

K-SEMISTABILITY OF SMOOTH FANO THREEFOLDS OF PICARD NUMBER ONE

ABSTRACT. We prove that every smooth complex Fano threefold of Picard number one is K-semistable. The remaining prime families have anticanonical degrees 18 and 22. We control their divisorial valuations by complete section refinements, using line projections and weighted point extractions, and treat the special members in each family. The main result of this paper was obtained using generative AI, particularly GPT-6 Astra and the Danus system.

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Line incidence and projection geometry	6
4. Curve centers	7
5. Special members and repeated contact	26
6. Points on at most one ordinary line	29
7. Ordinary-line incidence and reduced pairs	38
8. Three reduced lines in degree eighteen	46
9. A single doubled incidence point in degree eighteen	71
10. Mixed incidence points in degree eighteen	79
11. Curvilinear triple incidence in degree eighteen	89
12. Ordinary-line flags in degree twenty-two	101
13. Reduced incidence pairs and triples in degree twenty-two	107
14. Double and mixed incidence in degree twenty-two	115
15. Curvilinear triple incidence in degree twenty-two	124
16. The family theorems and the classification	134
References	135

1. INTRODUCTION

K-stability relates the birational geometry of a Fano variety to the existence and degeneration of Kähler–Einstein metrics. For smooth Fano threefolds of Picard number one, the classification leaves finitely many deformation families, but a statement about every member must also control special line configurations and members with positive-dimensional automorphism groups. The all-member results of Abban–Zhuang [AZ21, Theorem C], together with the known higher-index examples, leave the two prime families of anticanonical degrees 18 and 22.

Theorem 1.1. *Every smooth complex Fano threefold with Picard number one is K-semistable.*

The two remaining families are treated in the following theorems. Here the polarization is the anticanonical class itself.

Theorem 1.2. *Every smooth complex Fano threefold X with $\text{Pic}(X) = \mathbb{Z}[H]$, $H = -K_X$ and $H^3 = 18$ is K-semistable.*

Date: October 14, 2026.

2020 Mathematics Subject Classification. Primary 14J45; Secondary 14E30, 14J30, 32Q20.

Key words and phrases. Fano threefold, K-semistability, stability threshold, divisorial valuation.

Theorem 1.3. *Every smooth complex Fano threefold X with $\mathrm{Pic}(X) = \mathbb{Z}[H]$, $H = -K_X$ and $H^3 = 22$ is K -semistable.*

These assertions include every smooth member. The multiplicative members in degree twenty-two already admit Kähler–Einstein metrics [CS23, Corollary 1.7] and [Fuj23, Theorem 1.2]. Other special members require the semistable conclusion of Theorem 1.3. We use the classification of lines, conics and line projections in [KPZ14, KPS18, KP18, KP23], together with the all-member Mukai models in [BKM26a, BKM26b, Vir26]. The proof does not infer an all-member assertion from a statement about a general member.

The proof and its main ingredients. The valuative criterion reduces semistability to $A_X(F) \geq S_X(F)$ for prime divisors over X [BJ20, Theorem B]. The admissible-flag method of [AZ22] studies this inequality by refining the complete anticanonical section spaces along an extracted divisor and then along curves and points on it. A key feature here is that the refinement consists of actual restriction images. Fixed divisors, fractional terms from adjunction, and the two grade parities of a doubled-line extraction all contribute to the original threefold normalization.

The geometric partition is finite. At a point, the scheme of incident anticanonical lines has length at most three, and each local factor is curvilinear (Lemma 3.2). Line projections reduce curve centers and many point centers to explicit series on ruled surfaces. The nonreduced cases require the contact scheme with the inverse curve, including its infinitely near points. Lemmas 14.1 and 14.2 retain all the base fibers needed to pass from a sole doubled incidence to mixed incidence. Proposition 14.3 supplies the threefold transport, including the complete tail and both parities.

The remaining point cases use weighted extractions in the homogeneous models. For these, degree-one initial sections alone do not establish the required refinement. We prove equality of the full associated graded section ring in every degree by lattice counts and the threefold Hilbert polynomial. Further binary-form decompositions then bound all eigenbasis divisors, including their quotient and intersection points. A second filtration and lower semicontinuity of log canonical thresholds transfer these bounds to arbitrary valuations. The constants in Propositions 10.5, 11.4, 11.6 and 15.5 retain positive extraction coefficients.

Some configurations are better handled for the whole member. The conductor-crossing construction and the one-conic triple construction give smooth families with fixed original nonzero fibers and a smooth multiplicative central member (Propositions 5.2 and 15.3). Openness then proves semistability for those original members. For the final degree-twenty-two argument, optimal destabilization [LXZ22, Theorem 5.1, Corollary 3.4 and Lemma 3.5] selects one divisor computing the global threshold. We classify its actual center and keep that same divisor through all alternatives.

Structure of the paper. Section 2 gives the valuation and refinement tools, and Section 3 describes line incidence and the projection geometry. Sections 4 and 5 treat positive-dimensional centers and the first whole-member degenerations. Section 6 covers points on no line or one ordinary line. The subsequent degree-eighteen sections treat reduced pairs, reduced triples, doubled incidence, mixed incidence and curvilinear triples. The degree-twenty-two sections establish the ordinary-line flags, reduced incidence estimates, double and mixed extraction, and the one-conic and two-conic triple alternatives. Section 16 assembles the two family theorems and the classification.

2. PRELIMINARIES

We adopt the standard notation and terminology for the minimal model program from [Sho93, KM98, BCH⁺06] and use them freely. We work over the field of complex numbers $\mathbb{C}$. A prime Fano threefold is a smooth Fano threefold X with $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$. We write $H = -K_X$ and $d = H^3$. A line or conic on X is a subscheme with Hilbert polynomial $m + 1$ or $2m + 1$,

respectively, with respect to H . In particular, the term conic allows a nonreduced conic. A reduced irreducible curve of degree one is a line.

2.1. Valuations and complete refinements.

Definition 2.1. Let (Z, Δ) be a klt pair, let L be a big Cartier divisor on the projective variety Z , and let F be a prime divisor on a birational model $\mu: \tilde{Z} \rightarrow Z$. We use the normalization $v_F = \text{ord}_F$ and set

$$(2.1) \quad S(L; F) = \frac{1}{\text{vol}(L)} \int_0^\infty \text{vol}(\mu^*L - tF) dt, \quad A_{Z, \Delta}(F) = 1 + \text{ord}_F(K_{\tilde{Z}} - \mu^*(K_Z + \Delta)).$$

For a smooth Fano variety X , we write $S_X(F) = S(-K_X; F)$ and $\beta_X(F) = A_X(F) - S_X(F)$. If $T \subseteq Z$ is an irreducible subvariety, define

$$(2.2) \quad \delta_T(Z, \Delta; L) = \inf_{T \subseteq c_Z(F)} \frac{A_{Z, \Delta}(F)}{S(L; F)}.$$

The global threshold has the infimum over all prime divisors over Z . A subscript p denotes the threshold at the closed point p .

Thus the condition in (2.2) is that the center contains T . In particular, a bound at the generic point of a curve does not assert a bound for every valuation centered at a closed point of that curve.

Notation 2.2. For a prime Fano threefold, put $N_m = h^0(X, mH)$. Riemann–Roch and Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem] give

$$(2.3) \quad N_m = \frac{d}{6}m^3 + \frac{d}{4}m^2 + \left(\frac{d}{12} + 2\right)m + 1 \quad (m \geq 0).$$

An m -basis divisor is $(mN_m)^{-1} \sum_{i=1}^{N_m} \text{div}(s_i)$ for a basis of $H^0(X, mH)$ and $m > 0$. The maximal order of such a divisor at F is denoted by $S_m(X; F)$.

Suppose that $\pi: Y \rightarrow X$ extracts a prime divisor E of plt type. The refinement by E consists of the actual restriction images

$$(2.4) \quad W_{m,j} = \text{im} \left(H^0(Y, m\pi^*H - jE) \rightarrow H^0(E, (m\pi^*H - jE)|_E) \right), \quad j \geq 0.$$

For a non-Cartier E , the restriction in (2.4) is the divisorial restriction furnished by plt adjunction. We include its different and every fractional fixed part. If w is a divisorial valuation over E , then $S_m(W; w)$ is the sum of the w -jumping numbers in all the spaces $W_{m,j}$, divided by mN_m . We write $S(W; w)$ for its limit. All further refinements use this original denominator.

The images in (2.4) are the graded quotients of the order filtration, by [AZ22, Example 2.6]. Consequently their dimensions sum to N_m . A complete linear system on E may replace an image only after equality or a sufficient inclusion, together with its rank, has been proved. In the plt constructions considered here the refinements have bounded support and contain an ample series; the expected orders converge by [AZ22, Lemma 2.9].

Theorem 2.3 (Valuative criterion). *A smooth Fano variety X is K -semistable if and only if $\beta_X(F) \geq 0$ for every prime divisor F over X .*

This is [BJ20, Theorem B]. We will also use the following consequence of optimal destabilization.

Proposition 2.4 (Choice of a destabilizing divisor). *Let X be a smooth Fano threefold. If X is not K -semistable, there exists a prime divisor F over X such that*

$$(2.5) \quad \frac{A_X(F)}{S_X(F)} = \delta(X) < 1.$$

The divisor F can be chosen special, in the sense that its valuation induces a special test configuration.

Proof. Theorem 2.3 gives $\delta(X) < 1 < 4/3$. By [LXZ22, Theorem 5.1], a divisorial valuation computes this threshold. By [LXZ22, Corollary 3.4], this valuation is a monomial log canonical place of a special complement on a log smooth model. It is special by [LXZ22, Lemma 3.5]. These applications keep the divisor computing the global threshold fixed. $\square$

2.2. Adjunction with the extraction coefficient.

Lemma 2.5 (Two compatible filtrations). *Two finite filtrations of a finite-dimensional vector space admit a common compatible basis. In particular, an order filtration can be imposed together with one auxiliary filtration without decreasing its maximal total order.*

Proof. Index the filtrations decreasingly by integers, omitting repetitions. For each pair (i, j) lift a basis of

$$\frac{\mathcal{F}^i V \cap \mathcal{G}^j V}{\mathcal{F}^{i+1} V \cap \mathcal{G}^j V + \mathcal{F}^i V \cap \mathcal{G}^{j+1} V}$$

to $\mathcal{F}^i V \cap \mathcal{G}^j V$. Successive lifting along one filtration gives a basis of each quotient for the other filtration. The union is therefore compatible with both. This is the construction of [AZ22, Lemma 3.1]. $\square$

Lemma 2.6 (Retained coefficient). *Let $\pi: Y \rightarrow X$ be a plt extraction with exceptional divisor E over a smooth Fano threefold, and put*

$$(2.6) \quad (K_Y + E)|_E = K_E + \Delta_E, \quad a = A_X(E), \quad s = S_X(E).$$

Let W be the complete refinement in Notation 2.2. Suppose that $q > 0$ and that, for every $b < 1/q$ and every sufficiently large m , all m -basis divisors B_m of W satisfy that $(E, \Delta_E + bB_m)$ is log canonical. Then every prime divisor F over X whose center is contained in $\pi(E)$ satisfies

$$(2.7) \quad S_X(F) \leq qA_X(F) + (s - qa) \operatorname{ord}_F(E).$$

The assertion is also valid locally along an open subset of E if the stated log canonicity holds there and the center of F on Y meets that subset.

Proof. Fix F and $b < 1/q$. By Lemma 2.5, choose a basis compatible with F and E . Its basis divisor D_m satisfies

$$\pi^* D_m = S_m(X; E)E + \Theta_m, \quad \Theta_m \geq 0, \quad E \not\subseteq \operatorname{Supp} \Theta_m.$$

The divisor $\Theta_m|_E$ is an m -basis divisor of the actual refinement W . Inversion of adjunction gives log canonicity of $(Y, E + b\Theta_m)$ near E . Comparing it with the pullback of $K_X + bD_m$ gives

$$A_X(F) \geq bS_m(X; F) + (a - bS_m(X; E)) \operatorname{ord}_F(E).$$

Take $m \rightarrow \infty$ and then $b \rightarrow 1/q$. This proves (2.7). The local assertion follows by applying inversion of adjunction near the specified intersection of centers. This is the effective residual-divisor argument in [AZ22, Theorem 3.2, equation (3.6)]; no coefficient of a different component is subtracted from the residual boundary. $\square$

Lemma 2.7 (A flag on a surface). *Let (S, Δ) be a klt projective surface and let C be a prime curve whose coefficient in Δ is $c < 1$. Suppose that $(S, \Delta + (1 - c)C)$ is plt near C , with different Δ_C on its normalization. Let W be a bounded multigraded series with a specified normalization, and let $q > 0$. Suppose that its primary expected order along C is at most $q(1 - c)$, and that its further point expected orders are at most $qA_{C, \Delta_C}(z)$ for every point z in an open subset $U \subseteq C$. Assume that these estimates hold at finite degree with errors tending uniformly to zero over the compatible bases and the points of U . Then*

$$(2.8) \quad S(W; w) \leq qA_{S, \Delta}(w)$$

for every divisorial valuation w whose center meets U . For each $b < 1/q$, all sufficiently high degree basis divisors satisfy log canonicity after scaling by b near U .

Proof. Choose a basis compatible with w and C . Write its normalized zero divisor as $s_m C + \Gamma_m$, where Γ_m is effective without C . The restriction $\Gamma_m|_C$ is a basis divisor of the further image. For fixed $b < 1/q$, the two finite-degree estimates give

$$bs_m \leq 1 - c, \quad \text{mult}_z(b\Gamma_m|_C) \leq A_{C, \Delta_C}(z) \quad (z \in U)$$

for sufficiently large m . Inversion of adjunction gives log canonicity of $(S, \Delta + (1 - c)C + b\Gamma_m)$ near U . Reducing the coefficient of C to $c + bs_m$ preserves log canonicity. Its inequality for w , followed by passage to the limit, proves (2.8). The basis adapted to w maximizes its total order. Therefore its bound also bounds the order of every other basis divisor. Uniformity of the assumed errors gives the stated finite-degree assertion for every basis. $\square$

2.3. Ranks, initial spaces, and arbitrary valuations.

Lemma 2.8 (Ranks on a rational curve). *Let D be a nonnegative integer, and let $V \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D))$ have dimension r . At any point, the sum of the vanishing orders of a basis adapted to that point is at most*

$$(2.9) \quad rD - \frac{r(r-1)}{2}.$$

For the complete space this sum equals $D(D+1)/2$.

Proof. The dimension of the subspace vanishing to order at least j is at most $\min\{r, D - j + 1\}$. Sum this inequality over $1 \leq j \leq D$. For the complete space, powers of a parameter at the point give the orders $0, 1, \dots, D$. $\square$

Lemma 2.9 (Comparison with an asymptotically equal containing space). *Suppose that $B_m \subseteq U_m$ and $B_m \subseteq V_m$ are strict inclusions of filtered vector spaces: the filtration on B_m is induced from each ambient space, and both quotient filtrations are induced. Assume that*

$$(2.10) \quad \dim(U_m/B_m) + \dim(V_m/B_m) = O(m^2).$$

Assume that all relevant filtration jumps are $O(m)$ and that the common normalization is mN_m , where $N_m = cm^3 + O(m^2)$ and $c > 0$. Then the normalized total weights of U_m and V_m differ by $O(1/m)$. The conclusion is uniform over a family of filtrations if both assumed bounds are uniform over that family.

Proof. Total filtration weights add for a filtered subspace and its induced quotient. Each quotient in (2.10) contributes $O(m^3)$ to the weight. Division by mN_m gives the assertion. The argument applies to the direct sum of the graded pieces, so it retains their original grading and all fixed orders. $\square$

Lemma 2.10 (Degeneration of the complete series). *Let (S, Δ) be a fixed projective klt surface, let $q > 0$, and let a one-parameter group of automorphisms preserve Δ . In each section degree and each original refinement degree, let W^0 be the full initial space of a series W for this action. Assume that the rescaled sections define effective relative $\mathbb{Q}$ -Cartier divisors on $S \times \mathbb{A}^1$, with nonzero restriction to every fiber. Suppose that, for every $b < 1/q$ and every sufficiently large m , every basis divisor of W^0 formed from auxiliary eigenvectors in each retained graded piece gives a log canonical pair with Δ after scaling by b . The degree bound is required to be independent of the chosen eigenbases and any permitted parameters. All basis divisors use the original normalization of W . Then the identical log canonicity assertion holds for arbitrary basis divisors of W , and*

$$(2.11) \quad S(W; w) \leq qA_{S, \Delta}(w)$$

for every divisorial valuation w over S .

Proof. Fix w . In each original graded piece, use Lemma 2.5 to choose a basis compatible with w and the auxiliary weight filtration. Rescale each transformed section by its initial power of the parameter. The resulting polynomial family has a nonzero initial section, and its initial sections are independent auxiliary eigenvectors forming a basis of the full initial piece. After a common Cartier multiple, their zero divisors define an effective relative divisor on $S \times \mathbb{A}^1$ without a vertical component. Its central restriction is an eigenbasis divisor of W^0 ; all nonzero fibers are images of the original basis divisor under the given automorphisms.

Inversion of adjunction along $S \times \{0\}$ shows that log canonicity of the scaled central pair implies log canonicity near that fiber. Properness of S gives a neighborhood of zero where the nonzero fiber pairs are log canonical. The automorphisms then give log canonicity for the original basis divisor. Its value at w is the maximal basis weight. The uniform degree bound for the initial eigenbasis estimate is independent of w and of the chosen compatible basis. Every other basis divisor has at most this order at w . Since this holds for every w with this valuation-independent degree bound, all basis divisors of W satisfy the asserted log canonicity. Taking limits proves (2.11). $\square$

3. LINE INCIDENCE AND PROJECTION GEOMETRY

Setup 3.1. Let X be a prime Fano threefold of degree $d \in \{18, 22\}$. Its anticanonical model is an intersection of quadrics by [KP23, Theorem 2.8]. For $p \in X$, denote by $\mathcal{L}_p$ the Hilbert scheme of anticanonical lines containing p , with its scheme structure. We call a line ordinary if its normal bundle is $\mathcal{O} \oplus \mathcal{O}(-1)$, and special if its normal bundle is $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.

Lemma 3.2. *In Setup 3.1, the scheme $\mathcal{L}_p$ is finite of length at most three. Every nonempty local component is curvilinear. Its possible partitions by local lengths are*

$$(3.1) \quad 1, \quad 1+1, \quad 1+1+1, \quad 2, \quad 2+1, \quad 3.$$

A component of length one corresponds to an ordinary line; a component of length two or three corresponds to a special line. If $\mathcal{L}_p$ has length three, it is not contained scheme-theoretically in a line in $\mathbb{P}(T_p X)$.

Proof. The length bound is [KP23, Corollary A.6]. Adjunction gives $\deg N_{L/X} = -1$. Write $N_{L/X} = \mathcal{O}(a) \oplus \mathcal{O}(-1-a)$ with $a \geq 0$. The normal sequence in the anticanonical projective embedding injects this bundle into a sum of copies of $\mathcal{O}(1)$, so $a \leq 1$. The tangent space of $\mathcal{L}_p$ at $[L]$ is $H^0(L, N_{L/X}(-p))$. It has dimension zero for an ordinary line and one for a special line. A finite local algebra with zero cotangent space is the ground field; one with one-dimensional cotangent space is $\mathbb{C}[t]/(t^r)$ for some $r \geq 2$. This proves the assertions about local components and gives the exhaustive partitions.

By [KP23, Lemma 5.6], $\mathcal{L}_p \subseteq \mathbb{P}(T_p X)$ is the base of the cone $X \cap T_p X$ in the embedded projective tangent space, equivalently the common zero scheme of the second fundamental quadrics. If a length-three incidence scheme were contained in a line, every such quadratic would restrict to a degree-two polynomial vanishing on a length-three subscheme of that line. The restriction would be zero. That entire line would belong to $\mathcal{L}_p$, contrary to finiteness. $\square$

Proposition 3.3 (The line projection in degree eighteen). *Let X be as in Setup 3.1, with $d = 18$, and let $L \subseteq X$ be any line. Write $Y = \text{Bl}_L X$ and let E be its exceptional divisor. There exists an anticanonical flop from Y to $Y^+ = \text{Bl}_\Gamma Q$, where Q is a smooth quadric threefold and Γ is a smooth nondegenerate curve of degree seven and genus two. If h is the quadric hyperplane pullback and D the exceptional divisor over Γ , then the divisor identifications are*

$$(3.2) \quad h = H - 2E, \quad D = 2H - 5E, \quad H = 5h - 2D, \quad E = 2h - D.$$

The flopping curves are disjoint. They are the transforms of the other lines meeting L , together with the negative section of $E \simeq \mathbb{F}_3$ when L is special. Each has normal bundle $\mathcal{O}(-1)^{\oplus 2}$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. The image of E on Q is a normal quartic del Pezzo surface when L is ordinary. When L is special it is a nonnormal quartic with a double line, whose normalization is $\mathbb{F}_0$ or

$\mathbb{F}_2$. The strict transform of that double line in Y^+ is the inverse flopping curve corresponding to the negative section of E .

