170042>.
- [Sho92] V. V. Shokurov, *Threefold log flips*, with an appendix in English by Y. Kawamata, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 1, 105–203.
- [Sta26] The Stacks Project Authors, *The Stacks Project*, accessed 13 October 2026. <https://stacks.math.columbia.edu>; Section 33.25, Tag 04QM, Lemma 33.25.8; Lemma 15.39.4, Tag 0C35.
- [Ste13] J. Stevens, *On the classification of rational surface singularities*, J. Singul. **7** (2013), 108–133. DOI: <https://doi.org/10.5427/jsing.2013.7g>.
- [Tzi05] N. Tziolas, *Q-Gorenstein deformations of nonnormal surfaces*, arXiv:math/0504211.
- [XZ25] C. Xu and Z. Zhuang, *Stable degenerations of singularities*, J. Amer. Math. Soc. **38** (2025), no. 3, 585–626. DOI: <https://doi.org/10.1090/jams/1055>.
- [Zhu24] Z. Zhuang, *On boundedness of singularities and minimal log discrepancies of Kollár components*, J. Algebraic Geom. **33** (2024), no. 3, 521–565. DOI: <https://doi.org/10.1090/jag/822>.