The all-line assertion, including the converse that starts with the specified X and L , is [KPZ14, Theorem 2.6]. The statements about the flopping curves and normality are [KPZ14, Proposition 2.12 and Theorem 2.13]; the normalization is [KPZ14, Proposition 2.2]. The identities in (3.2) follow from double projection and $H - E = -K_Y = 3h - D$.

Lemma 3.4. *In Proposition 3.3, the volume is*

$$(3.3) \quad \text{vol}(H - tE) = \begin{cases} 18 - 3t^2 - t^3, & 0 \leq t \leq 1, \\ 18 - 3t^2 - t^3 + 4(t - 1)^3, & 1 \leq t \leq 2, \\ 2(5 - 2t)^3, & 2 \leq t \leq 5/2, \\ 0, & t \geq 5/2. \end{cases}$$

In particular $S_X(E) = 101/72$. For $j/m = t$ away from the two walls, the actual restriction ranks satisfy

$$(3.4) \quad \dim W_{m,j} = \frac{m^2}{2}w(t) + O(m), \quad w(t) = \begin{cases} t^2 + 2t, & 0 \leq t < 1, \\ -3t^2 + 10t - 4, & 1 < t < 2, \\ 4(5 - 2t)^2, & 2 < t < 5/2. \end{cases}$$

The error is uniform in j . If $j \geq 2m$, the full section space factors by D^{j-2m} , with residual space $H^0(Q, \mathcal{O}_Q(5m - 2j))$.

Proof. The intersection tables on the two blowups are

$$\begin{aligned} H^3 &= 18, & H^2E &= 0, & HE^2 &= -1, & E^3 &= 1, \\ h^3 &= 2, & h^2D &= 0, & hD^2 &= -7, & D^3 &= -23. \end{aligned}$$

The divisors $H - tE$ are nef on Y for $0 \leq t \leq 1$. For $1 \leq t \leq 2$ they are the nef combination $(2 - t)(H - E) + (t - 1)h$ on Y^+ . Their cubes give the first two pieces of (3.3).

For $t \geq 2$, the class is $(5 - 2t)h + (t - 2)D$ on Y^+ . For nonnegative integers q, k , every section of $qh + kD$ is the pullback of a section of $\mathcal{O}_Q(q)$ times the canonical section of kD : off Γ it descends to Q , and it extends across Γ because Q is normal and Γ has codimension two. If $q < 0$, there exist no sections, since the pushforward of an effective divisor cannot have negative hyperplane degree. Let D also denote the effective strict transform on Y . The flop identifies the full section spaces: their rational functions satisfy identical divisorial inequalities under the correspondence of prime divisors, since the map is an isomorphism in codimension one. Thus the factorization also holds on Y , before restriction to E . Since E is not a component of D , multiplication by the nonzero section $s_D|_E$ is injective, and preserves the actual residual image rank. This proves the factorization, the support bound, and the last two volume pieces. The integrals on the three chambers are $67/4$, $33/4$, and $1/4$, giving $S_X(E) = 101/72$.

On either nef model, Kawamata–Viehweg vanishing, in the boundary-free case of [DEL00, p. 142, Nadel Vanishing Theorem], identifies the section dimension with its Riemann–Roch polynomial: subtracting the canonical divisor adds a nef and big anticanonical divisor. The difference between the values at j and $j + 1$ is $-m^2v'(t)/6 + O(m)$, which gives (3.4). In the terminal chamber use the exact quadric Hilbert polynomial instead. The finitely many wall grades have ranks $O(m^2)$ and contribute $O(m^3)$ to any order sum whose jumps are $O(m)$. $\square$

4. CURVE CENTERS

We first treat curves whose degree makes a local colength estimate sufficient. The remaining curves will be treated by their line or conic projections.

Lemma 4.1. *There exists an absolute number $\alpha > 1/3$ such that every ideal of positive colength at most 20 on a smooth complex surface germ has log canonical threshold at least α . We may choose $\alpha \leq 1/2$.*

Proof. By [Zhu18, Definition 3.2, Lemmas 3.4 and 5.2], an ideal whose threshold is at most $1/3$ on a smooth surface germ has colength at least 21. We explain why the strict inequality for colength at most 20 is uniform.

An ideal of colength ℓ contains the ℓ th power of the maximal ideal: successive powers in its Artinian quotient strictly decrease until they vanish, since equality would force vanishing by Nakayama's lemma [Sta26, Tag 00DV]. Thus each ideal under consideration contains the twentieth power of the maximal ideal. In local coordinates it is represented by a polynomial ideal containing $(u, v)^{20}$. A monomial degeneration preserves its colength and does not increase its log canonical threshold, as in [Zhu18, Lemma 3.4]. There exist finitely many possible monomial ideals containing $(u, v)^{20}$, because their standard monomials belong to the finite set $u^i v^j$ with $i + j < 20$. The minimum of their thresholds, capped by $1/2$, is therefore a number $\alpha > 1/3$ with the required property. The unit ideal has infinite threshold and satisfies this bound as well. $\square$

Proposition 4.2. *There exists an absolute constant $\epsilon_0 > 0$ with the following property. Let X be a prime Fano threefold with very ample $H = -K_X$ and $d = H^3 \geq 6$. If $C \subseteq X$ is an integral curve of degree $e = H \cdot C$ satisfying*

$$(4.1) \quad \left\lfloor \frac{2d+2}{e} \right\rfloor \leq 20,$$

then $\delta_C(X) \geq 1 + \epsilon_0$. In particular, the conclusion holds for $d = 18$, $e \geq 2$, and for $d = 22$, $e \geq 3$.

Proof. We refine by a Picard number one K3 surface and a curve avoiding its intersection with C . A multiplier ideal then gives one point where the residual basis divisor has a uniform threshold.

Step 1. We describe the complete refined spaces. By [AZ21, Lemma 5.10 and Theorem 5.8], a very general anticanonical surface through a prescribed point has Picard group generated by the restriction of H . Choose the prescribed point outside C and intersect this very general set with the Bertini and transversality open sets. We obtain a smooth K3 surface $T \in |H|$ with $\text{Pic}(T) = \mathbb{Z}[L]$, where $L = H|_T$, such that $Z = T \cap C = \{p_1, \dots, p_e\}$ consists of distinct smooth points of C . Adjunction gives $K_T = 0$; the anticanonical exact sequence and vanishing give $H^1(T, \mathcal{O}_T) = 0$. Choose a smooth $G \in |L|$ avoiding Z . The flag $X \supset T \supset G$ is Cartier.

Put $h_n = h^0(T, nL)$. K3 Riemann–Roch gives $h_0 = 1$ and $h_n = dn^2/2 + 2$ for $n \geq 1$. The refinement by T is exactly

$$W_{m,j} = H^0(T, (m-j)L) \quad (0 \leq j \leq m).$$

For $k = m - j \geq 1$, surjectivity follows from $H^1(X, (k-1)H) = H^1(X, K_X + kH) = 0$. For $k = 0$, it follows from $H^1(X, K_X) = 0$, by Serre duality [Sta26, Tags 0FVV and 0FW0] and $H^2(X, \mathcal{O}_X) = 0$. Thus $N_m = \sum_{n=0}^m h_n$.

For a basis divisor D of W compatible with G , write

$$D = a_m G + \Gamma, \quad \Gamma \geq 0, \quad G \not\subseteq \text{Supp } \Gamma.$$

Its numerical data, independent of the chosen compatible basis, are

$$\begin{aligned} D &\sim_{\mathbb{Q}} c_m L, & c_m &= \frac{\sum_{n=0}^m n h_n}{m N_m} \rightarrow \frac{3}{4}, \\ a_m &= \frac{\sum_{n=0}^m \sum_{q=1}^n h_{n-q}}{m N_m} = \frac{\sum_{\ell=0}^{m-1} (m-\ell) h_\ell}{m N_m} \rightarrow \frac{1}{4}, & \Gamma &\sim_{\mathbb{Q}} \lambda_m L, \quad \lambda_m = c_m - a_m \rightarrow \frac{1}{2}. \end{aligned}$$

The leading terms of the three sums are $dm^3/6$, $dm^4/8$, and $dm^4/24$, respectively.

Step 2. We bound the residual multiplier ideal. Every integral curve $C_0 \subseteq T$ is linearly equivalent to rL for an integer $r \geq 1$. The maximal basis coefficient along it is

$$\frac{1}{mN_m} \sum_{n=0}^m \sum_{q=1}^{\lfloor n/r \rfloor} h_{n-qr} \leq a_m,$$

since h_n is nondecreasing. This inequality bounds the coefficient in every basis divisor, whether or not it is compatible with C_0 . It therefore bounds $\text{ord}_{C_0}(\Gamma)$.

For all sufficiently large m , set

$$\eta_m = \left(1 - \frac{1}{m}\right) \min \left\{1, \frac{1}{4a_m}, \frac{1}{2\lambda_m}\right\}.$$

Then $\eta_m \rightarrow 1$, $4\eta_m a_m < 1$, and $4\eta_m \lambda_m < 2$. The divisor $B = 4\eta_m \Gamma$ has coefficients less than one along every curve, so its multiplier ideal $\mathcal{J} = \mathcal{J}(T, B)$ has finite cosupport. Moreover

$$2L - (K_T + B) \sim_{\mathbb{Q}} (2 - 4\eta_m \lambda_m)L$$

is ample. Nadel vanishing [DEL00, p. 142, Nadel Vanishing Theorem], with line bundle $2L - K_T$ and boundary B , makes $H^0(T, 2L) \rightarrow H^0(T, \mathcal{O}_T(2L) \otimes \mathcal{O}_T/\mathcal{J})$ surjective. It follows that

$$\text{length}(\mathcal{O}_T/\mathcal{J}) \leq h^0(T, 2L) = 2d + 2.$$

Hence some $p \in Z$, allowed to depend on m and the basis, has local colength at most $\lfloor (2d + 2)/e \rfloor \leq 20$. Lemma 4.1 gives $\text{lct}_p(T; \mathcal{J}) \geq \alpha$.

Step 3. We pass from the ideal to the divisor. For every prime divisor F over T with center containing p , the definition of the multiplier ideal and the preceding threshold imply

$$\text{ord}_F(\mathcal{J}) + A_T(F) > \text{ord}_F(B), \quad A_T(F) \geq \alpha \text{ord}_F(\mathcal{J}).$$

Consequently $\text{lct}_p(T; B) \geq \alpha/(1 + \alpha)$ and

$$\text{lct}_p(T; \Gamma) \geq \kappa \eta_m, \quad \kappa = \frac{4\alpha}{1 + \alpha} \in (1, 4/3].$$

This argument also covers the case $\mathcal{J}_p = \mathcal{O}_{T,p}$. Because G avoids Z , the divisors D and Γ agree near p . Thus every G -compatible basis divisor has threshold at least $\kappa \eta_m$ at some point of Z .

Step 4. We recover the generic curve estimate. Fix a prime divisor F over X with $C \subseteq c_X(F)$. Choose a basis of $H^0(X, mH)$ compatible with F and the lexicographic filtration of the full flag. These are two filtrations, so Lemma 2.5 applies. Write its normalized divisor as

$$D_{X,m} = s_m T + \Theta_m, \quad s_m = S_m(X; T) = 1 - c_m \rightarrow \frac{1}{4}.$$

The restriction $\Theta_m|_T$ is a G -compatible basis divisor of W . Put $q_m = \min\{1/s_m, \kappa \eta_m\}$. Step 3 gives a point $p \in C \cap T$ at which $(T, q_m \Theta_m|_T)$ is log canonical. Inversion of adjunction and $q_m s_m \leq 1$ imply that $(X, q_m D_{X,m})$ is log canonical in a neighborhood of p . The irreducible center $c_X(F)$ contains p , so its generic point belongs to this neighborhood. Therefore

$$A_X(F) \geq q_m S_m(X; F).$$

Since $q_m \rightarrow \kappa$ and $S_m(X; F) \rightarrow S_X(F)$, we obtain $A_X(F) \geq \kappa S_X(F)$. The constant $\epsilon_0 = \kappa - 1 > 0$ is independent of X and C . Finally, $38/2 = 19 \leq 20$ and $\lfloor 46/3 \rfloor = 15 \leq 20$, which give the two stated specializations. $\square$

4.1. Lines in degree eighteen.

Proposition 4.3. *Let X be a prime Fano threefold of degree 18, and let L be an ordinary line. Then*

$$(4.2) \quad \delta_L(X) \geq \frac{63000}{62779} > 1.$$

Proof. Let $E \simeq \mathbb{F}_1$ be the exceptional surface of $Y = \text{Bl}_L X$, with negative section s and ruling class f . Then

$$H|_E = f, \quad -E|_E = s + f, \quad (mH - jE)|_E = js + (m + j)f.$$

We estimate every horizontal prime curve $C \subseteq E$, keeping the actual ranks in Lemma 3.4.

If $a = C \cdot f \geq 2$, effectivity gives $\text{ord}_C(u) \leq j/a$ for every nonzero $u \in W_{m,j}$. Hence $S(W; C) \leq S_X(E)/a \leq 101/144$. If $a = 1$, write $C \sim s + bf$ with $b \geq 0$. For $0 \leq a_0 \leq b_0$ we obtain

$$h^0(\mathbb{F}_1, a_0s + b_0f) = (a_0 + 1)(b_0 + 1) - \frac{a_0(a_0 + 1)}{2}.$$

Subtracting kC imposes at least as many conditions as subtracting ks . For $0 \leq k \leq j \leq 2m$, the dimension of the vanishing subspace is therefore bounded both by $\dim W_{m,j}$ and by

$$\frac{m^2}{2}P(j/m, k/m) + O(m), \quad P(t, u) = (t - u)(2 + t + u) = t^2 + 2t - 2u - u^2.$$

For $0 \leq t \leq 1$,

$$\int_0^1 \int_0^t P(t, u) du dt = \frac{1}{2}.$$

For $1 \leq t \leq 2$, set $v = t - 1$ and $c = 6/5$. The interval $0 \leq u \leq cv^2$ lies in $[0, t]$. Use the actual rank density $w(t)$ there and $P(t, u)$ elsewhere. Since $P(t, u) - w(t) = 4v^2 - 2u - u^2$, the saving is

$$\int_0^1 \int_0^{cv^2} (4v^2 - 2u - u^2) du dv = \frac{4c - c^2}{5} - \frac{c^3}{21} = \frac{516}{875}.$$

Thus the middle contribution to the double integral is at most $29/6 - 516/875$.

For the tail, write $q = 5m - 2j$ and $k_0 = j - 2m$. The exact factorization in Lemma 3.4 gives

$$W_{m,j} = (s_D|_E)^{k_0} U_q, \quad U_q \subseteq H^0(E, q(2s + 3f)),$$

with $\dim U_q = \dim W_{m,j}$. Since $D|_E \sim 5s + 7f$, its coefficient b_C along C is at most five, by intersection with f . This fixed contribution to $S(W; C)$ is

$$\frac{b_C}{6} \int_2^{5/2} (t - 2) 4(5 - 2t)^2 dt = \frac{b_C}{72} \leq \frac{5}{72}.$$

For the moving part put $r = 5 - 2t$. The containing-system polynomial is $P_{\text{tail}}(r, u) = 8r^2 - 2ru - u^2$, while its rank density is $4r^2$. Using the cap for $0 \leq u \leq cr$ gives

$$\int_0^{2r} \min\{4r^2, P_{\text{tail}}(r, u)\} du \leq \left(\frac{28}{3} - \frac{348}{125} \right) r^3.$$

Its integral on $2 \leq t \leq 5/2$ is at most $7/6 - 87/250$. The total moving double integral is at most

$$\frac{1}{2} + \frac{29}{6} - \frac{516}{875} + \frac{7}{6} - \frac{87}{250} = \frac{4867}{875}.$$

Because $N_m = 3m^3 + O(m^2)$ and each dimension has leading coefficient $m^2/2$, division by mN_m supplies the factor $1/6$. Therefore

$$S(W; C) \leq \frac{4867}{5250} + \frac{5}{72} = \frac{62779}{63000}.$$

There exist $O(m^2)$ terms in the double order sum, each with error $O(m)$. The wall grades also contribute $O(m^3)$. The bounds are uniform in the horizontal section because all its containing systems were compared with s .

The only proper centers on E dominating L are these horizontal curves. Their generic thresholds are $1/S(W; C)$. Since $A_X(E) = 2$, [AZ22, Theorem 3.2] gives

$$\delta_L(X) \geq \min \left\{ \frac{144}{101}, \frac{63000}{62779} \right\} = \frac{63000}{62779}.$$

□

Proposition 4.4. *Let L be a special line on a prime Fano threefold X of degree 18. Blow up L , and then blow up the negative section of the exceptional $\mathbb{F}_3$. Let F be the second exceptional divisor, regarded as a divisor over X . Then $A_X(F) = 3$ and*

$$(4.3) \quad \text{vol}(H - tF) = \begin{cases} 18 - \frac{3}{2}t^2, & 0 \leq t \leq 2, \\ 18 - \frac{3}{2}t^2 + \frac{3}{4}(t-2)^3, & 2 \leq t \leq 4, \\ 0, & t \geq 4. \end{cases} \quad S_X(F) = \frac{43}{18}.$$

This holds for both normalization types of the quartic surface and for all contact configurations.

Proof. We first construct the initial nef model from the doubled line. We then reverse the two blowups on the quadric side and verify global generation for every contact scheme.

Step 1. We identify the valuation and its first chamber. The quotient $N_{L/X}^* = \mathcal{O}(-1) \oplus \mathcal{O}(2) \rightarrow \mathcal{O}(-1)$ defines the planar ribbon supported on L . In transverse smooth coordinates (x, y) its ideal is (y, x^2) ; its Hilbert polynomial is $2m + 1$ because $0 \rightarrow \mathcal{O}_L(-1) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_L \rightarrow 0$. On the first blowup, with $y = xu$, its ideal is $x(x, u)$, and the negative section is $x = u = 0$. Its blowup has $v_F(x) = 1$, $v_F(y) = 2$, and discrepancy three.

The conic C spans a plane Π . Since X is cut out by quadrics and contains no plane, $X \cap \Pi = C$ scheme-theoretically: every restricted quadratic is a scalar multiple of the equation of C , and some restriction is nonzero. A plane in X is impossible because its class would be aH with degree $18a$, rather than one. Hence $\mathcal{I}_C(H)$ is globally generated. On $U = \text{Bl}_C X$, the charts $y = x^2t$ and $x^2 = yt$ show that U is normal, with an A_1 surface singularity along the curve in the second chart. Its Cartier exceptional divisor is $B = 2F$. The powers of (y, x^2) are precisely the even valuation ideals, since $a + 2b \geq 2j$ is equivalent to $\lfloor a/2 \rfloor + b \geq j$. Thus this blowup computes the F -filtration.

The divisor $H - B$ is globally generated. The regular-immersion blowup formulas give

$$H^3 = 18, \quad H^2B = 0, \quad HB^2 = -2, \quad B^3 = 0.$$

For the last identity, adjunction gives $\det N_{C/X} = \omega_C \otimes \omega_X^{-1}|_C = \mathcal{O}_C$. These are Segre-class formulas for the locally complete-intersection conic and do not require it to be reduced. Therefore $H - (t/2)B$ is nef for $0 \leq t \leq 2$, and its cube is $18 - 3t^2/2$.

Step 2. We construct the reversed blowup. Use Proposition 3.3, and let $\Lambda \subseteq Q$ be the double line of its nonnormal quartic surface. Its strict transform is the inverse flopping curve corresponding to the negative section $s \subseteq \text{Bl}_L X$, by that proposition. The ordinary blowup valuation of Λ is also the blowup valuation of its strict transform in $\text{Bl}_\Gamma Q$, since these spaces agree at the generic point of Λ . It has discrepancy two on this smooth threefold, and hence on $\text{Bl}_L X$ by crepancy. Its center on $\text{Bl}_L X$ lies in the negative section and cannot be a closed point: a divisor centered at a smooth threefold point has discrepancy at least three. At a smooth codimension-two center the unique discrepancy-two divisor is its ordinary blowup divisor. It is therefore the valuation F .

Set $V = \text{Bl}_\Lambda Q$, with exceptional surface G_0 . Projection from Λ is a $\mathbb{P}^1$ -bundle $p: V \rightarrow \mathbb{P}^2$. For an explicit description take $Q = (xv - yw - z^2 = 0)$ and $\Lambda = (z = w = v = 0)$. Over $[z : w : v]$, its resolved fiber is $xv - yw - tz^2 = 0$ in $\mathbb{P}_{[x:y:t]}^2$; the three coefficients do not vanish

simultaneously. Put $\ell = h - G_0 = p^*\mathcal{O}_{\mathbb{P}^2}(1)$. The surface $G_0 \simeq \mathbb{F}_1$ has negative section g , ruling f , and

$$h|_{G_0} = f, \quad G_0|_{G_0} = -g, \quad (-K_V)|_{G_0} = g + 3f.$$

The restriction of p contracts g to $o = [1 : 0 : 0]$.

The quartic is the intersection of Q with a quadratic equation in z, w, v . To see this, subtract a multiple of the equation of Q : proportionality of its normal linear forms along $[x : y] \in \Lambda$ forces a constant proportionality factor, leaving an equation in $\mathcal{I}_\Lambda^2$. Its strict transform is $T = p^{-1}(K) \sim 2\ell$, for an integral plane conic K , which is smooth. It is the normalization of the quartic. By [KPZ14, Proposition 2.2], its classes are

T	$h _T$	$G_0 _T$	Γ_T
$\mathbb{F}_0$	$\sigma + 2f_T$	σ	$2\sigma + 3f_T$
$\mathbb{F}_2$	$\sigma + 3f_T$	$\sigma + f_T$	$2\sigma + 5f_T$

Here $\Gamma_T \simeq \Gamma$ is smooth: blowing up the finite Cartier ideal induced by Λ on Γ does not change Γ . Its intersection $Z_0 = \Gamma_T \cap G_0$ has length three and maps isomorphically to $\Gamma \cap \Lambda$.

Blow up Γ_T to obtain the smooth threefold W , with exceptional divisor D . Use h, G for the pullbacks of h, G_0 . The strict transform T_W is isomorphic to T , and

$$A = -K_W = 3h - G - D, \quad \ell = h - G, \quad T_W = 2h - 2G - D.$$

Step 3. We prove that A is globally generated. The subscheme $Z_0 \subseteq G_0$ embeds in the base Λ of the ruling. Moreover $\text{length}(Z_0 \cap g) \leq 2$. If $o \notin K$, the curves g and T are disjoint; if $o \in K$, then g is a fiber of $T \rightarrow K$, and Γ_T has degree two on that fiber.

For the ruling $r: G_0 \rightarrow \Lambda$,

$$r_*\mathcal{O}_{G_0}(g + 3f) = \mathcal{O}_\Lambda(3) \oplus \mathcal{O}_\Lambda(2).$$

Its evaluation kernel at Z_0 has rank two and degree two. Write it as $\mathcal{O}(a) \oplus \mathcal{O}(2-a)$ with $a \geq 1$. Its injection into $\mathcal{O}(3) \oplus \mathcal{O}(2)$ gives $a \leq 3$. If $a = 3$, the entire $\mathcal{O}(3)$ summand lies in the kernel, forcing $Z_0 \subseteq g$, contrary to $\text{length}(Z_0 \cap g) \leq 2$. The kernel is therefore $\mathcal{O}(2) \oplus \mathcal{O}$ or $\mathcal{O}(1)^{\oplus 2}$ and is globally generated. Its pullback generates $\mathcal{I}_{Z_0/G_0}(g + 3f)$. Locally, write Z_0 as $(u - P(t), Q(t))$ with t a base parameter and u a fiber parameter. The kernel of evaluation of fiber-linear forms supplies $u - P(t)$ and $Q(t)$ times a form nonzero at the point; these generate the required ideal.

There exist exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{I}_{\Gamma_T}(3h - 2G_0) \rightarrow \mathcal{I}_{\Gamma_T}(3h - G_0) \rightarrow \mathcal{I}_{Z_0/G_0}(g + 3f) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_V(h) \rightarrow \mathcal{I}_{\Gamma_T}(3h - 2G_0) \rightarrow \mathcal{O}_T((3h - 2G_0)|_T - \Gamma_T) \rightarrow 0. \end{aligned}$$

The last term has degree -1 on every ruling fiber of T , so all its cohomology vanishes. Also $H^1(V, h) = 0$. These sequences give surjectivity on global sections onto the ideal on G_0 . A further sequence is

$$0 \rightarrow \mathcal{O}_V(h + G_0) \rightarrow \mathcal{I}_{\Gamma_T}(3h - G_0) \rightarrow \mathcal{O}_T(3f_T) \rightarrow 0.$$

Here $H^1(V, h + G_0) = 0$, since its quotient by $\mathcal{O}_V(h)$ is $\mathcal{O}_{G_0}(f - g)$, of fiber degree -1 . Thus the residual system on T is generated as well.

Along G_0 , generation follows from the first surjection and Nakayama's lemma [Sta26, Tag 00DV]; the equation of G_0 is a non-zero-divisor on Γ_T . At a point of $\Gamma_T \setminus G_0$, sections from $\mathcal{O}_V(h + G_0)$ supply the local equation of T , while lifts of $\mathcal{O}_T(3f_T)$ supply the equation of Γ_T within T . They generate its ideal. On $T \setminus \Gamma_T$ the latter system has no base point. Outside $T \cup G_0$, multiply a section of h by the equations of T and G_0 . This proves generation of $\mathcal{I}_{\Gamma_T}(3h - G_0)$ everywhere, and hence of A on W .

Step 4. We identify the complete section filtration. The prime divisors on W exceptional over X are T_W , of discrepancy two, and G , of discrepancy three. The remaining divisors correspond

to divisors on X under the inverse link. For the strict transform M of a general anticanonical divisor, chosen not to contain L , the canonical-divisor identity gives

$$K_W = -M + T_W + 2G, \quad M = 5h - 2D - G.$$

Consequently

$$H^0(X, mH)^{v_F \geq j} = H^0(W, mM - jG).$$

This equality uses the inequalities at all prime divisors on the common function field. It does not require a morphism $W \rightarrow X$. At nonexceptional divisors the inequalities express that the rational section is global on normal X ; its orders at the two exceptional divisors are nonnegative. Imposing order j at G is exactly the extra condition displayed.

For $2 \leq t \leq 4$ put

$$P_t = \frac{4-t}{2}A + \frac{t-2}{2}\ell. \quad M - tG = P_t + \frac{t}{2}T_W.$$

The divisor P_t is nef by Step 3. A general ruling fiber of T_W has intersections $T_W \cdot f_T = -2$, $G \cdot f_T = 1$, $h \cdot f_T = 1$, and $D \cdot f_T = 2$. Thus $(M - tG) \cdot f_T = -t$, so any effective multiple contains T_W with coefficient at least $t/2$. Removing that factor identifies its section space with the complete space of the corresponding multiple of P_t . For $t > 4$, intersection with a general fiber of $W \rightarrow \mathbb{P}^2$, away from K , is $4 - t < 0$. These moving fibers exclude pseudoeffectivity.

Step 5. We compute the volume. On V , $h^3 = 2$, $h^2G_0 = 0$, $hG_0^2 = -1$, and $G_0^3 = -1$. The curve Γ_T has h -degree seven and G_0 -degree three. Thus $\deg N_{\Gamma_T/V} = 20$, and on W ,

$$hD^2 = -7, \quad GD^2 = -3, \quad D^3 = -20.$$

A product of D with two classes pulled back from V is zero. Expansion yields

$$A^3 = 12, \quad A^2\ell = 8, \quad A\ell^2 = 2, \quad \ell^3 = 0.$$

For $z = (t-2)/2$, the cube of $P_t = (1-z)A + z\ell$ is

$$12(1-z)^3 + 24(1-z)^2z + 6(1-z)z^2 = 18 - \frac{3}{2}t^2 + \frac{3}{4}(t-2)^3.$$

This proves (4.3), first at rational t and then at all t by continuity. Its two nonzero integrals are 32 and 11, so $S_X(F) = 43/18$. Every length-three contact scheme was retained in Step 3. $\square$

The following local criterion is [CRZ19, Proposition 3.3]; we include its proof.

Lemma 4.5. *Let C be a smooth curve through a point p on a smooth surface, and let $\Omega \geq 0$ have no component C . If $0 \leq a \leq 1$ and*

$$(4.4) \quad \text{mult}_p \Omega \leq 1, \quad (C \cdot \Omega)_p \leq 2 - a,$$

then $(S, aC + \Omega)$ is log canonical at p .

Proof. Put $r = \text{mult}_p \Omega$. If $a + r \leq 1$, restrict to a general smooth curve through p transverse to the tangent cones of the boundary components. Its boundary has coefficient $a + r \leq 1$, and inversion of adjunction applies.

Suppose that $a + r > 1$. Blow up p , with exceptional curve J and strict transforms denoted by a tilde. The transformed boundary is

$$a\tilde{C} + \tilde{\Omega} + (a + r - 1)J, \quad 0 < a + r - 1 \leq 1.$$

Away from $\tilde{C} \cap J$, the intersection of $\tilde{\Omega}$ with J is at most its total intersection $r \leq 1$. Increasing the coefficient of J to one and applying inversion of adjunction proves log canonicity there. At $\tilde{C} \cap J$, the residual intersection with $\tilde{C}$ is

$$(C \cdot \Omega)_p - r + (a + r - 1) = (C \cdot \Omega)_p + a - 1 \leq 1.$$

Increase the coefficient of $\tilde{C}$ to one and apply inversion of adjunction. This proves log canonicity everywhere above p . The argument works over any characteristic-zero field after passage to an algebraic closure. $\square$

Proposition 4.6. *Let X be a prime Fano threefold of degree 18, and let $L \subseteq X$ be a special line. Then $\delta_L(X) \geq 72/71$.*

Proof. Write $\pi: Y = \text{Bl}_L X \rightarrow X$, with exceptional surface $E \simeq \mathbb{F}_3$, negative section s , and ruling f . For the actual refinement by E , put

$$b_m(C) = \frac{1}{mN_m} \sum_j \sum_{k \geq 1} \dim W_{m,j}^{\text{ord}_C \geq k}$$

for each prime horizontal curve $C \subseteq E$.

Step 1. We bound these horizontal orders. By Lemma 3.4, $S_X(E) = 101/72$ and the terminal fixed factor is D^{j-2m} . On E ,

$$D|_E = 5s + 12f, \quad h|_E = 2s + 5f.$$

The coefficient of any horizontal prime section in $D|_E$ is at most five. Its fixed contribution to $b_m(C)$ is therefore at most

$$\frac{5}{6} \int_2^{5/2} (t-2) 4(5-2t)^2 dt = \frac{5}{72}.$$

All remaining order spaces are contained in the complete systems of class $ts + (1+2t)f$ for $0 \leq t \leq 2$, or $(5-2t)(2s+5f)$ in the terminal chamber.

For $as + bf$ on $\mathbb{F}_3$, the positive part is the divisor itself if $b \geq 3a \geq 0$. If $0 \leq b < 3a$, the negative part is $(a-b/3)s$ and the positive square is $b^2/3$. Integration after subtracting successive multiples of s gives the following complete-system order integrals, where $r = 5-2t$:

	$0 \leq t \leq 1$	$1 \leq t \leq 2$	$2 \leq t \leq 5/2$
s	$t^2 + t^3$	$(1+2t)^2(7t-1)/27$	$325r^3/27$
$s + 3f$	t^2	$(1+2t)^3/27$	$125r^3/27$

For example, in the middle entry of the first row, the fixed part contributes $(t-1)(1+2t)^2/9$, and the remaining moving integral is $2(1+2t)^3/27$. The integrated rows are respectively

$$\left(\frac{7}{12}, \frac{329}{54}, \frac{325}{216} \right), \quad \left(\frac{1}{3}, \frac{68}{27}, \frac{125}{216} \right).$$

Every prime section other than s has class $s + bf$ with $b \geq 3$, so its order spaces are bounded by the second row. Dividing by six and including the fixed part gives

$$\limsup_m b_m(s) \leq \frac{619}{432}, \quad \limsup_m b_m(C) \leq \frac{277}{432} \quad (C \neq s, C \cdot f = 1).$$

For $C \cdot f = a \geq 2$, fiber degree bounds its order in $W_{m,j}$ by j/a , and hence

$$\limsup_m b_m(C) \leq \frac{S_X(E)}{a} \leq \frac{101}{144}.$$

The bounds are uniform in C . For sections we used one containing class, and for multisections one fiber-degree inequality. The formula for sections of a ruled surface is a finite sum of linear terms, giving an $O(m)$ error for each order space and an $O(m^3)$ error in the total order sum.

Step 2. We retain the second extraction. Let F be the exceptional divisor of $\text{Bl}_s Y \rightarrow Y$. Proposition 4.4 gives $S_X(F) = 43/18$. Fix a prime divisor G over X centered at L , with $G \neq E$. Choose an m -basis divisor D_m compatible with G and E , and write

$$\pi^* D_m = S_m(X; E)E + \Omega_m.$$

Every basis has F -order at most $S_m(X; F)$. Since $\text{ord}_F(E) = 1$,

$$\text{mult}_s \Omega_m \leq S_m(X; F) - S_m(X; E).$$

This inequality does not require a basis compatible with three filtrations.

Fix a rational $\lambda < 72/71$ sufficiently close to $72/71$. If the center of G on Y is a horizontal curve other than s , Step 1 and inversion of adjunction show that $(Y, (\lambda S_m(X; E) - 1)E + \lambda \Omega_m)$ is log canonical at its generic point for large m . The coefficient of E is less than one.

If that center is s , localize at its generic point. This gives a smooth surface over $\mathbb{C}(L)$ with smooth curve E . Set $a_m = \lambda S_m(X; E) - 1$. For large m , $0 \leq a_m \leq 1$,

$$\lambda \operatorname{mult}_s \Omega_m \leq \lambda (S_m(X; F) - S_m(X; E)) < 1,$$

because $43/18 - 101/72 = 71/72$. The residual intersection with E is at most $\lambda b_m(s)$, and

$$\lambda \left(\frac{101}{72} + \frac{619}{432} \right) = \lambda \frac{1225}{432} < 3.$$

Thus $\lambda b_m(s) \leq 2 - a_m$ for large m . Lemma 4.5 gives the required log canonicity.

It follows that $A_X(G) \geq \lambda S_m(X; G)$. First let $m \rightarrow \infty$, then let $\lambda \rightarrow 72/71$. For $G = E$, the ratio is $144/101 > 72/71$. If the center is a prime divisor $T \subseteq X$ containing L , write $T \sim aH$ with $a \geq 1$. Direct integration gives $S_X(T) = 1/(4a)$ and $A_X(T) = 1$. These cases exhaust the centers containing L , and prove the assertion. $\square$

Corollary 4.7. *Let X be a prime Fano threefold of degree 18. Every prime divisor over X whose center has positive dimension has positive beta invariant.*

Proof. For prime divisors on X the ratio A_X/S_X is at least four. Integral curves of degree at least two are covered by Proposition 4.2. Lines are covered by Propositions 4.3 and 4.6. $\square$

4.2. Low-degree curves in degree twenty-two.

Proposition 4.8. *Let X be a prime Fano threefold of degree 22, and let $C \subseteq X$ be an integral curve of degree $e \in \{1, 2\}$. Then $C \simeq \mathbb{P}^1$. Write $\pi: Y = \operatorname{Bl}_C X \rightarrow X$, with exceptional surface E . The possible normal bundles and restrictions are*

$$(4.5) \quad \begin{array}{c|c|c|c|c} e & N_{C/X} & E & H|_E & -E|_E \\ \hline 1 & \mathcal{O} \oplus \mathcal{O}(-1) & \mathbb{F}_1 & f & s + f \\ 1 & \mathcal{O}(1) \oplus \mathcal{O}(-2) & \mathbb{F}_3 & f & s + 2f \\ 2 & \mathcal{O}^{\oplus 2} & \mathbb{F}_0 & 2f & s \\ 2 & \mathcal{O}(1) \oplus \mathcal{O}(-1) & \mathbb{F}_2 & 2f & s + f. \end{array}$$

Here f is a ruling fiber, and s is the negative section, or a section of square zero on $\mathbb{F}_0$.

The line link flops Y to $\operatorname{Bl}_Z V_5$, where Z is a rational normal quintic in the smooth quintic del Pezzo threefold. The conic link flops Y to $\operatorname{Bl}_Z Q$, where Z is a smooth rational sextic in a smooth quadric. Writing B for the target hyperplane pullback and R for its exceptional divisor, the class identities are

$$(4.6) \quad \begin{array}{c|cccc} & B & R & H & E \\ \hline e = 1 & H - 2E & H - 3E & 3B - 2R & B - R \\ e = 2 & H - 2E & 2H - 5E & 5B - 2R & 2B - R. \end{array}$$

The anticanonical class $A = H - E$ is base point free on both sides. For the line link the flopping curves are disjoint, have normal bundle $\mathcal{O}(-1)^{\oplus 2}$ or $\mathcal{O} \oplus \mathcal{O}(-2)$, and their inverse images map to bisecant lines of Z . They include the negative section of E precisely when C is special.

The volume functions, zero beyond the indicated intervals, are

$$(4.7) \quad \begin{array}{c|c|c} & e = 1 & e = 2 \\ \hline 0 \leq t \leq 1 & 22 - 3t^2 - t^3 & 22 - 6t^2 \\ t = 1 + u, \ 0 \leq u \leq 1 & 18 - 9u - 6u^2 + 2u^3 & 16 - 12u - 6u^2 + 4u^3 \\ t \geq 2 & 5(3 - t)^3 \ (t \leq 3) & 2(5 - 2t)^3 \ (t \leq 5/2). \end{array}$$

In particular $S_X(E) = 17/11$ for a line and $117/88$ for a conic. Every prime divisor F centered at C , except possibly those centered at s on Y when $E \neq \mathbb{F}_0$, satisfies

$$(4.8) \quad \beta_X(F) \geq (2 - S_X(E)) \operatorname{ord}_F(E) > 0.$$

Proof. The anticanonical embedding identifies integral curves of degrees one and two with a line and a smooth plane conic. For the conic assertion, a nondegenerate integral curve of degree two spans at most a plane, and an integral plane quadratic over $\mathbb{C}$ is smooth. Adjunction gives $\deg N_{C/X} = e - 2$. For a line, injection into the ambient normal bundle gives the first two splitting types, as in Lemma 3.2. For a conic, the Hilbert scheme of conics is $\mathbb{P}^2$ by [KPS18, Proposition 2.3.6(v)]. Its tangent space is $H^0(C, N_{C/X})$, so $\mathcal{O}(a) \oplus \mathcal{O}(-a)$ must have $a = 0$ or 1. The tautological bundles on their projectivizations give (4.5).

The line link and its curve description are [KPS18, Theorem 5.2.2, Remark 5.2.4, and Lemma 5.2.5]. The conic link and base point freeness are [KP18, Theorem 2.2 and Lemma 2.4]. These statements apply to every specified smooth X and line or smooth conic. For a line, its ideal in the anticanonical embedding is generated by the ambient linear equations. Their pullbacks after removal of the exceptional factor generate $H - E$ on $\text{Bl}_L X$, proving its base point freeness there as well.

Step 1. We compute the complete volume and its fixed tail. The divisor $H - tE$ is nef on Y for $0 \leq t \leq 1$, and on the flopped model for $1 \leq t \leq 2$, since there it equals $(2 - t)A + (t - 1)B$. The intersection data are

$$H^3 = 22, \quad H^2E = 0, \quad HE^2 = -e, \quad E^3 = 2 - e,$$

and

$$(B^3, B^2R, BR^2, R^3) = \begin{cases} (5, 0, -5, -8), & e = 1, \\ (2, 0, -6, -16), & e = 2. \end{cases}$$

For $j \geq 2m$, the complete space factors by R^{j-2m} , leaving degree $q = 3m - j$ on V_5 or $q = 5m - 2j$ on Q . To prove equality, intersect an effective divisor $qB + pR$ with a general exceptional ruling of R . Its degree is $-p$, forcing multiplicity at least p on R . The remaining sections descend and extend across the smooth codimension-two center. Negative q gives no section.

Taking cubes proves (4.7). The integrals in the three chambers are $(83/4, 12, 5/4)$ for a line and $(20, 9, 1/4)$ for a conic. Their division by 22 gives the stated expected orders. The mean fixed-tail coefficient is

$$J = \frac{1}{22} \int_2^\tau \text{vol}(H - tE) dt = \begin{cases} 5/88, & e = 1, \\ 1/88, & e = 2, \end{cases}$$

where $\tau = 3$ or $5/2$.

Step 2. We bound all horizontal curves other than the negative section. For the actual images $W_{m,j}$, put

$$b_m(\Gamma) = \frac{1}{mN_m} \sum_j \sum_{k \geq 1} \dim W_{m,j}^{\text{ord}_\Gamma \geq k}.$$

For $j > 2m$, fixed-factor removal contributes $(j - 2m) \text{ord}_\Gamma(R|_E) \dim W_{m,j}$. Its normalized limit is $\text{ord}_\Gamma(R|_E)J$, since

$$\sum_{j > 2m} (j - 2m) \dim W_{m,j} = \sum_{j > 2m} h^0(Y, mH - jE).$$

The remaining spaces embed in complete systems of class

$$M(t) = \begin{cases} (H - tE)|_E, & 0 \leq t \leq 2, \\ (3 - t)B|_E, & e = 1, \ 2 \leq t \leq 3, \\ (5 - 2t)B|_E, & e = 2, \ 2 \leq t \leq 5/2. \end{cases}$$

Therefore

$$\limsup_m b_m(\Gamma) \leq \frac{3}{22} \int_0^\tau \int_0^\infty \text{vol}_E(M(t) - v\Gamma) dv dt + \text{ord}_\Gamma(R|_E)J.$$

On $\mathbb{F}_n$, a horizontal prime curve other than its negative section dominates the effective class $s + nf$; on $\mathbb{F}_0$ it dominates s . These give upper bounds for all the order spaces. The resulting inner integrals are

E	$0 \leq t \leq 2$	tail	$\int I(t) dt$	$\limsup b_m$
$\mathbb{F}_1$	$t^3/3 + t^2$	$20(3-t)^3/3$	$17/3$	$83/88$
$\mathbb{F}_3$	t^2 ($t \leq 1$) $(1+2t)^3/27$ ($t \geq 1$)	$125(3-t)^3/27$	$433/108$	$71/99$
$\mathbb{F}_0$	$2t^2$	$8(5-2t)^3$	$19/3$	$81/88$
$\mathbb{F}_2$	$2t^2 - t^3/3$	$16(5-2t)^3/3$	$14/3$	$61/88$

For the last column we used $\text{ord}_\Gamma(R|_E) \leq 3$ for lines and at most five for conics. Each is strictly below one. The volume formulas follow by removing the negative part of $as + bf$: it is zero for $b \geq na$, and otherwise it is $(a - b/n)s$. The positive square is then b^2/n .

These integrations bound the actual finite-dimensional spaces. Explicitly,

$$h^0(\mathbb{F}_n, \mathcal{O}(as + bf)) = \sum_{i=0}^a \max\{b - ni + 1, 0\}$$

for $a \geq 0$, and it is zero for $a < 0$. In bounded ranges of coefficients of size $O(m)$, this differs from half the volume by $O(m)$ uniformly. There exist $O(m^2)$ pairs (j, k) and the normalization has size m^4 , so the total normalized error tends to zero.

Step 3. We apply adjunction at the generic center. Apart from $F = E$, the center on Y is a horizontal curve $\Gamma \subseteq E$. Choose a basis compatible with E and F , and write $\pi^*D_m = S_m(X; E)E + \Delta_m$. For every curve covered in Step 2 the coefficient of $\Delta_m|_E$ is less than one for large m . Inversion of adjunction gives

$$A_X(F) - S_m(X; F) \geq (2 - S_m(X; E)) \text{ord}_F(E).$$

Taking limits proves (4.8). For $F = E$ it is equality. $\square$

Proposition 4.9. *If L is an ordinary line on a prime Fano threefold of degree 22, then every prime divisor F centered at L satisfies*

$$(4.9) \quad \beta_X(F) \geq \frac{5}{11} \text{ord}_F(E) > 0, \quad E = \text{Exc}(\text{Bl}_L X \rightarrow X).$$

Proof. By Proposition 4.8, it remains to treat the negative section s of $E = \mathbb{F}_1$. We retain the base cluster of the restricted V_5 projection.

Step 1. We identify its three simple base conditions. The restrictions are

$$H|_E = f, \quad -E|_E = s + f, \quad A|_E = s + 2f, \quad B|_E = 2s + 3f.$$

Since $A|_E$ is ample, no flopping curve is contained in E . The restricted B -series has no fixed curve and maps birationally to an integral hyperplane section of V_5 , of degree five. Resolve its base points, using total-transform exceptional classes e_i , and write its moving divisor as

$$M = p^*(2s + 3f) - \sum b_i e_i.$$

Then $M^2 = 5$ gives $\sum b_i^2 = 3$. There exist exactly three successive base points of multiplicity one. The proximity inequality $b_i \geq \sum_{j \succ i} b_j$ excludes both branching and satellite points. Each connected cluster is a free chain.

A free chain of length r has completed ideal $(z - P(u), u^r)$ in suitable smooth coordinates, and its w th power imposes equal multiplicity w at its r points. After replacing z by $z - P(u)$, this is the monomial condition $ri + j \geq rw$. Thus the cluster is a curvilinear length-three scheme Z_0 whose ideal powers give these equal-multiplicity conditions. Nefness of M on the transform of s shows that at most one successive point is on s or its transforms. A nonreduced component supported on s is consequently transverse to s .

These conditions persist in all section degrees. On a common resolution of the flop,

$$p_Y^* B = q_Y^* B^+ + G, \quad p_Y^* A = q_Y^* A^+,$$

where G is the effective fixed divisor of the pulled-back complete B -series. For $m \leq j \leq 2m$ its coefficient in $mH - jE$ is $j - m$. Hence every restricted section lies in $\mathcal{I}_{Z_0}^{j-m}$. In the tail, after the fixed factor R^{j-2m} , its residual image of degree $q = 3m - j$ lies in $H^0(E, q(2s + 3f) \otimes \mathcal{I}_{Z_0}^q)$.

Step 2. We compare all clusters with one triple point. On $E \setminus s$, viewed as the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$, use coordinates (z, y) . Automorphisms $y \mapsto ay + c_0 + c_1 z$ allow the finitely many support points off s to have nonzero, distinct new y -values and make $d(1/y)$ nonzero on each curvilinear tangent. Each failure imposes a proper linear condition on the parameters. At the possible point on s , transversality gives this last property already. Thus $u = 1/y$ is a cyclic coordinate on $\mathcal{O}_{Z_0}$ and

$$\mathcal{I}_{Z_0} = (z - P(u), Q(u)), \quad \deg P \leq 2, \quad Q \text{ monic of degree } 3.$$

The global torus action $(z, u) \mapsto (\lambda^3 z, \lambda u)$ preserves s . Its flat limit has ideal (z, u^3) . More precisely, put

$$z' = z - \lambda^3 P(u/\lambda), \quad Q_\lambda = \lambda^3 Q(u/\lambda).$$

For every $w \geq 1$, the quotient by $(z', Q_\lambda)^w$ is free over $\mathbb{C}[\lambda]$, with basis

$$(z')^i u^j, \quad 0 \leq i < w, \quad 0 \leq j < 3(w - i).$$

For a fixed divisor D and further s -order k , the required section space is the kernel of evaluation from $H^0(E, D - ks)$ to this finite flat scheme, after multiplication by the canonical section of ks . Kernel dimension can only increase at the limit. This proves the comparison for every order space, including all powers.

Step 3. We bound the fixed part and sum the rows. The divisor $R|_E$ has class $3s + 4f$. Its s -coefficient is at most two. Otherwise the residual components would all be ruling fibers. On the flopped model, $E^+ \cap R^+$ has a component dominating the quintic Z : at its generic point, the hyperplane section containing Z has multiplicity one and cuts a point in the exceptional ruling. This component is not a flop curve, since the latter map to lines. Its inverse transform cannot be s or a ruling fiber, whose B -degrees are one and two. Neither can dominate a quintic. This proves the asserted coefficient bound.

In the triple-point model, the coefficient of u^ℓ in a section of $as + bf$ has z -degree at most $b - a + \ell$. Membership in $(z, u^3)^w$ imposes

$$\text{ord}_{z=0} \geq \max\{0, \lceil w - \ell/3 \rceil\}.$$

In the needed ranges the resulting dimension is exactly $b - a + \ell + 1 - \max\{0, \lceil w - \ell/3 \rceil\}$. Weight it by ℓ to sum all further orders along s .

Without the cluster conditions, the double surface-volume integral is

$$\int_0^2 \left(t^2 + \frac{2}{3} t^3 \right) dt + \int_2^3 \frac{28}{3} (3 - t)^3 dt = \frac{23}{3}.$$

For $1 \leq t \leq 2$, put $w = t - 1$ and $z_0 = \min\{t, 3w\}$. The loss is

$$2 \int_0^t v \max\{w - v/3, 0\} dv = wz_0^2 - \frac{2}{9} z_0^3.$$

Its integral is $3/64 + 337/576 = 91/144$, splitting at $t = 3/2$. In the tail, with $r = 3 - t$, the loss is $2 \int_0^{2r} v(r - v/3) dv = 20r^3/9$, of integral $5/9$. Thus the remaining integral is $311/48$. Ceiling errors contribute $O(m^3)$ to the total weight. The limiting s -coefficient is at most

$$\frac{3}{22} \frac{311}{48} + 2 \frac{5}{88} = \frac{351}{352} < 1.$$

Choose an E, F -compatible basis and apply inversion of adjunction at the generic point of s . The coefficient just bounded is strictly below one for large m , and the discrepancy term retained from E is $2 - S_X(E) = 5/11$. This proves (4.9). $\square$

Proposition 4.10. *Let C be an integral conic on a prime Fano threefold of degree 22. Every prime divisor F centered at C satisfies*

$$(4.10) \quad \beta_X(F) \geq \frac{59}{88} \text{ord}_F(E) > 0, \quad E = \text{Exc}(\text{Bl}_C X \rightarrow X).$$

Proof. Proposition 4.8 leaves only the negative section s of $E = \mathbb{F}_2$ for a conic with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Step 1. We retain two base conditions away from s . Here $A|_E = s + 3f$ is ample and $B|_E = 2s + 4f$. The restricted quadric projection has no fixed curve, since no flopping curve is contained in E . Its base scheme is disjoint from s : the restriction to s has degree zero, so vanishing of every section at one point would make s a fixed component. The image is an integral quadric section of Q , of degree four. Resolving the finite base locus gives

$$M = p^*(2s + 4f) - \sum b_i e_i, \quad M^2 = 4, \quad \sum b_i^2 = 4.$$

Thus there exist four successive simple points or one point of multiplicity two. In the first case retain the first two simple conditions. In the second case a condition of order $2w$ at the point implies order w at that point and at an infinitely near point in its fiber direction: the difference of the corresponding exceptional conditions is $w(e_1 - e_2) \geq 0$.

For every middle piece these two conditions have weight $w = j - m$, and for every residual tail piece they have weight $q = 5m - 2j$. This follows by pulling back $(2m - j)A + (j - m)B$ to a common resolution, where the fixed divisor of B has coefficient $j - m$. It applies to every section, without requiring generation by products.

Step 2. We find one upper model for the two conditions. The complement of s is the total space of $\mathcal{O}_{\mathbb{P}^1}(2)$. Base automorphisms, fiber scalings, and translations by global sections act transitively on ordered pairs on distinct fibers, on distinct pairs on a common fiber, and on pairs consisting of a point and a nonfiber tangent direction, respectively. The last assertion uses the freedom to prescribe a value and a first derivative. A pair on distinct fibers specializes to a pair on one fiber; those points may then collide in the fiber direction. A nonfiber infinitely near direction may also specialize to the fiber direction. Make each specialization in a family of first blowups followed by a blowup along the extending second section. These are smooth projective families, and the divisor classes and total-transform exceptional curves extend. Upper semicontinuity of section dimension therefore bounds every configuration by the two-point fiber cluster.

On this model put $\ell = f - e_1 - e_2$. Its square is -2 . The surface is toric, with invariant curves

$$s, \quad f, \quad s + 2f - e_1, \quad e_1 - e_2, \quad e_2, \quad \ell.$$

For

$$D = as + bf - w(e_1 + e_2),$$

in either range

$$(a, b, w) = (t - v, t + 2, t - 1), \quad 1 \leq t \leq 2, \quad 0 \leq v \leq t,$$

or

$$(a, b, w) = (2r - v, 4r, r), \quad 0 \leq r \leq 1, \quad 0 \leq v \leq 2r,$$

the volume is

$$\text{vol}(D) = 2ab - 2a^2 - 2w^2 + \frac{1}{2}(2w - a)_+^2.$$

If $a \geq 2w$, its intersections with the six curves are $b - 2a, a, b - w, 0, w, a - 2w$, all nonnegative. If $a < 2w$, subtract $N = (2w - a)\ell/2$. The intersections of $D - N$ are $b - w - 3a/2, a, b - w, 0, a/2, 0$,

again nonnegative in both ranges. Thus this is the Zariski decomposition. For $a < 0$ there exist no sections, by intersection with a general ruling fiber.

Step 3. We include the fixed tail and conclude. Without the two retained conditions, the double volume integral for further s -order is

$$\int_0^2 \left(2t^2 + \frac{t^3}{3} \right) dt + \int_2^{5/2} \frac{32}{3} (5 - 2t)^3 dt = 8.$$

For the middle interval its loss is

$$\int_0^t \left(2(t-1)^2 - \frac{1}{2}(v - (2-t))_+^2 \right) dv = 2(t-1)^2 + \frac{2}{3}(t-1)^3,$$

whose integral is $5/6$. In the tail, put $r = 5 - 2t$. The loss is $\int_0^{2r} (2r^2 - v^2/2) dv = 8r^3/3$, with integral $1/3$. The free integral is consequently at most $41/6$. The fixed divisor has restriction $R|_E = 5s + 9f$, so its s -multiplicity is at most five. Its expected coefficient is $1/88$. Hence the complete limiting s -coefficient is at most

$$\frac{3}{22} \frac{41}{6} + \frac{5}{88} = \frac{87}{88} < 1.$$

On the fixed toric model, section dimensions count lattice points in polygons of diameter $O(m)$. The discrepancy from area is $O(m)$ uniformly, since the boundary meets $O(m)$ unit squares. Summing over the $O(m^2)$ pairs of grades gives a negligible $O(m^3)$ total error. The finite-degree estimate therefore holds for every compatible basis.

Apply inversion of adjunction to an E, F -compatible basis at the generic point of s . The retained coefficient is $2 - S_X(E) = 2 - 117/88 = 59/88$, proving (4.10). $\square$

4.3. The second extraction over a special line.

Lemma 4.11 (Pullbacks through a pagoda). *Let a flop of a smooth rational curve be a Reid pagoda of width h , where h counts the successive curve blowups, including the final blowup with exceptional surface $\mathbb{F}_0$. Denote its exceptional prime divisors on the common resolution by $P_1, \dots, P_h$, and the two maps by p and q . If D is a Cartier divisor of degree d on the original curve and D^+ is its transform, then*

$$(4.11) \quad p^*D - q^*D^+ = -d \sum_{i=1}^h iP_i, \quad (D^+)^3 - D^3 = -hd^3.$$

For a line link of degree 22, the sum of the widths of its disjoint flopping curves is three.

Proof. The successive blowups and the reverse contractions are the pagoda resolution of [Rei83, Theorem (5.4) and diagram (5.8)]; the description by $\mathbb{F}_2$ surfaces followed by $\mathbb{F}_0$ is also in [BO95, §3, pp. 40–41]. We give the divisor calculation to retain the width.

Write the pullback difference as $\sum c_i P_i$. On an intermediate $\mathbb{F}_2$, let σ_i be the negative section and f_i a ruling fiber. After the following blowup its normal class is $-2\sigma_i - 2f_i$. Its preceding neighbor meets it in $\sigma_i + 2f_i$, and its following neighbor in σ_i . These classes follow from the blowup normal class $-\sigma_i - 2f_i$ and subtraction of the next center σ_i . On the final $\mathbb{F}_0$, the preceding neighbor has class $u + f$, and the normal class is $-u - f$.

The transformed curve degree is $-d$. On the first $\mathbb{F}_2$, restriction of the pullback difference gives

$$(-2c_1 + c_2)\sigma_1 - 2c_1f_1 = 2df_1,$$

so $c_1 = -d$ and $c_2 = -2d$. On each later intermediate surface the fiber coefficient gives $c_i = c_{i-1} - d$. On the final surface it gives $(c_{h-1} - c_h)(u + f) = d(u + f)$. For $h = 1$ this last equation alone gives $c_1 = -d$. This proves the first identity.

For $h = 1$, the exceptional cube is two, while $q^*D^+ \cdot P^2 = d$ and $(q^*D^+)^2P = 0$. Cubing $p^*D = q^*D^+ - dP$ gives $D^3 = (D^+)^3 + d^3$. For $h > 1$, blow up the first curve on both sides.

The transform of the first pullback under the remaining width- $(h-1)$ pagoda is $q_1^* D^+ - dP_1^+$. Its restriction to the first exceptional surface is $d(\sigma + f)$, as follows either from the first identity or from its two ruling degrees. On the inverse first blowup its cube is $(D^+)^3 + d^3$. Applying the width- $(h-1)$ identity proves the second identity by induction.

In the degree-22 line link, $B = H - 2E$ has degree -1 on every flopping curve, including the horizontal negative section when present; this is the curve description in [KPS18, Lemma 5.2.5]. Its cubes before and after the flop are two and five. The curves are disjoint, so their changes add, giving $\sum h_i = 3$. $\square$

Proposition 4.12. *Let L be a special line on a prime Fano threefold X of degree 22. Write $Y = \text{Bl}_L X$, $E = \mathbb{F}_3$, and let $s \subseteq E$ be its negative section. Let $p: T = \text{Bl}_s Y \rightarrow Y$ have exceptional surface P , and let E' be the strict transform of E . Then*

$$(4.12) \quad A_X(P) = 3, \quad S_X(P) \leq 61/22.$$

If $N_{s/Y} = \mathcal{O}(-1)^{\oplus 2}$, then every prime divisor G centered at s on Y satisfies

$$(4.13) \quad \beta_X(G) \geq \frac{5}{22} \text{ord}_G(P) > 0.$$

If $N_{s/Y} = \mathcal{O} \oplus \mathcal{O}(-2)$, then $P = \mathbb{F}_2$, and (4.13) holds for $G = P$ and for every such G whose center on T differs from the negative section σ of P . The horizontal pagoda then has width two or three.

Let R be the strict transform on T of the divisor $H - 3E$ on Y . The complete P -filtration has no sections above degree $5m$, and for $4m \leq k \leq 5m$ it has the exact factorization

$$(4.14) \quad H^0(T, mH - kP) = E'^{k-2m} R^{k-4m} H^0(Q, \mathcal{O}_Q(5m - k)),$$

as rational section spaces, for a smooth quadric Q .

Proof. We treat both normal types with one calculation, keeping their distinct restrictions on P .

Step 1. We describe the inverse projection. On Y , put $A = H - E$, $B_Y = H - 2E$, and $R_Y = H - 3E$. Their degrees on s are $0, -1, -2$, while $H \cdot s = E \cdot s = 1$. Let $M \subseteq V_5$ be the inverse bisecant corresponding to s . The other flopping curves are disjoint from s and do not affect its neighborhood.

The first exceptional divisor of the pagoda is ordinary order along the inverse flopping curve as well. The first coefficient of Lemma 4.11 gives multiplicity two of R_Y along s , since the inverse curve is not contained in the exceptional divisor over the quintic Z . Therefore

$$R = H - 3E' - 5P.$$

It also gives coefficient two of P in the pullback of E^+ . Hence the hyperplane section $S \subseteq V_5$ containing Z has multiplicity two along M .

For every line M on V_5 , its blowup U maps to a smooth quadric Q as the blowup of a twisted cubic, by [Fuj26a, Theorem 3.6(4)–(6)]. If B is the V_5 hyperplane pullback and D the exceptional divisor over M , then

$$C = B - D, \quad J = B - 2D$$

are the quadric hyperplane pullback and its prime exceptional divisor. The space $H^0(U, J)$ is one-dimensional: intersecting with the exceptional rulings forces every effective representative to contain J . Thus S is this unique hyperplane singular along M .

Write u, f for the ruling classes on $P = \mathbb{F}_0$ in the balanced case, and σ, f for the negative section and ruling on $P = \mathbb{F}_2$ in the unbalanced case. The restrictions are

P	$H _P$	$-P _P$	$E' _P$	$R _P$
$\mathbb{F}_0$	f	$u + f$	$u + 2f$	$2u$
$\mathbb{F}_2$	f	$\sigma + 2f$	$\sigma + 3f$	$2\sigma + 2f$

The curve $C_0 = E' \cap P$ is smooth and irreducible: blowing up the Cartier curve s on E leaves E unchanged. Its class is the one displayed in the E' column. The normal sequence identifies it with the section induced by $N_{s/E} = \mathcal{O}(-3) \subseteq N_{s/Y}$.

Step 2. We bound $S_X(P)$ through the first refinement. For $m \leq j \leq 2m$, the first pagoda coefficient in $(2m - j)A + (j - m)B_Y$ forces order at least $j - m$ along s . Thus its leading restriction to E has that fixed s part. The residual qB_Y -series in the terminal chamber has order at least q along s .

Moreover, the s -multiplicity of $R_Y|_E = 3s + 7f$ is exactly two. The lower bound follows from its multiplicity along s . If it were three, all residual components would be ruling fibers. But $E^+ \cap R^+$ has a component dominating the quintic Z , and this component is not an inverse flopping curve. No ruling fiber can dominate Z , since its B_Y -degree is two. This excludes multiplicity three.

After removing these prescribed parts, the containing classes on $\mathbb{F}_3$ are

$t = j/m$	fixed s coefficient	remaining class
$[0, 1]$	0	$ts + (1 + 2t)f$
$[1, 2]$	$t - 1$	$s + (1 + 2t)f$
$[2, 3]$	$2(t - 2) + (3 - t)$	$(3 - t)(s + 5f)$.

All remaining classes are nef. For $as + bf$ with $b \geq 3a$,

$$\int_0^\infty \text{vol}(as + bf - vs) dv = a^2b - a^3.$$

The three integrals in t are $7/12, 3, 1$. Their normalized sum is $55/88$. Using the actual rank density $-\frac{1}{6}V'(t)$ from (4.7), the prescribed fixed contributions are

$$\begin{aligned} \frac{1}{22} \int_1^2 (t - 1)(-V'(t)) dt &= \frac{28}{88}, \\ \frac{1}{22} \int_2^3 (3 - t)(-V'(t)) dt &= \frac{15}{88}, \quad \frac{1}{22} \int_2^3 2(t - 2)(-V'(t)) dt = \frac{10}{88}. \end{aligned}$$

Thus the limiting s -coefficient of the E -refinement is at most $27/22$.

Choose a basis compatible with E and P . Locally put $E = (x = 0)$ and $s = (x = y = 0)$. If the E -order is j and its leading restriction has s -order k , its P -order is at most $j + k$, as the term $x^j y^k$ shows. Averaging gives

$$S_X(P) \leq S_X(E) + \frac{27}{22} = \frac{61}{22}.$$

The canonical identity $K_T - (\pi p)^* K_X = E' + 2P$ gives $A_X(P) = 3$.

Step 3. We determine the complete tail. At the generic point of L , P is the weight-(1, 2) valuation in two transverse coordinates. Its order is at most twice ordinary order. A section of P -order at least k therefore contains E' to order at least $\lceil k/2 \rceil$.

The line link identifies $H^0(X, mH)$ with $H^0(V_5, \mathcal{O}(3m) \otimes \mathcal{I}_Z^{2m})$. By the first pagoda coefficient for H , its P -order is its M -multiplicity on V_5 minus m . On $U = \text{Bl}_M V_5$ the condition is

$$3mB - (m + k)D = (5m - k)C + (k - 2m)J.$$

For $k \geq 4m$, the fixed power J^{k-2m} already imposes the entire multiplicity $2m$ along Z , since its image is $S \supseteq Z$. Extension across the twisted cubic gives exactly $H^0(Q, \mathcal{O}_Q(5m - k))$. This proves the support bound and (4.14), because

$$mH - kP = (k - 2m)E' + (k - 4m)R + (5m - k)(H - 2E' - 4P).$$

The residual restriction class is $2u + f$ or $2\sigma + 3f$.

Let $n_{m,k}$ be the rank of the actual P -refinement. In this tail,

$$n_{m,k} = m^2(5 - k/m)^2 + O(m).$$

The mean prescribed R -coefficient tends to

$$J_P = \frac{3}{11} \int_4^5 (t-4)(5-t)^2 dt = \frac{1}{44}.$$

Let A_m be the mean prescribed E' -coefficient, using $\lceil k/2 \rceil$ up to $4m$ and $k-2m$ afterwards. Then

$$\limsup_m A_m \leq \frac{1}{2} S_X(P) + \frac{1}{88} \leq \frac{123}{88}.$$

Step 4. We control the horizontal centers on P . Fix G centered at s on Y , other than P , and choose a P, G -compatible basis divisor. Remove the prescribed parts to write

$$(\pi p)^* D_m = S_m(X; P)P + A_m E' + J_m R + B_m, \quad B_m \geq 0, \quad P \not\subseteq \text{Supp } B_m, \quad J_m \rightarrow 1/44.$$

Further E' or R components are allowed in B_m and are included in the estimates.

First let $P = \mathbb{F}_0$. The free containing classes are $(t/2)u + f$ for $0 \leq t \leq 4$ and $(5-t)(2u+f)$ for $4 \leq t \leq 5$. For any horizontal curve $\Gamma \neq C_0$, its class dominates u . The integral along u is $a^2 b$ for $au + bf$. Its two t -integrals are $16/3$ and 1 , so the free coefficient is at most $19/22$. Since $R|_P = 2u$, its addition raises this to at most $10/11 < 1$.

For $C_0 = u + 2f$, integrate $2(a-v)(b-2v)$ up to $\min(a, b/2)$. The result is $a^2 b - 2a^3/3$ if $a \leq b/2$, and $ab^2/2 - b^3/12$ otherwise. For the containing classes, the t -integrals are $27/16$ and $11/48$, yielding the free bound $23/88$. The divisor $R|_P$ has no C_0 component.

Now let $P = \mathbb{F}_2$, and exclude $\Gamma = \sigma$. Every other horizontal curve dominates $\sigma + 2f$. The free containing classes are $(t/2)\sigma + (1+t/2)f$ up to $t = 4$ and $(5-t)(2\sigma + 3f)$ afterwards. For $a\sigma + bf$, the integral along $\sigma + 2f$ is

$$\begin{cases} a^2 b - \frac{4}{3} a^3, & b \geq 2a, \\ b^3/12, & b \leq 2a. \end{cases}$$

The integrals over $[0, 2]$, $[2, 4]$, $[4, 5]$ are $1/2, 65/24, 9/16$, respectively, giving free coefficient at most $181/352$. Along $\Gamma \neq \sigma, C_0$, the coefficient of $R|_P = 2\sigma + 2f$ is at most one, so the bound becomes $189/352 < 1$. Along $C_0 = \sigma + 3f$ it is zero.

In either normal type define the effective boundary

$$\Theta_m = \max\{A_m - 1, 0\} E' + J_m R + B_m.$$

It dominates $(\pi p)^* D_m - S_m(X; P)P - E'$. Its limiting restricted coefficient at C_0 is at most $58/88$ in the balanced case and $35/88 + 181/352 = 321/352$ in the unbalanced case. Every other covered horizontal curve has coefficient bounded strictly below one as well.

All bounds are uniform finite-degree inclusions into the complete ruled-surface systems just used. Their section formulas give $O(m)$ errors per order space, and bounded rounding of $\lceil k/2 \rceil$ contributes a normalized error tending to zero.

Step 5. We retain the discrepancy surplus. Inversion of adjunction at the generic point of the center of G gives $(T, P + \Theta_m)$ log canonical there for sufficiently large m . Therefore

$$\begin{aligned} A_X(G) - S_m(X; G) &\geq A_T(G) + \text{ord}_G(E') + 2 \text{ord}_G(P) \\ &\quad - S_m(X; P) \text{ord}_G(P) - \text{ord}_G(E') - \text{ord}_G(\Theta_m) \\ &\geq (3 - S_m(X; P)) \text{ord}_G(P). \end{aligned}$$

Taking limits proves (4.13); the case $G = P$ follows directly from (4.12). The unbalanced horizontal pagoda has width at least two, and Lemma 4.11 bounds it by three. $\square$

Proposition 4.13. *In the unbalanced case of Proposition 4.12, suppose that the inverse bisecant M meets the rational normal quintic Z in two distinct points. Then*

$$(4.15) \quad S_X(P) = 117/44.$$

Every prime divisor G whose center on T is the negative section σ of $P = \mathbb{F}_2$ satisfies

$$(4.16) \quad \beta_X(G) \geq \frac{15}{44} \operatorname{ord}_G(P) > 0.$$

Proof. We retain both points of contact through the inverse projection and compute the coefficient on σ .

Step 1. We transport the two base points. On T put

$$A = H - E' - 2P = -K_T, \quad C = H - 2E' - 4P.$$

Their restrictions to P are $\sigma + 2f$ and $2\sigma + 3f$. Blow up the inverse flopping curve M' in $\operatorname{Bl}_Z V_5$ to obtain T_a . The shorter pagoda between T and T_a identifies their first exceptional $\mathbb{F}_2$ surfaces. The induced map is an isomorphism: each is the minimal resolution of the quadric cone of the contraction, and the only possible exceptional surface curve is its negative section.

Let $U = \operatorname{Bl}_M V_5$ and let $Z' \subseteq U$ be the strict transform of Z . It is smooth and isomorphic to Z , since the ideal induced on Z by M is invertible. Set $T_b = \operatorname{Bl}_{Z'} U$. Exchanging the order of these two curve blowups gives two disjoint Atiyah flops between T_a and T_b , one over each reduced contact, by [Fuj26a, Example 2.10]. Locally the two curves have equations $(x = y = 0)$ and $(y = z = 0)$. The two orders of blowup induce the elementary transformation of the exceptional ruled surfaces: blow up one point, then contract its ruling fiber. There exists one simple base point because the target fiber has degree one for the quadric hyperplane.

The class C becomes the quadric hyperplane pullback on T_b . Its restriction to P first has a fixed σ of multiplicity one. Its degree on the next pagoda center is -1 , so Lemma 4.11 gives coefficient one on the first further exceptional divisor. Restriction to P subtracts exactly σ ; later exceptional surfaces are disjoint from its strict transform. The two blowup-order exchanges then impose simple points p_1, p_2 on distinct ruling fibers. They lie over the two distinct points of $M \cap Z$, and the ruling bases are identified by the non-Atiyah pagoda. There exists no other base point: after removing σ , the class is $\sigma + 3f$, of square four, and these two points leave the degree-two moving image in the quadric. At most one point lies on σ , since $(\sigma + 3f) \cdot \sigma = 1$ and the resolved moving divisor is nef.

On a common resolution, the pullbacks of A agree, whereas the pullback of C has the effective fixed divisor consisting of this σ and the two point conditions. Consequently, for $m \leq j \leq 2m$, every section of

$$mH - jE' - 2jP = (2m - j)A + (j - m)C$$

has fixed $(j - m)\sigma$ in its restriction and, after its removal, multiplicity at least $j - m$ at both points. This proves the conditions for the complete series.

Step 2. We determine the rank distribution. Put $t = k/m$, and let $n_{m,k}$ be the actual rank of the P -refinement. For $0 \leq t \leq 2$, after removal of $\lceil k/2 \rceil E'$ the containing class is $(t/2)\sigma + (1 + t/2)f$, of square t . Hence $n_{m,k} \leq m^2 t/2 + O(m)$.

For $2 \leq t \leq 4$ put $w = t/2 - 1$. Step 1 gives the free class $\sigma + (2 + w)f$, with multiplicity mw at both points. For odd k , set $a = \lceil k/2 \rceil$; the remaining divisor is $(2m - a)A + (a - m)C + \varepsilon P$ with $\varepsilon \in \{0, 1\}$. On the fixed common resolution this bounded last twist changes every prescribed order by a bounded amount. This leading estimate holds for both parities with uniform $O(m)$ error.

We may bound all σ -filtered spaces by the configuration with p_1 on σ and p_2 off σ on another fiber. If both were off σ , a translation in $\mathcal{O}_{\mathbb{P}^1}(2)$ places p_2 on its zero section and p_1 at a nonzero fiber coordinate. Scaling that coordinate to infinity sends p_1 to σ while fixing p_2 . Each order space is an evaluation kernel at two flat fat points, after multiplication by a power of the σ section. Upper semicontinuity gives the claimed comparison.

Place the resulting pair at torus fixed points and blow them up. The divisor

$$\sigma + (2 + w)f - w(e_1 + e_2)$$

has intersections $0, 1-w, w, 1-w, w, 2$ with the six invariant curves $\sigma-e_1, f-e_1, e_1, f-e_2, e_2, \sigma+2f-e_2$. It is nef and has square $2(1+w-w^2)$. Therefore the complete rank bounds, with the exact tail from (4.14), are

$$n_{m,k} \leq m^2 \rho(t) + O(m), \quad \rho(t) = \begin{cases} t/2, & 0 \leq t \leq 2, \\ 1+w-w^2, & 2 \leq t \leq 4, \\ (5-t)^2, & 4 \leq t \leq 5. \end{cases} \quad w = t/2 - 1,$$

The three integrals of ρ are $1, 7/3, 1/3$. Their sum is $11/3$, exactly the leading coefficient of N_m . The nonnegative rank deficits therefore have total $O(m^2)$ after absorbing the uniform error. For every bounded continuous weight, the limiting integrals for the actual and containing ranks agree. In particular,

$$S_X(P) = \frac{3}{11} \left(\frac{4}{3} + 7 + \frac{17}{12} \right) = \frac{117}{44}.$$

Step 3. We compute the residual σ coefficient. After the prescribed E' and σ factors, the free classes are

$$(t/2)\sigma + (1+t/2)f \quad (0 \leq t \leq 2), \quad \sigma + (1+t/2)f \quad (2 \leq t \leq 4),$$

and $q(\sigma + 3f)$ in the tail, where $q = 5 - t$. Ignoring the point conditions, their double order integrals along σ are $5/6, 11/3, 7/12$, giving normalized coefficient $61/88$.

For the two-point conditions, write the free class as $a\sigma + bf$ with equal point order w . The relevant parameters satisfy either $(a, b) = (1, 2+w)$ or $(a, b, w) = (q, 3q, q)$. After further σ -order v the divisor is

$$D_v = (a-v)\sigma + bf - (w-v)e_1 - we_2.$$

Its negative part is

$$(v-w)_+e_1 + (w-a+v)_+(f-e_2).$$

The two curves are disjoint; after their subtraction, intersection with each of the six invariant curves is nonnegative, using $a \geq w$ and $b = 2a + w$. The loss of volume relative to the unconditioned class is therefore

$$(w-v)_+^2 + w^2 - (w-a+v)_+^2.$$

Its v -integral is w^2 in the middle range and q^3 in the tail. Their t -integrals are $2/3$ and $1/4$. Thus the free coefficient is at most

$$\frac{61}{88} - \frac{3}{22} \left(\frac{2}{3} + \frac{1}{4} \right) = \frac{25}{44}.$$

The fixed σ parts contribute w in the middle and $q+2(t-4) = t-3$ in the tail, since $R|_P = 2\sigma+2f$. With the actual rank density from Step 2, their expected coefficients are

$$\frac{3}{11} \int_2^4 w(1+w-w^2) dt = \frac{14}{44}, \quad \frac{3}{11} \int_4^5 (t-3)(5-t)^2 dt = \frac{5}{44}.$$

The total residual bound is consequently one.

The toric section polygons have a uniform $O(m)$ lattice-point error, and all bounded twists and rounding errors are also $O(m)$. After summing over the two grades, division by mN_m makes the error tend to zero. Thus the coefficient is at most $1 + \epsilon_m$ for every compatible basis, where $\epsilon_m \geq 0$ tends to zero.

Step 4. We pass through the coefficient-one limit. Fix G centered at σ , and choose a P, G -compatible basis divisor. The divisor E' is absent near the generic point of σ , because $C_0 = E'|_P$ is a different irreducible curve. Locally write its pullback as $S_m(X; P)P + \Delta_m$. Set $\lambda_m = (1 + \epsilon_m)^{-1}$. Inversion of adjunction gives $(T, P + \lambda_m \Delta_m)$ log canonical there, and hence

$$A_X(G) - \lambda_m S_m(X; G) \geq (3 - \lambda_m S_m(X; P)) \text{ord}_G(P).$$

Taking limits proves (4.16). $\square$

5. SPECIAL MEMBERS AND REPEATED CONTACT

Theorem 5.1. *Every smooth prime Fano threefold of degree 22 admitting a faithful $\mathbb{G}_m$ -action is K-semistable.*

Proof. The classification in [KPS18, Theorem 1.1.2 and Proposition 5.4.6], with the parameter convention of [CS23, Introduction], identifies these threefolds with V_u for $u \in \mathbb{C} \setminus \{0, 1\}$. The Mukai–Umemura member is $V_{-1/4}$. For $u \notin \{3/4, 2\}$, the existence of a Kähler–Einstein metric is [CS23, Theorem 1.6 and Corollary 1.7]. Both excluded parameters are covered by [Fuj23, Theorem 1.2]. A smooth Fano variety admitting such a metric is K-semistable by [Ber16, Theorem 1.1]. This proves the assertion for every parameter. $\square$

Proposition 5.2. *Let M be a special line on V_5 , with $N_{M/V_5} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$, and let S be its unique singular hyperplane section. Its normalization is $\mathbb{F}_3$, with polarization $s + 4f$ and conductor $s + f_0$. Suppose that a smooth rational normal quintic $Z \subseteq S$ lifts to a curve passing through $s \cap f_0$. Let X be the smooth prime Fano threefold obtained by the inverse line link from (V_5, Z) . Then X is K-semistable.*

More precisely, X belongs to a smooth projective family over $\mathbb{A}^1$ with relatively ample anticanonical bundle, all nonzero fibers isomorphic to X , and central fiber a smooth prime Fano threefold with a faithful $\mathbb{G}_m$ -action.

Proof. We construct the family from the quintic and then reconstruct its Fano fibers.

Step 1. We choose coordinates compatible with the actual torus on V_5 . The normalization of S and its polarization follow from [Fuj26a, Theorem 3.6(4),(5)]. The conductor gluing for the fixed hyperplane $\langle x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5 \rangle$ is given by [KPS18, Lemma 5.4.2 and its proof]. By [KPS18, Lemma 5.1.4 and Corollary 5.1.5], the special lines form one PGL_2 -orbit. Uniqueness of the singular hyperplane along such a line therefore transports this fixed-surface description to S . Only its surface and normalization are used here. The complete system $|s + 4f|$ embeds $\mathbb{F}_3$ as the scroll with sections

$$ar, at, br^4, br^3t, br^2t^2, brt^3, bt^4.$$

Here $[r : t]$ is the ruling coordinate and the displayed expressions incorporate the twists of the fiber coordinates a, b . Put $s = (b = 0)$ and $f_0 = (t = 0)$. The finite normalization map identifies s and f_0 onto the singular line of S . Its projection center lies in their spanning plane and on neither of these two lines. Coordinate changes preserving their intersection put the map in the form

$$(5.1) \quad \nu = [ar, at - br^4, br^3t, br^2t^2, brt^3, bt^4].$$

Away from $bt = 0$ these coordinates recover the scroll point; on $b = 0$ and $t = 0$ both restrictions parametrize the conductor line. This verifies the stated conductor.

The coordinates can be chosen compatibly with a torus induced from V_5 . By [KPS18, Theorem 5.1.1, Lemma 5.1.4 and Corollary 5.1.5], $\mathrm{Aut}(V_5) = \mathrm{PGL}_2$ in its action on $\mathbb{P}(\mathrm{Sym}^6 \mathbb{C}^2)$, and the stabilizer of a special line is a Borel subgroup. Its maximal torus has six consecutive distinct weights on the displayed fixed hyperplane. It lifts to the normalization, preserves s, f_0 , and fixes the projection center. Choose r, t as base eigenvectors. The two nonzero coordinates at and br^4 at the center must have equal weight. After a common character and reversal of the parameter if needed, the action is

$$(5.2) \quad (r, t, a, b) \longmapsto (\lambda r, t, \lambda^4 a, b).$$

The six sections in (5.1) then have weights $5, 4, 3, 2, 1, 0$, agreeing with the actual torus on V_5 . The distinct weight of ar also forces its coordinate at the projection center to vanish.

Step 2. We construct a smooth family of rational normal quintics. The lift $\tilde{Z}$ has class $as + bf$ with $a + b = 5$ by its degree. Since it is integral and is neither s nor a ruling fiber, $a \geq 1$ and

$b \geq 3a$, giving $(a, b) = (1, 4)$. Thus its equation is

$$a\ell(r, t) + bp_4(r, t) = 0.$$

The polynomials ℓ and p_4 have no common zero, because the curve is a smooth section. Passing through $s \cap f_0$ forces $\ell = t$ after scaling. Write

$$p_4 = cr^4 + c_1r^3t + c_2r^2t^2 + c_3rt^3 + c_4t^4.$$

Its image under (5.1) is parametrized by

$$[-rp_4, -tp_4 - tr^4, r^3t^2, r^2t^3, rt^4, t^5].$$

Modulo the last four degree-five monomials, the first two entries are $-cr^5 - c_1r^4t$ and $-(c+1)r^4t$. Their coefficient matrix is triangular, so rational normality is exactly the condition $c(c+1) \neq 0$.

Replace p_4 by

$$p_{4,\lambda} = cr^4 + \lambda c_1r^3t + \lambda^2 c_2r^2t^2 + \lambda^3 c_3rt^3 + \lambda^4 c_4t^4.$$

The determinant just computed is nonzero for every λ . These parameterizations form a closed family $\mathcal{Z} \simeq \mathbb{P}^1 \times \mathbb{A}^1$ in $V_5 \times \mathbb{A}^1$. For $\lambda \neq 0$ they are the images under (5.2) of the given quintic. The central quintic is

$$[-cr^5, -(c+1)r^4t, r^3t^2, r^2t^3, rt^4, t^5],$$

which is smooth and invariant under that torus.

Step 3. We reconstruct the threefolds in a family. Blow up $\mathcal{Z}$ in $V_5 \times \mathbb{A}^1$, obtaining a smooth projective morphism $f: \mathcal{Y} \rightarrow \mathbb{A}^1$. Let B be the pulled-back hyperplane, R the exceptional divisor, and $D = 3B - 2R$. The inverse line link of [KPS18, Theorem 5.2.2 and Remark 5.2.4] applies to every smooth rational normal quintic here. It identifies

$$\bigoplus_{m \geq 0} H^0(\mathcal{Y}_\lambda, mD_\lambda)$$

with the anticanonical section algebra of a smooth degree-22 prime Fano threefold X_λ . Its dimensions are independent of λ by (2.3). Cohomology and base change therefore show that $\mathcal{A}_m = f_*\mathcal{O}_{\mathcal{Y}}(mD)$ is locally free and commutes with base change.

There exists also a uniform finite generation bound. For each anticanonical embedding, $\mathcal{O}_{X_\lambda}$ is 3-regular: Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem] gives $H^i(X_\lambda, (3-i)H) = 0$ for $i = 1, 2, 3$. The Castelnuovo–Mumford multiplication theorem, in [DEL00, Lemma 1.4 and its proof], gives generation of the section algebra in degrees at most three. The relative multiplication maps generated by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are consequently surjective on every fiber in every degree, and hence are surjective by Nakayama’s lemma [Sta26, Tag 00DV].

Set $\mathcal{X} = \text{Proj}_{\mathbb{A}^1} \bigoplus_m \mathcal{A}_m$. It is projective and flat: on homogeneous affine charts, its coordinate modules are degree-zero localizations, hence filtered colimits of flat graded pieces. Base change identifies every fiber with X_λ . These fibers are geometrically smooth, so the flat morphism of finite presentation is smooth. Its relative anticanonical bundle is Cartier and ample on every fiber, and therefore relatively ample.

For $\lambda \neq 0$, the automorphisms of V_5 identify the marked quintic pairs, and functoriality of the link identifies X_λ with X . The central torus acts faithfully on X_0 by [KPS18, Remark 5.2.6], which identifies the automorphism groups of the marked pairs.

Step 4. We apply openness in the direction furnished by the family. The central fiber is K-semistable by Theorem 5.1. The family is smooth projective over a normal base and has relatively ample Cartier anticanonical bundle. It thus satisfies the hypotheses of [BLX19, Corollary 1.2]. Its K-semistable locus is open and contains zero, so it contains a nonzero parameter. All nonzero fibers are isomorphic to X , proving the assertion. $\square$

Lemma 5.3. *In the unbalanced special-line situation of Proposition 4.12, the inverse bisecant $M \subseteq V_5$ is special. On the $\mathbb{F}_3$ normalization of its singular hyperplane, the quintic lifts to a*

smooth curve of class $s_0 + 4f$. Its two contacts with M coincide if and only if this lift passes through the conductor crossing $s_0 \cap f_0$. In that case X is K-semistable.

Proof. Let M' be the inverse flopping curve in $\mathrm{Bl}_Z V_5$. Symmetry of the non-Atiyah pagoda gives $N_{M'/\mathrm{Bl}_Z V_5} = \mathcal{O} \oplus \mathcal{O}(-2)$. Its image M meets Z in length two, since $-K_{\mathrm{Bl}_Z V_5} = 2B - R^+$ has degree zero on M' . The hyperplane S containing Z has multiplicity two along M , by the first pullback coefficient in Lemma 4.11.

On $U = \mathrm{Bl}_M V_5$, let D be the line exceptional divisor and $J = B - 2D$ the exceptional divisor of the quadric projection. The map $J \rightarrow S$ is its normalization. For $J = \mathbb{F}_n$, $n \in \{1, 3\}$, its polarization and adjunction give

$$B|_J = s_0 + \frac{n+5}{2}f, \quad D|_J = -K_J - B|_J = \begin{cases} s_0, & n = 1, \\ s_0 + f_0, & n = 3. \end{cases}$$

The lift Z' is isomorphic to Z , since the finite intersection ideal on Z is invertible. It meets D in a length-two scheme W .

Suppose that M is ordinary on V_5 . Then $D = \mathbb{F}_0$ and $B|_D = f$, $-D|_D = u$, so $J|_D = 2u + f$. This intersection is the irreducible negative section of $J = \mathbb{F}_1$. Every constant-normal-direction section $\Gamma_v \sim u$ on D meets it in length one.

The normal map

$$N_{M'/\mathrm{Bl}_Z V_5} \rightarrow N_{M/V_5} = \mathcal{O}^{\oplus 2}$$

is injective with torsion cokernel of length two. A constant normal section v lifts precisely when Γ_v contains W . At a reduced transverse contact, this says that the displaced normal direction agrees with the tangent of Z . For a repeated contact, write

$$M = (x = y = 0), \quad Z = (x = f(z), y = g(z)), \quad \min\{\mathrm{ord} f, \mathrm{ord} g\} = 2.$$

Choose the normal trivialization with $f = z^2$ times a unit and put $a(z) = g(z)/f(z)$. The ratio on the relevant blowup chart for a first-order displacement is

$$\frac{-g + \varepsilon v_y}{-f + \varepsilon v_x} = a(z) + \varepsilon \frac{a(z)v_x - v_y}{f(z)}.$$

It is regular exactly when $v_y - a(z)v_x$ is divisible by $f(z)$, which is the condition that the length-two scheme $(z^2 = 0, [x : y] = [1 : a(z)])$ lie on Γ_v . This also computes the colength of the normal inclusion.

A nonzero global section of $\mathcal{O}^{\oplus 2}$ is nowhere zero. Such a lift would require a length-two subscheme of $J \cap D$ on Γ_v , contradicting their intersection number one. Hence $H^0(N_{M'/\mathrm{Bl}_Z V_5}) = 0$. Its degree is -2 , forcing $\mathcal{O}(-1)^{\oplus 2}$, contrary to the pagoda normal type. Thus M is special.

Now $J = \mathbb{F}_3$. Write $Z' \sim as_0 + bf$. Its degree is $a + b = 5$, while integrality gives $a \geq 1$ and $b \geq 3a$. Hence $Z' \sim s_0 + 4f$, and its intersection with each conductor component has length one. Because $Z' \rightarrow Z$ is an isomorphism, these two intersection points have coinciding images exactly when they are the single point $s_0 \cap f_0$. The nonreduced contact is therefore precisely the hypothesis of Proposition 5.2, which proves K-semistability. $\square$

Corollary 5.4. *Let X be a prime Fano threefold of degree 22. Every prime divisor over X whose center has positive dimension has nonnegative beta invariant.*

Proof. A prime divisor on X has class aH with $a \geq 1$, so its ratio A_X/S_X is $4a$. Curves of degree at least three are covered by Proposition 4.2. Conics and ordinary lines are covered by Propositions 4.10 and 4.9. For a special line, Proposition 4.8 reduces to the negative section on its first blowup. Proposition 4.12 treats every higher center except the negative section of the second $\mathbb{F}_2$. In this remaining case the inverse contacts are distinct or repeated. Proposition 4.13 treats distinct contacts, and Lemma 5.3 gives K-semistability of X for repeated contact. These alternatives exhaust all curve centers. $\square$

6. POINTS ON AT MOST ONE ORDINARY LINE

Lemma 6.1 (Weights of subspaces of plane forms). *Fix a prime divisorial valuation v over $\mathbb{P}^2$ and put $a = A_{\mathbb{P}^2}(v)$. For every r -dimensional subspace $U \subseteq H^0(\mathbb{P}^2, \mathcal{O}(j))$, its total filtration weight satisfies*

$$(6.1) \quad w_v(U) \leq a \left(jr - \frac{2\sqrt{2}}{3} r^{3/2} \right) + o(j^3).$$

For fixed v , the error is uniform over r and U . For the complete space,

$$(6.2) \quad w_v(H^0(\mathbb{P}^2, \mathcal{O}(j))) \leq \frac{a}{3} j h^0(\mathbb{P}^2, \mathcal{O}(j)) + o(j^3).$$

Proof. The valuation filtration on the complete section ring has bounded normalized support, and its jumping measures converge by [BJ20, Theorem 2.8]. Let $f(t) = \text{vol}(\mathcal{O}(1); v \geq t)$ be the limiting survival function. On a smooth model $q: Z \rightarrow \mathbb{P}^2$ carrying the prime divisor Q of v , it equals $\text{vol}_Z(q^*\mathcal{O}(1) - tQ)$ by [BJ20, §3.4 and Corollary 3.9]. Continuity of divisor volume makes f continuous down to zero at its finite pseudoeffective threshold T . Equivalently, the limiting measure has no terminal atom for a divisorial valuation, by [BJ20, Theorem 3.3(iii) and Remark 3.4]. Since $\mathcal{O}(1)$ is nef, f is strictly decreasing by [BJ20, Theorem 3.3(v)], and $\phi(t) = \sqrt{f(t)}$ is concave by [BJ20, Proposition 2.3 and Theorem 3.3(ii)]. Here the endpoint assertion uses the complete divisorial series; it is not asserted for an arbitrary filtration. Its decreasing inverse g is concave: applying the decreasing inverse to

$$\phi(\theta g(s_1) + (1 - \theta)g(s_2)) \geq \theta s_1 + (1 - \theta)s_2$$

gives the concavity inequality for g . Moreover $g(1) = 0$, so $R(s) = g(s)/(1 - s)$ is nondecreasing.

The complete mean and the total weight of its largest ρ fraction are

$$S_0 = \int_0^1 2sg(s) ds, \quad C(\rho) = \int_0^{\sqrt{\rho}} 2sg(s) ds.$$

The average of the nondecreasing function R over an initial interval with respect to $2s(1 - s) ds$ is at most its average over the whole interval. Hence

$$C(\rho) \leq S_0(3\rho - 2\rho^{3/2}).$$

The anticanonical polytope of $\mathbb{P}^2$ has vertices $(-1, -1), (2, -1), (-1, 2)$ and barycenter zero. By [BJ20, Theorem B and Corollary 7.17], $\mathbb{P}^2$ is K-semistable. Scaling its polarization gives $S_0 \leq a/3$. Thus

$$C(\rho) \leq a \left(\rho - \frac{2}{3} \rho^{3/2} \right).$$

At every filtration level, $\dim \mathcal{F}^\lambda U \leq \min\{r, \dim \mathcal{F}^\lambda H^0(\mathcal{O}(j))\}$. Integration bounds $w_v(U)$ by the largest r weights of the complete space. Weak convergence on a common bounded support gives convergence of the upper-quantile integrals, uniformly in ρ : they are uniformly Lipschitz in ρ , and pointwise convergence follows by integrating the minimum of ρ and the survival function. Put $h_j = h^0(\mathcal{O}(j))$ and $\rho = r/h_j$. The resulting bound is

$$w_v(U) \leq ajh_j \left(\frac{r}{h_j} - \frac{2}{3} \left(\frac{r}{h_j} \right)^{3/2} \right) + o(jh_j).$$

Since $j/\sqrt{h_j} = \sqrt{2} + O(1/j)$, this proves (6.1). The complete mean gives (6.2). $\square$

Proposition 6.2. *Let X be a prime Fano threefold of degree 18 or 22, and let $x \in X$ lie on no line. Then $\delta_x(X) \geq 33/32$. For the exceptional plane E of $\text{Bl}_x X$, the actual refinement satisfies*

$$(6.3) \quad S(W; v) \leq c_d A_E(v), \quad c_{18} = \frac{126 - 10\sqrt{30}}{81}, \quad c_{22} = \frac{174 - 13\sqrt{39}}{99},$$

for every prime divisor v over E .

Proof. We use the uniform point link to compute the mass of each block of grades, then apply Lemma 6.1.

Step 1. We determine the volume and support. By [KP23, Lemma 5.7 and Theorem 5.17(iii),(iv)], the ordinary blowup $Y = \text{Bl}_x X$ has base point free anticanonical divisor $A = H - 2E$, whose morphism is small. Its flop Y^+ is smooth. In degree 18 the second contraction is a degree-six del Pezzo fibration with fiber class $B = H - 3E$. In degree 22 it is the blowup of a smooth rational sextic in $\mathbb{P}^3$.

On Y ,

$$H^3 = d, \quad H^2E = HE^2 = 0, \quad E^3 = 1, \quad A^3 = d - 8, \quad A^2E = 4, \quad AE^2 = -2.$$

Products involving A are unchanged by the flop: a sufficiently divisible general member of $|kA|$ avoids the finitely many images of the flopping curves and restricts the other classes on the common surface. Since $H - tE$ is nef up to $t = 2$, its volume on this interval is $d - t^3$.

For $d = 18$, $B^3 = AB^2 = 0$ and $A^2B = 6$. With $s = t - 2$, its volume on $2 \leq t \leq 3$ is

$$((1 - s)A + sB)^3 = 10 - 12s - 6s^2 + 8s^3.$$

For $t > 3$ its restriction to a general del Pezzo fiber is a negative multiple of an ample class. A moving ample curve in such fibers separates this divisor from the pseudoeffective cone. The support ends at three, and the two volume integrals are 32 and 4. Thus $S_X(E) = 2$.

For $d = 22$, write L for the hyperplane pullback on $Y^+ \rightarrow \mathbb{P}^3$ and D for the sextic exceptional divisor. Then

$$(L^3, L^2D, LD^2, D^3) = (1, 0, -6, -22), \quad A = 4L - D.$$

Both (A, E) and (L, D) are integral bases of the Picard group. Writing $E = aL - bD$ gives $a - 4b = \varepsilon \in \{1, -1\}$. The identity $A^2E = 4$ gives $10a - 26b = 4$. Hence $7b = 2 - 5\varepsilon$, forcing $\varepsilon = -1$, $a = 3$, and $b = 1$. Therefore

$$E = 3L - D, \quad L = H - 3E, \quad D = 3H - 10E.$$

For $2 \leq t \leq 3$ the nef cube is $14 - 12s - 6s^2 + 5s^3$. For $3 \leq t \leq 10/3$ the class is $(10 - 3t)L + (t - 3)D$. The exceptional factor does not change sections, by extension across the sextic. Its volume is $(10 - 3t)^3$. Beyond $10/3$ its pushforward has negative degree, excluding pseudoeffectivity and sections. The three integrals are 40, $29/4$, $1/12$, and hence $S_X(E) = 71/33$.

Step 2. We bound the sum of powers of the actual ranks. Put $n_{m,j} = h^0(Y, mH - jE)$ and $r_{m,j} = n_{m,j} - n_{m,j+1}$. The ranks sum to N_m . For $0 \leq j < 2m$ the restriction image is complete:

$$(mH - (j + 1)E) - K_Y = (m + 1)H - (j + 3)E$$

is nef and big, so vanishing gives surjectivity to $H^0(E, \mathcal{O}_E(j))$. Consequently

$$\lim_m m^{-4} \sum_{j < 2m} r_{m,j}^{3/2} = \int_0^2 (t^2/2)^{3/2} dt = \sqrt{2}.$$

The middle block has total rank

$$\sum_{2m \leq j < 3m} r_{m,j} = \begin{cases} (5/3)m^3 + o(m^3), & d = 18, \\ (13/6)m^3 + o(m^3), & d = 22. \end{cases}$$

This follows by telescoping at $2m$ and $3m$ and using the semiample endpoint classes. In degree 18 the terminal fiber-class space has dimension $O(m)$.

For nonnegative numbers r_i , convexity of $t^{3/2}$, evaluated at their arithmetic mean, gives $\sum_{i=1}^k r_i^{3/2} \geq (\sum r_i)^{3/2} / \sqrt{k}$. There exist m middle grades. In degree 22, the remaining block has $m/3 + O(1)$ grades and mass $h^0(\mathbb{P}^3, \mathcal{O}(m)) = m^3/6 + O(m^2)$. Writing $J_m = m^{-4} \sum_j r_{m,j}^{3/2}$, we obtain

$$\liminf J_m \geq \begin{cases} \sqrt{2} + (5/3)^{3/2}, & d = 18, \\ \sqrt{2} + (13/6)^{3/2} + 1/(6\sqrt{2}), & d = 22. \end{cases}$$

Step 3. We control every valuation on the plane. Fix v and apply (6.1) to each actual image. Its errors sum to $o(m^4)$, because all grades are $O(m)$ and the error is uniform in the rank and subspace for this fixed valuation. Dividing by mN_m gives

$$\frac{S(W; v)}{A_E(v)} \leq S_X(E) - \frac{4\sqrt{2}}{d} \liminf J_m.$$

Substitution gives (6.3). The valuation was fixed before taking the limit; the final constant is independent of it. Both constants are positive and less than $32/33$: use $5 < \sqrt{30} < 6$ and $6 < \sqrt{39} < 7$.

The ordinary point extraction has discrepancy three and zero different. By [AZ22, Theorem 3.2],

$$\delta_x(X) \geq \min \left\{ \frac{3}{S_X(E)}, \inf_v \frac{A_E(v)}{S(W; v)} \right\}.$$

The first terms are $3/2$ and $99/71$; the second terms exceed $33/32$. This proves the assertion. $\square$

Lemma 6.3 (The first two point-volume chambers). *Let X have degree $d \in \{18, 22\}$, and suppose that $\mathcal{L}_x$ is reduced of length $k \in \{1, 2, 3\}$. On $Y = \text{Bl}_x X$, with exceptional plane E , we obtain*

$$(6.4) \quad \text{vol}(H - tE) = \begin{cases} d - t^3, & 0 \leq t \leq 1, \\ d - t^3 + \frac{k}{2}(t - 1)^3, & 1 \leq t \leq 2. \end{cases}$$

For the actual restriction ranks,

$$(6.5) \quad \dim W_{m,j} = \frac{j^2}{2} - \frac{k}{4}(j - m)^2 + O(m) \quad (m \leq j < 2m),$$

uniformly in j . The images are complete for $0 \leq j \leq m - 2$.

Proof. The fixed exceptional planes in the second chamber must be retained, including their parity.

Step 1. We construct the smooth nef model. Each incident line is ordinary by Lemma 3.2. Its strict transform on Y has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, since blowing up the marked point twists the normal bundle by $\mathcal{O}(-1)$. Blow up these disjoint curves to obtain $q: Z \rightarrow Y$, with exceptional surfaces $F_i = \mathbb{F}_1$. Let $F = \sum F_i$. By [KP23, Lemma 5.11(i)],

$$A = H - 2E - F = -K_Z$$

is base point free and $A^3 = d - 8 > 0$. The class $B = H - E$ is the resolved linear projection from x , and is also base point free. On F_i , with negative section s_i and ruling f_i ,

$$-F_i|_{F_i} = s_i + 2f_i, \quad B|_{F_i} = 0, \quad A|_{F_i} = s_i + f_i.$$

The normal sequence for s_i has both outer terms $\mathcal{O}(-1)$ and splits, so $N_{s_i/Z} = \mathcal{O}(-1)^{\oplus 2}$.

The only curves contracted by B on Y are the transforms of the incident lines. A fiber of projection is contained in a projective line through x , and a positive-dimensional fiber forces that line into X ; $B|_E = \mathcal{O}_{\mathbb{P}^2}(1)$ excludes curves in E . Projection is generically injective because X is cut out by quadrics: a line not contained in X meets it in length at most two, with one point already x . It follows that the Stein factorization of $|A + B|$ is a projective small birational contraction $g: Z \rightarrow Z_0$ of precisely the disjoint curves s_i .

The line bundles A, B descend through its normal Stein target, and $K_Z = -A$ is a Cartier pullback. Thus the contraction is crepant and the target is terminal. The Cartier divisor $-F$ is relatively ample, since it has positive degree on each exceptional fiber. On the open set obtained by deleting the images of the other s_j , the contraction is indecomposable: a factorization can only contract its sole curve at one step, and the remaining proper birational map with finite fibers is an isomorphism. Apply [Kol89, Theorem 2.4] with this Cartier divisor. Its unique F -flops glue over the disjoint exceptional points; the transforms of F supply relative ampleness,

so the glued morphism is projective. The theorem preserves analytic singularities, hence the new threefold Z^+ is smooth.

Locally these are the Atiyah flops of the curves with normal bundle $\mathcal{O}(-1)^{\oplus 2}$. Each transform of F_i contracts its negative section to give a plane. On it A restricts to $\mathcal{O}(1)$ and B to $\mathcal{O}$. Adjunction gives its normal bundle $\mathcal{O}(-2)$. The planes remain disjoint.

Step 2. We identify the fixed part exactly. On Z^+ put $Q = A + F/2$. It has degree zero on curves contained in the planes, and nonnegative degree on every other curve because A is nef and F effective. Thus Q is nef. For $1 \leq t \leq 2$,

$$P_t = H - tE - \frac{t-1}{2}F = (2-t)B + (t-1)Q$$

is nef. For integral $m \leq j \leq 2m$, put $\ell = j - m$ and $a = \lceil \ell/2 \rceil$. The restriction of $mH - jE - bF$ to each plane is $\mathcal{O}(-\ell + 2b)$. Successive restriction sequences force exactly the prescribed factor F^a as a factor of every section. After division, the space is the complete space of

$$P_{m,j} = mH - jE - aF = \begin{cases} (m-\ell)B + \ell Q, & \ell \text{ even,} \\ (m-\ell)B + (\ell-1)Q + A, & \ell \text{ odd.} \end{cases}$$

These are integral nef divisors. All section identifications use pullback, strict transform across a small map, and multiplication by the displayed canonical factor.

Step 3. We compute the volume and uniform ranks. For $0 \leq t \leq 1$, the divisor is nef on Y and its cube is $d - t^3$. For the next interval put $u = t - 1$ and $a = u/2$. Before the flops, the curve-blowup formulas give

$$(H - tE)^2 F_i = 0, \quad (H - tE)F_i^2 = u, \quad F_i^3 = 3.$$

Hence $(H - tE - aF)^3 = d - t^3 + 3ku^3/8$. An Atiyah flop changes the cube of a divisor of degree c on its curve by $-c^3$, as in Lemma 4.11. Here that degree is $-u/2$. The k disjoint corrections give the second expression in (6.4).

To retain parity in the ranks, write

$$P_0 = mH - jE - \frac{\ell}{2}F, \quad P_{m,j} = P_0 - \varepsilon F, \quad \varepsilon \in \{0, 1/2\}.$$

On each plane P_0 restricts to zero. Therefore

$$P_0^2 F_i = P_0 F_i^2 = P_0 A F_i = 0, \quad F_i^3 = 4, \quad A F_i^2 = -2,$$

and

$$P_{m,j}^3 = P_0^3 - 4k\varepsilon^3, \quad P_{m,j}^2 A = P_0^2 A - 2k\varepsilon^2.$$

The linear Chern-class term changes only by a bounded parity-dependent constant. Since $P_{m,j} + A$ is nef and big, vanishing and Riemann–Roch give

$$h^0(Y, mH - jE) = R(m, j) + c_\varepsilon,$$

where R is a fixed polynomial with homogeneous cubic part

$$\frac{1}{6} \left(dm^3 - j^3 + \frac{k}{2}(j - m)^3 \right).$$

Taking the difference at j and $j + 1$ proves (6.5) with uniform $O(m)$ error.

Finally, for $j \leq m - 2$, $(m + 1)H - (j + 3)E$ is a nef and big combination of H and B on Y . Vanishing gives the claimed completeness. The single omitted grade $j = m - 1$ has rank $O(m^2)$ and does not affect normalized Riemann sums. $\square$

Proposition 6.4. *Let X be a prime Fano threefold of degree 18, and let $\mathcal{L}_x$ consist of one reduced point. Then*

$$(6.6) \quad \delta_x(X) \geq \frac{1944}{1903} > 1.$$

On the ordinary point blowup there exist no sections of $mH - jE$ for $j > 3m$.

Proof. The tail support follows from a conic fibration model; its mass then suffices for the plane estimate.

Step 1. We pass from the incident line to a five-secant conic. Let L be the unique incident line. It is ordinary. Its line link from Proposition 3.3 is an isomorphism near the entire fiber f over x of the line exceptional surface J : no other line passes through x , and there exists no horizontal flopping section. On $\mathrm{Bl}_\Gamma Q$, this fiber has h -degree two and exceptional degree five. It cannot lie in the exceptional divisor, since a nonvertical curve there has h -degree a positive multiple of seven. It maps isomorphically to an integral degree-two curve $C \subseteq Q$, which is a smooth plane conic. The scheme $\Gamma \cap C$ has length five, including any contact multiplicities.

The curve Γ is nondegenerate in Q . An effective member of $h - D$ would transform to $-H + 3J$, whose pushforward on X has negative anticanonical class, an impossibility. Blow up C to obtain W , with exceptional divisor G_C . The pencil of hyperplanes through C gives a quadric-surface fibration

$$p: W \rightarrow \mathbb{P}^1, \quad B = h - G_C.$$

Its general fiber is smooth. Each fiber is the corresponding hyperplane quadric, since its conic center is Cartier within that surface. The strict transform Γ' is smooth of genus two and satisfies $h \cdot \Gamma' = 7$, $G_C \cdot \Gamma' = 5$, so it is a finite flat bisection.

Step 2. We obtain a fiberwise nef test class. Set

$$\mathcal{V} = p_* \mathcal{O}_W(h), \quad \mathcal{U} = p_* \mathcal{O}_{\Gamma'}(h), \quad \mathcal{K} = \ker(\mathcal{V} \rightarrow \mathcal{U}).$$

The evaluation is surjective on every fiber: a hyperplane system separates every length-two subscheme of an embedded quadric, including a cone. Also H^1 of its hyperplane bundle vanishes. Thus these are bundles of ranks four, two, and two. The moving equation of the hyperplane pencil gives $\mathcal{V} = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)$. Riemann–Roch on Γ' gives $\deg \mathcal{U} = 4$, so $\deg \mathcal{K} = -3$. Nondegeneracy of Γ gives $H^0(\mathcal{K}) = 0$. Consequently

$$\mathcal{K} = \mathcal{O}(-1) \oplus \mathcal{O}(-2), \quad \mathcal{K}(2) = \mathcal{O}(1) \oplus \mathcal{O}.$$

Now blow up Γ' in W to obtain M , with exceptional divisor D' , and let G be the pullback of G_C . Put

$$R = 3h - 2G - D' = h + 2B - D', \quad F = 2h - G - D'.$$

The divisor F is prime and effective: it is the transform of J after blowing up its fiber f . These models identify in codimension one with $\mathrm{Bl}_f(\mathrm{Bl}_\Gamma Q)$, since their only exceptional valuations over Q are the ordinary exceptional valuations along Γ and C . On a chart of $\mathrm{Bl}_L X$ with coordinates $(z, u, v/u)$ and $f = (z = u = 0)$, blowing up f gives order one to z, u, v . Thus G represents ordinary order at x . The model M is identified in codimension one with the model obtained by blowing up L and then its marked fiber; its only prime divisors exceptional over X are F and G . Consequently,

$$(6.7) \quad H^0(\mathrm{Bl}_x X, mH - jE) = H^0(M, 5mh - 2mD' - jG).$$

The other exceptional inequality over the generic point of L is automatic for a regular section on X . This verifies the equality for rational sections on a common model.

The bundle computation gives

$$h^0(M, R) = 3, \quad h^0(M, R - F) = h^0(M, B) = 2, \quad h^0(M, R - G) = 0.$$

For the last equality, the class is $3B - D'$, and on a general quadric fiber a constant must vanish at two points. Hence neither F nor G is fixed in $|R|$. Multiplication by F identifies this three-dimensional space with an anticanonical subspace on X vanishing to order at least three at x , since $R + F = 5h - 3G - 2D'$. Any other fixed prime divisor would push to a fixed prime divisor of that subspace. As $\mathrm{Pic}(X) = \mathbb{Z}[H]$, a nonzero anticanonical subspace with a fixed prime divisor has dimension at most one. Thus $|R|$ has no fixed divisor.

On a general smooth quadric fiber, the two points of Γ' are distinct. Their joining line is not a ruling line. Otherwise, since $\mathcal{K}(2)$ is generated, the ruling line would be fixed in the restricted system on every general fiber and its closure would be a fixed surface of $|R|$. On the two-point blowup of this general fiber, the divisor

$$N = h_S - e_1 - e_2$$

is therefore base point free: hyperplanes through the secant line give a pencil whose base is exactly these points.

Step 3. We prove the support bound and apply the plane estimate. The restriction of the class in (6.7) to this blown-up fiber is $(5m - j)h_S - 2m(e_1 + e_2)$. Its intersection with N is $2(3m - j)$. An effective divisor restricts effectively to a general fiber and has nonnegative intersection with N . Hence there exist no sections for $j > 3m$.

Lemma 6.3 with $k = 1$ gives tail mass

$$\sum_{2m \leq j \leq 3m} \dim W_{m,j} = \frac{7}{4}m^3 + O(m^2), \quad N_m = 3m^3 + O(m^2).$$

For fixed v put $a = A_E(v)$. By (6.2), the grades below $2m$ have total weight at most $(2a/3)m^4 + o(m^4)$. For every $r \geq 0$ and $j \geq 2m$,

$$jr - \frac{2\sqrt{2}}{3}r^{3/2} \leq \frac{2m}{3}r + \frac{1}{6}(j - 2m/3)^3.$$

The maximum after subtracting $2mr/3$ occurs at $r = (j - 2m/3)^2/2$. Apply (6.1) and sum: the limiting raw tail weight divided by am^4 is at most

$$\frac{2}{3} \frac{7}{4} + \frac{1}{6} \int_2^3 (t - 2/3)^3 dt = \frac{1471}{648}.$$

The errors sum to $o(m^4)$ for the fixed v . Thus

$$S(W; v) \leq \frac{1903}{1944} A_E(v).$$

The initial and tail masses are $5/4$ and $7/4$, supported below grades $2m$ and $3m$, respectively. Hence $S_X(E) \leq [2(5/4) + 3(7/4)]/3 = 31/12$. Point adjunction [AZ22, Theorem 3.2] gives

$$\delta_x(X) \geq \min\{36/31, 1944/1903\} = 1944/1903.$$

□

Proposition 6.5. *Let X be a prime Fano threefold of degree 22, and let $\mathcal{L}_x$ consist of one reduced point. Then $\delta_x(X) \geq 1$. On the ordinary point blowup, the threshold is $7/2$ and*

$$(6.8) \quad \text{vol}(H - tE) = 2(7 - 2t)^3 \quad (3 \leq t \leq 7/2).$$

There exists a unique section, up to scalar, of $2H$ with order at least seven at x . Its order is exactly seven, and its initial divisor B_7 on $E = \mathbb{P}^2$ satisfies that $(E, B_7/3)$ is klt. For the actual refinement, every divisorial valuation v over E satisfies $S(W; v) \leq A_E(v)$.

Proof. We identify the complete terminal series and use its septic initial divisor to bound its fixed contribution.

Step 1. We describe the two curve projections. Let L be the sole incident line and J the exceptional surface of $\text{Bl}_L X$. It is an ordinary line, and Proposition 4.8 gives the link to $\text{Bl}_\Gamma V_5$ with

$$h = H - 2J, \quad D = H - 3J, \quad H = 3h - 2D, \quad J = h - D.$$

The flop is an isomorphism near the fiber $f = J_x$, since no other line passes through x . Its h -degree is two and its D -degree is three. It is not contained in D , since any nonvertical curve there has h -degree a positive multiple of five. Thus it maps isomorphically to a smooth conic $C \subseteq V_5$ meeting Γ in length three. Its normal bundle on $\text{Bl}_\Gamma V_5$ is $\mathcal{O} \oplus \mathcal{O}(-1)$.

By [PCS19, Remark 6.4 and diagram (6.14)], the blowup $W = \text{Bl}_C V_5$ is also the blowup of $\mathbb{P}^3$ along a smooth rational quartic B . These geometric statements apply to every smooth conic before any automorphism condition. Let Q be the exceptional surface over C , a the hyperplane pullback from $\mathbb{P}^3$, and R_B the quartic exceptional divisor. Then

$$h = 3a - R_B, \quad Q = 2a - R_B, \quad a = h - Q, \quad R_B = 2h - 3Q.$$

The unique quadric Q_0 through B is smooth by [PCS19, Lemma 6.12], and $Q \simeq Q_0$. Choose ruling classes A, B_0 on Q with A a fiber over C . Then

$$a|_Q = A + B_0, \quad B \sim A + 3B_0.$$

The latter follows from the quartic degree and $R_B \cdot A = 3$.

The strict transform Γ' remains smooth and has $a \cdot \Gamma' = 2$, $R_B \cdot \Gamma' = 1$. It is not contained in R_B , by the quartic degree, so it maps isomorphically to a smooth plane conic $T \subseteq \mathbb{P}^3$ meeting B in exactly one reduced point. It is not contained in Q_0 : otherwise its intersection with the quartic on that quadric would be four, contrary to $R_B \cdot \Gamma' = 1$. The plane section $C_0 = Q_0 \cap \langle T \rangle$ is either a smooth conic or two distinct ruling lines; in both cases it is a reduced simple normal crossing divisor. The length-three scheme $Z = \Gamma' \cap Q$ lies on C_0 .

Step 2. We keep all contact lengths in the exceptional-surface correspondence. Blow up f in $\text{Bl}_\Gamma V_5$. Its exceptional surface is $F = \mathbb{F}_1$, with negative section s and ruling f_0 . The contraction $b: F \rightarrow E$ has $b^* \mathcal{O}_E(1) = s + f_0$. It represents the ordinary point valuation, by the local chart $(z, u, v/u)$ with center $(z = u = 0)$.

Resolve the surface map between F and Q by

$$p_1: S_1 \rightarrow F, \quad q: S_1 \rightarrow Q.$$

At a contact of C and Γ of length k , choose coordinates

$$C = (u = v = 0), \quad \Gamma = (u = 0, v = t^k).$$

For $k > 1$ the common tangent supplies t ; for $k = 1$ choose it nonzero on both tangent directions. Straightening C and using the principal ideal of the two normal displacements on the smooth parameter curve gives this form. On the blowup of Γ , use $u = (v - t^k)z$. The strict transform of C has $z = v = 0$ and the map of its normal bundle to that of C is $\text{diag}(-t^k, 1)$. On the blowup of C the scheme Z has ideal (w, t^k) , with $w = u/v$. In particular it is curvilinear in the base direction and transverse to the ruling fiber.

The induced ruled-surface map is

$$(t, [z : v]) \longmapsto (t, [-t^k z : v]).$$

Its inverse is resolved by k successive point blowups of the cluster (w, t^k) . Each new point is on the newest exceptional curve, away from the older one. The charts $w = tw_1, \dots, w = t^k w_k$ show that the map to F contracts the original ruling fiber and then the first $k - 1$ exceptional curves in reverse order. This also describes the restriction of the small modification that exchanges the two curve blowups in [Fuj26a, Example 2.10]; no tangency is replaced by a transverse flop.

Set

$$G = K_{S_1} - p_1^* K_F, \quad R_q = K_{S_1} - q^* K_Q.$$

At a length- k contact, let e_i be the strict q -exceptional curves and ℓ the strict original fiber. The local charts give

$$R_q = \sum_{i=1}^k i e_i, \quad G = k\ell + \sum_{i=1}^{k-1} (k-i) e_i, \quad q^* A = \ell + \sum_{i=1}^k e_i$$

for that fiber. Summing over lengths totaling three gives the class identity $G + R_q \sim 3q^* A$. As $p_1^* f_0 = q^* A$, comparison of $K_F = -2s - 3f_0$ and $K_Q = -2A - 2B_0$ gives

$$p_1^* s \sim q^* (A + B_0) - R_q.$$

Define the actual divisor $E_0 = q^*C_0 - R_q$. It is effective, reduced, and has simple normal crossings. At each point blowup its weak transform is the pullback of the preceding weak transform minus the new exceptional curve. The center remains on it, since the residual cluster is contained in it. A smooth point gives coefficient zero on the new exceptional curve, and a crossing gives coefficient one; the support remains simple normal crossing. This inductive description includes a cluster at the node of a reducible C_0 . Since $E_0 \sim p_1^*s$ and $h^0(F, s) = 1$, we have equality of divisors

$$(6.9) \quad E_0 = p_1^*s.$$

The total pullback q^*B is also reduced with simple normal crossings. If a cluster point lies on B , the equality $R_B \cdot \Gamma' = 1$ makes that intersection transverse. For a cluster of length greater than one its tangent direction is therefore transverse to B on Q ; the first blowup separates the strict transform of B from all later centers. Its pullback consists of the strict transform and a chain with coefficient one. The length-one case is a single ordinary point blowup.

Step 3. We identify every terminal section. Reverse the two blowups over $\mathbb{P}^3$, first forming $U = \text{Bl}_T \mathbb{P}^3$ with exceptional divisor R_T , then blowing up the strict transform of B . Both orders have exactly the ordinary exceptional valuations along B and T , so they identify in codimension one. On this model,

$$H = 9a - 3R_B - 2R_T, \quad E = 2a - R_B.$$

The strict transform of J is the plane class $P_0 = a - R_T$. Equality of rational sections subject to all prime-divisor inequalities gives, for $j \geq 3m$,

$$H^0(\text{Bl}_x X, mH - jE) = R_B^{j-3m} H^0(U, (9m - 2j)a - 2mR_T).$$

The nonnegative exceptional coefficient imposes no further condition when pushed to U .

The plane P_0 has normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$, with $a|_{P_0} = \mathcal{O}(1)$ and $R_T|_{P_0} = \mathcal{O}(2)$. For a class $Aa - BR_T$, successive restriction sequences force P_0^{2B-A} when $2B - A \geq 0$, leaving $(A - B)(2a - R_T)$. Thus the complete identity is

$$(6.10) \quad H^0(\text{Bl}_x X, mH - jE) = R_B^{j-3m} P_0^{2j-5m} H^0(U, (7m - 2j)L_0), \quad L_0 = 2a - R_T.$$

The class L_0 is base point free, since quadrics generate the ideal of the plane conic after the blowup, and $L_0^3 = 2$. It is nef and big, so negative multiples have no section. This proves the support bound; equality occurs at $j = 7m/2$ for even m . It also proves (6.8). Since $-K_U = 2a + L_0$ is nef and big, vanishing and Riemann–Roch give a fixed polynomial $h^0(U, qL_0) = q^3/3 + O(q^2)$. In particular

$$h^0(\text{Bl}_x X, mH - 3mE) = m^3/3 + O(m^2), \quad \dim W_{m,j} = 2(7m - 2j)^2 + O(m)$$

in the terminal range.

At $(m, j) = (2, 7)$, (6.10) has residual space $\mathbb{C}$, and at order eight it is zero. It is the section of the prime divisor $R_X \in |2H|$ obtained by pushing R_B to X . On the model blowing up f , the divisor identity is

$$R_B = 2H - 4F_L - 7E_x,$$

where F_L is the transform of J and E_x the point exceptional surface. The actual factors in (6.10) show its multiplicities four along L and seven at x , with nonzero leading restriction. Its septic initial divisor satisfies

$$(6.11) \quad b^*B_7 = D_R + 4s,$$

where $D_R = R_B|_F \sim 3s + 7f_0$.

Step 4. We prove the klt estimate for the septic. On Q , the leading equation of the divisor defining R_B in the normal directions to C is a cubic whose zero divisor is B . In the length- k chart of Step 2, its equation becomes $P_B(t, -t^k z, v)$. Resolution of $[z : v] \mapsto [-t^k z : v]$ contributes the

base divisor G with multiplicity three. Higher normal terms stay of higher order and do not change this leading equation. Thus the equality of actual divisors is

$$p_1^* D_R = q^* B + 3G.$$

For $p = bp_1$, (6.9) and (6.11) yield

$$p^* B_7 = q^* B + 3G + 4E_0, \quad K_{S_1} - p^* K_E = G + E_0.$$

Hence

$$(6.12) \quad p^*(K_E + B_7/3) - K_{S_1} = \frac{1}{3}(q^* B + E_0).$$

Each summand $q^* B$ and E_0 is a reduced simple normal crossing divisor. For every divisorial valuation w over S_1 its order is at most $A_{S_1}(w)$. Their average in (6.12) has order at most $2A_{S_1}(w)/3 < A_{S_1}(w)$. This proves klt without requiring the union of their supports to have simple normal crossings. In particular, $v(B_7) \leq 3A_E(v)$ for every v over E .

Step 5. We estimate the full plane refinement. For $3m \leq j \leq \lfloor 7m/2 \rfloor$, put $q_0 = 7m - 2j \geq 0$ and $b_0 = j - 3m$. Then $(m, j) = q_0(1, 3) + b_0(2, 7)$. The exact factors in (6.10) give

$$(6.13) \quad W_{m,j} = s_{B_7}^{b_0} U_{q_0}, \quad U_q = W_{q,3q} \subseteq H^0(E, \mathcal{O}_E(3q)), \quad \dim U_q = 2q^2 + O(q + 1).$$

Order of a product is additive, so passage to initial forms retains the kernels and proves equality of the actual images.

Fix v and let $\alpha = A_E(v)$. Lemma 6.3 gives $h^0(mH - 2mE) = 29m^3/12 + O(m^2)$, so the middle block $2m \leq j < 3m$ has mass $25m^3/12 + O(m^2)$. The grades $j < 2m$ contribute at most $(2\alpha/3)m^4 + o(m^4)$ by (6.2). For the middle block use

$$jr - \frac{2\sqrt{2}}{3}r^{3/2} \leq \frac{m}{2}r + \frac{1}{6}(j - m/2)^3.$$

The limiting raw middle contribution divided by αm^4 is at most

$$\frac{1}{2} \frac{25}{12} + \frac{1}{6} \int_2^3 (t - 1/2)^3 dt = \frac{59}{24}.$$

Since $N_m = 11m^3/3 + O(m^2)$, the first two blocks give $75\alpha/88$ after normalization.

Apply (6.1) to U_q , of degree $3q$ and rank $2q^2 + O(q + 1)$:

$$w_v(U_q) \leq \frac{10\alpha}{3}q^3 + o(q^3) + O(q^2 + 1).$$

Its terminal normalized contribution is at most $10\alpha/88$. The fixed contribution in (6.13) is exactly $v(B_7)/88$, since

$$\int_3^{7/2} 2(t - 3)(7 - 2t)^2 dt = \frac{1}{24}.$$

Thus

$$S(W; v) \leq \frac{85}{88}\alpha + \frac{1}{88}v(B_7) \leq \alpha.$$

For fixed v , all errors sum to $o(m^4)$; the bounded endpoint grades contribute negligibly.

The three block masses are $5/4, 25/12, 1/3$, with supports bounded by $2, 3, 7/2$ after division by m . Therefore

$$S_X(E) \leq \frac{2(5/4) + 3(25/12) + (7/2)(1/3)}{11/3} = \frac{119}{44} < 3.$$

The point extraction has discrepancy three and zero different. Applying [AZ22, Theorem 3.2] with its actual refinement proves $\delta_x(X) \geq 1$. $\square$

7. ORDINARY-LINE INCIDENCE AND REDUCED PAIRS

Lemma 7.1. *Let X be a smooth prime Fano threefold of degree $d \in \{18, 22\}$, let L be an ordinary line, and let $\pi : Y = \text{Bl}_L X \rightarrow X$ have exceptional surface $E = \mathbb{F}_1$. Write s, f for its negative section and ruling, and put $A = H - E$, $B = H - 2E$. Let Z be the base scheme of the image of $H^0(Y, B)$ in $H^0(E, 2s + 3f)$. Then the incidence scheme*

$$(7.1) \quad \mathcal{I}_L = \{(M, y) : M \neq L \text{ is a line, } y \in M \cap L\}$$

is canonically isomorphic to Z , compatibly with the maps to L . At a line M whose transform has pagoda width n , the local algebra is $\mathbb{C}[t]/(t^n)$. The sum of the widths is four for $d = 18$ and three for $d = 22$. The resolution of Z consists of that many simple free point blowups.

For $x \in L$, write f_x for the exceptional fiber. If a local coordinate of L pulls back to $\psi(t)$, then

$$(7.2) \quad \widehat{\mathcal{O}}_{F_1(X, x), [M]} = \mathbb{C}[t]/(t^n, \psi(t)), \quad \text{length}_{[M]} F_1(X, x) = \min\{n, \text{ord}_t \psi\}.$$

Here $\text{ord}_t 0 = \infty$, and $[L]$ is an isolated reduced point of $F_1(X, x)$. For distinct ordinary lines L, M meeting at x , one also has

$$(7.3) \quad n_L(M) > 1 \iff \text{the point representing } L \text{ on } \text{Bl}_M X \text{ lies on its negative section.}$$

Proof. We compute the two deformation functors in the local flop and then determine the total length from the global projection.

Step 1. We compute embedded deformations of a flopping curve. The line links in Proposition 3.3 and [KPS18, Theorem 5.2.2, Lemma 5.2.5 and the following paragraph] have disjoint flopping curves, exactly the transforms of the lines meeting L , with normal bundles $\mathcal{O}(-1)^{\oplus 2}$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. None is contained in E , since $A|_E = s + 2f$ is ample. The transform C of M meets E transversely in one point: the ideal of L restricts to $\mathcal{O}_M(-x)$.

By [Rei83, Remark (5.13)(b)], a formal neighborhood is a small resolution of

$$R = \mathbb{C}[[X_0, Y_0, Z_0, T]] / (X_0 Y_0 - Z_0(Z_0 - T^n)).$$

Its charts (a, t, λ) and (b, t, μ) have transition and contraction maps

$$\begin{aligned} \mu &= \lambda^{-1}, & b &= \lambda^2 a - \lambda t^n, \\ (X_0, Z_0, Y_0, T) &= (a, \lambda a, \lambda^2 a - \lambda t^n, t), \\ (Y_0, Z_0, X_0, T) &= (b, \mu b + t^n, \mu^2 b + \mu t^n, t). \end{aligned}$$

The coordinates λ, μ define a projection to $\mathbb{P}^1$. Over a local Artin algebra R_0 , a flat embedded deformation C_{R_0} maps isomorphically to $\mathbb{P}_{R_0}^1$. The map is proper and quasi-finite, hence finite; its algebra map is surjective by Nakayama's lemma [Sta26, Tag 00DV], and flatness over R_0 followed by Nakayama's lemma makes its kernel zero. Thus $t = \tau$ is constant, and the deformation is a graph with

$$b(\lambda^{-1}) = \lambda^2 a(\lambda) - \lambda \tau^n.$$

Comparison of Laurent coefficients gives $a = b = 0$ and $\tau^n = 0$. Conversely, each such τ defines a flat embedded curve. The local Hilbert algebra is therefore $\mathbb{C}[\tau]/(\tau^n)$.

Step 2. We compute the base ideal of the actual global series. The reflexive rank-one module $J = (X_0, Z_0 - T^n)$ has chart generators

$$(a, \lambda a - t^n), \quad (\mu b + t^n, b).$$

It is reflexive because $R/J = \mathbb{C}[[Y_0, T]]$ has depth two and the depth sequence gives $\text{depth } J = 3$. The corresponding line bundle has degree -1 on C and degree 1 on the other resolution's curve C^+ . These are the formal restrictions of B and B^+ : the conormal bundle of either exceptional curve is $\mathcal{O}(1)^{\oplus 2}$ or $\mathcal{O} \oplus \mathcal{O}(2)$, and all its symmetric powers have vanishing H^1 . The exact sequences of units on successive infinitesimal neighborhoods show that a formal line bundle is determined by its degree.

The inverse curve C^+ maps isomorphically to a projective line in Q or V_5 . Global hyperplane sections span $H^0(C^+, \mathcal{O}(1))$, hence span J modulo the maximal ideal of R . Nakayama's lemma [Sta26, Tag 00DV] implies that they generate J itself. Equality of section spaces across the small modification now gives the exact base ideal (a, t^n) on the first chart. Transversality writes E as $\lambda = \phi(a, t)$, so its restriction to E has algebra $\mathbb{C}[t]/(t^n)$. Resolving this ideal takes n successive simple free blowups.

The universal curve intersects E in $a = 0$, $t = \tau$, $\lambda = \phi(0, \tau)$, identifying its deformation functor with that of the base point. A marked deformation of M meeting L lifts uniquely to $\text{Bl}_L X$: one equation of L has a simple zero on the closed line, and the other equations are multiples of its parameter over the Artin base. The restricted ideal is consequently the invertible ideal of the marked section. The lift is a closed immersion by its graph in the fiber product over X . Conversely, a deformation of C is a relative $\mathbb{P}^1$ with relative H -degree one. Anticanonical sections span its free rank-two module by Nakayama's lemma [Sta26, Tag 00DV], so its image is a relative projective line meeting L in the section obtained from E . These constructions are inverse by the universal property of the blowup. They prove the canonical incidence identification, including the intersection-point map.

Step 3. We determine the length and the marked fibers. There exists no other base support, since the opposite hyperplane series is base point free off the flopping curves. Let $p : T \rightarrow E$ resolve all base points, with total exceptional classes e_i . The moving divisor is

$$M_T = p^*(2s + 3f) - \sum_{i=1}^N e_i.$$

It maps birationally onto the integral quartic surface in Q when $d = 18$, and onto the hyperplane surface of degree five in V_5 when $d = 22$. Thus $8 - N$ is respectively four or five. The local isomorphisms from Step 2 assemble globally, and passage to the fiber of $E \rightarrow L$ gives (7.2). The ordinary line L itself is reduced in the pointed Hilbert scheme, since $H^0(N_{L/X}(-x)) = 0$.

Finally, at an ordinary M the Hilbert scheme of lines is a smooth curve with tangent space $H^0(N_{M/X})$; its nonzero normal section spans the trivial summand and never vanishes. The differential of universal evaluation is evaluation of this section, by [KPS18, Corollary 2.1.6(ii) and equation (2.2.2)]. The line can continue to meet fixed L to first order precisely when this normal vector is proportional to $T_x L$ modulo $T_x M$. A zero normal section cannot move the intersection, since the tangent lines are distinct. The trivial summand determines the negative section of $\mathbb{P}(N_{M/X})$, whose normal bundle is $\mathcal{O}(-1)$. The algebra $\mathbb{C}[t]/(t^{n_L(M)})$ has a nonzero tangent space precisely when $n_L(M) > 1$. This proves (7.3). $\square$

Lemma 7.2. *Let X have degree 18, and suppose that $F_1(X, x)$ consists of two reduced points $[L], [M]$. On $E = \mathbb{F}_1$ over L , let p be the base point representing M , and let $D_0 = D|_E$ for the divisor $D = 2H - 5E$ of Lemma 3.4. Then f_x is not a component of D_0 , and*

$$(7.4) \quad \deg(D_0|_{f_x}) = 5, \quad D_0|_{f_x} \geq 3p.$$

For the full refinement we obtain

$$(7.5) \quad W_{m,j} \subseteq H^0(E, \mathcal{O}_E(js + (m+j)f) \otimes \mathcal{I}_Z^{j-m}) \quad (m \leq j \leq 2m),$$

$$(7.6) \quad W_{m,j} = s_{D_0}^{j-2m} U_{5m-2j}, \quad U_r \subseteq H^0(E, r(2s + 3f) \otimes \mathcal{I}_Z^r) \quad (2m \leq j \leq 5m/2).$$

The mean fixed coefficient in the second line tends to $1/72$. At most one of the four successive base centers is on the successive strict transforms of s .

Proof. Both lines are ordinary by Lemma 3.2. The moving divisor M_T of Lemma 7.1 is nef. Its intersection with the strict transform of s is $1 - k_s$, so $k_s \leq 1$. In particular, a chain rooted on s is transverse to s .

On a common resolution of the line link, the pullbacks of A agree, whereas the pullback of B is the opposite pullback plus the effective fixed divisor resolving the base scheme. The identity

$mH - jE = (2m - j)A + (j - m)B$ imposes multiplicity $j - m$ at every simple base center. For a free chain with local ideal (a, b^n) , its w th power is described by $ni + h \geq nw$ on $a^i b^h$; these are exactly those successive equal-multiplicity conditions. This proves (7.5). The complete terminal section identity in Lemma 3.4 proves (7.6). Its actual ranks give fixed mean $1/72$.

Since $D = 3B - A$, the divisor $D_T = p^*D_0 - 3\sum e_i$ is effective and $M_T D_T = 19 - 12 = 7$. The quartic surface contains the smooth degree-seven genus-two curve Γ with generic multiplicity one. Therefore D_T has a component mapping onto Γ . The degree budget forces this component to have coefficient one and map with degree one; every other component has M_T -degree zero.

The residual incidence on f_x is the reduced point p . By (7.2), the chain rooted at p is transverse to the fiber; just one center lies on its strict transforms. Hence $M_T f_{x,T} = 2 - 1 = 1$. This curve neither maps onto Γ nor has degree zero, and cannot occur in D_T . Thus D_0 has no fiber component. Effectivity of D_T gives multiplicity at least three at p . Since $D_0 \sim 5s + 7f$, its restriction has degree five, proving (7.4). $\square$

Proposition 7.3. *In the setting of Lemma 7.2, suppose that $p \notin s$. Then every prime divisor F over X whose center contains x satisfies*

$$(7.7) \quad A_X(F) \geq \frac{144}{143} S_X(F) + \frac{84}{143} \text{ord}_F(E).$$

In particular, $\delta_x(X) \geq 144/143$.

Proof. We estimate the first order filtration along f_x and the induced point filtrations on its actual rows. Put $q = s \cap f_x$.

Step 1. We bound the first filtration by a finite collection of degenerations. Choose a base coordinate z with $f_x = (z = 0)$ and infinity outside the base supports. On $E \setminus s$ use the fiber coordinate u of $\mathcal{O}_{\mathbb{P}^1}(1)$, translated so that $p = (0, 0)$. The marked width- n ideal is $(u - P(z), z^n)$, with $P(0) = 0$. Under

$$z \mapsto \lambda^{-1}z, \quad u \mapsto \lambda^B u, \quad B \in \mathbb{Z}_{>0},$$

it limits to (u, z^n) . At infinity put $\zeta = 1/z$, $u_\infty = u/z$, $v_\infty = 1/u_\infty$. Their weights are $1, B + 1, -B - 1$. An off-section point tends to $r = (\zeta, u_\infty) = (0, 0)$ and a point on s tends to $t = s \cap f_\infty$.

Two retained off-section units give either the horizontal double (u_∞, ζ^2) or the vertical double (ζ, u_∞^2) . For distinct base coordinates this follows by linear interpolation: the slope and constant of the interpolating graph acquire positive weights B and $B + 1$. A double with nonzero base tangent limits to (u_∞, ζ^2) . Two points on one fiber, or a vertical double, give the second ideal by their quadratic fiber polynomial. A double rooted on s is transverse to s and has ideal $(\zeta - \zeta_0 - \alpha v_\infty, v_\infty^2)$; its limit is (ζ, v_∞^2) . A retained off-section point and one on s give $r + t$.

These degenerations are flat for all powers of the ideals. Locally they have form $(a - P_\lambda(b), Q_\lambda(b))$, with Q_λ monic of degree ℓ . After changing a , a free basis of the quotient by the w th power is

$$a^i b^h, \quad 0 \leq i < w, \quad 0 \leq h < \ell(w - i).$$

The marked and residual supports remain disjoint. For every fixed fiber order k , sections are the kernel of the evaluation map from $H^0(E, C - k f_x)$ into the locally free quotient by this ideal power, after multiplication by the defining section of $k f_x$. Upper semicontinuity bounds each original filtered dimension by its limit. We use it only for this first filtration.

The exhaustive choices are the following. If $n \geq 3$, retain three marked units. If $n = 2$ and there exists residual off-section support, retain one such point; otherwise retain the transverse double at t . If $n = 1$, retain two off-section units when available, giving either double at r ; if there exists just one, retain $r + t$; if none, retain the first two units of the unique transverse chain at t . Uniqueness in the last case follows from $k_s \leq 1$.

Step 2. We give the first moments explicitly. For $\mathcal{O}_E(as + bf)$ the monomials are $z^i u^\ell$, with $0 \leq \ell \leq a$ and $0 \leq i \leq b - \ell$. The marked condition is $i + n\ell \geq nw$. In the middle chamber set

$$w = j/m - 1, \quad y = \ell/m, \quad a/m = 1 + w, \quad b/m = 2 + w.$$

The lower bound for i/m is $L = \max\{0, n(w - y)\}$. The upper bound is the minimum of $2 + w - y$ and the applicable entries:

r	2
r horizontal double	$2 - w + y$
r vertical double	$2 - y/2$
t	$3 + w - 2y$
t vertical double	$5/2 + w/2 - 3y/2$

For $r + t$ use both simple-point bounds. These follow from the infinity exponent $b - \ell - i$ and fiber exponents ℓ at r , $a - \ell$ at t . Integrate $(U^2 - L^2)/2$ where $L \leq U$, over $0 \leq y \leq 1 + w$ and $0 \leq w \leq 1$. The initial moment is $11/24$. For the free tail use $(a, b, w) = (2, 3, 1)$ and multiply by $\int_2^{5/2} (5 - 2t)^3 dt = 1/8$. The resulting values are

n	residual	middle	tail	$\frac{1}{3}(11/24 + \text{middle} + \text{tail})$
3	$\emptyset$	$13/6$	$17/48$	$143/144$
2	r	2	$5/16$	$133/144$
2	t vertical double	$47/24$	$7/24$	$65/72$
1	r horizontal double	$11/6$	$13/48$	$41/48$
1	r vertical double	$11/6$	$13/48$	$41/48$
1	$r + t$	$47/24$	$5/16$	$131/144$
1	t vertical double	$25/12$	$17/48$	$139/144$

For example, the four middle cells for the third row have vertices

(w, y) vertices	L	U	integral
$(0, 0), (1/2, 1/2), (0, 1)$	0	$2 + w - y$	$17/48$
$(1/2, 1/2), (1, 1), (1, 2), (0, 1)$	0	$5/2 + w/2 - 3y/2$	$31/64$
$(0, 0), (1, 0), (1/2, 1/2)$	$2w - 2y$	$2 + w - y$	$29/48$
$(1, 0), (1, 1), (1/2, 1/2)$	$2w - 2y$	$5/2 + w/2 - 3y/2$	$33/64$

Their sum is $47/24$; setting $w = 1$ gives the displayed tail. The other rows are obtained from these explicit endpoints, split at $y = w, 1, 2w$ as applicable. Lattice boundary and rounding errors are $O(m^2)$ in each of $O(m)$ grades. The fixed tail has no f_x component by Lemma 7.2. Thus $S(W; f_x) \leq 143/144$.

Step 3. We bound the nested rows directly. Retain only the simple condition at p in (7.5) and (7.6). After division by z^k and restriction to f_x , the actual row embeds in

$$(7.9) \quad \langle u^h, \dots, u^d \rangle, \quad h = \max\{0, w - k\}, \quad d = \min\{a, b - k\}.$$

Here the symbols a, b, w, k are the unscaled integral degrees of that row. Its total order at q is bounded by $\sum_{\ell=h}^d (a - \ell)$. The initial moment is $1/4$, and the middle moment is the integral over $0 \leq w \leq 1$ of

$$\int_0^w (1 + w - y)(2 + w - y - (w - y)) dy + \int_w^{1+w} (1 + w - y)(2 + w - y) dy = w^2 + 2w + 5/6.$$

It equals $13/6$; the free tail is $23/48$. The moving contribution is consequently $139/144$. Since $p \neq q$, (7.4) bounds the fixed contribution by $2/72$, and $S(V; q) \leq 143/144$.

At any finite point of the fiber discard all base conditions. The row has polynomial degree at most $\min\{a, b - k\}$, and Lemma 2.8 bounds its total point order by $d(d + 1)/2$. Summing gives initial, middle and tail moments $5/24, 43/24, 5/12$. The moving contribution is $29/36$; the fixed contribution is at most $5/72$. Thus $S(V; a_0) \leq 7/8$ for $a_0 \neq q$. These are injections of the

actual associated graded rows; no semicontinuity of quotient filtrations is required. The errors are again $O(m^3)$ before division by mN_m .

The surface flag lemma gives a uniform bound $143/144$ along the entire fiber. Apply Lemma 2.6 with $A_X(E) = 2$, $S_X(E) = 101/72$, and scale by $144/143$. Its retained coefficient is $2 - (144/143)(101/72) = 84/143$, proving (7.7). $\square$

Proposition 7.4. *Let X be a smooth prime Fano threefold of degree 18, and suppose that $F_1(X, x)$ consists of two reduced points. One of the two incident lines L can be chosen so that, for the exceptional surface E of $\text{Bl}_L X$, every prime divisor F with $x \in c_X(F)$ satisfies*

$$(7.10) \quad A_X(F) \geq \frac{216}{215} S_X(F) + \frac{127}{215} \text{ord}_F(E).$$

Consequently, $\delta_x(X) \geq 216/215$.

Proof. The off-section case follows from the surface bounds in Proposition 7.3, using the smaller constant $216/215$ in its final adjunction. Suppose henceforth that both marked roots are on their negative sections. By (7.3), either line has marked width $n \in \{2, 3, 4\}$. Fix one, and put $q = s \cap f_x$.

Step 1. We preserve the slope of the marked cluster. The cluster is transverse to f_x by (7.2), and transverse to s by Lemma 7.2. All remaining length $4 - n$ lies off both curves. In the negative-section chart $v = 1/u$, a scaling of u writes the marked ideal as

$$(v - z - a_2 z^2 - \cdots - a_{n-1} z^{n-1}, z^n).$$

It is contained in the sloped-double ideal $J = (v - z, z^2)$. This uses only its tangent jet, without a formal coordinate change in the global section system.

The torus $z \mapsto \lambda^{-1}z$, $u \mapsto \lambda u$ takes the marked ideal to $(v - z, z^n)$ at zero. At infinity its weights on (ζ, u_∞) are $(1, 2)$. The residual length is zero for $n = 4$, one simple point r for $n = 3$, and either a horizontal or vertical double at r for $n = 2$. The graph, monic-polynomial and flat-power arguments in Proposition 7.3 apply: the relevant coefficients have weights one and two. Therefore each first fiber-filtered dimension can only increase in the limit.

Step 2. We compute the first fiber moment. In $\mathcal{O}_E(as + bf)$ the negative-chart monomials satisfy

$$z^i v^\ell, \quad 0 \leq \ell \leq a, \quad 0 \leq i \leq b - a + \ell.$$

At total degree $d = i + \ell$, write the section as $z^d P(v/z)$. Membership in $(v - z, z^n)^w$ forces divisibility by $(T - 1)^h$, where

$$h = \max\{0, \lceil (nw - d)/(n - 1) \rceil\}, \quad \ell_{\min} = \max\{0, \lceil (d - b + a)/2 \rceil\}, \quad \ell_{\max} = \min\{a, d\}.$$

Since T and $T - 1$ are coprime, the possible leading T -exponents are exactly $\ell_{\min} + h, \dots, \ell_{\max}$. The corresponding fiber order is $d - \ell$. The residual infinity conditions become

$$\ell \geq d - b + w, \quad d \leq a + b - 2w, \quad \ell \geq (2d - 2b + a + 2w)/3$$

for the simple point, horizontal double, and vertical double, respectively.

Scale by m , set $w = j/m - 1$, and put $a = 1 + w$, $b = 2 + w$. The interval for ℓ/m is $L \leq \ell/m \leq U$, where

$$L = \max\{0, (d - 1)/2, \text{residual lower bound}\} + \max\{0, (nw - d)/(n - 1)\}, \\ U = \min\{1 + w, d\}.$$

The residual lower bound is $d - 2$ for the simple point and $(2d - 3 + w)/3$ for the vertical double; the horizontal double also imposes $d \leq 3$. Integrating $d(U - L) - (U^2 - L^2)/2$ over this feasible

region gives

n	residual	middle	tail	normalized total
4	$\emptyset$	129/64	29/96	533/576
3	r	133/72	1/4	23/27
2	horizontal double	55/32	7/32	115/144
2	vertical double	991/576	2/9	461/576

The initial contribution is 11/24, the tail uses $(a, b, w) = (2, 3, 1)$ with scale 1/8, and the final normalization is 1/3. The rational cells are recorded in Step 5. There exists no fixed fiber contribution, so $S(W; f_x) \leq 533/576$.

Step 3. We impose the actual rank on the secondary rows at q . Keep only J^w in the actual middle and free terminal series. In the terminal range the notation $V_{m,j,k}$ in this step denotes the actual rows after division by $s_{D_0}^{j-2m}$. This factor has zero fiber order and preserves the row ranks; its point orders are restored after estimating these moving rows. The preceding formula with $n = 2$ and no residual condition gives the containing fiber rows, monomial in v . Their q -order is ℓ . For each nonnegative integer h the actual rows satisfy

$$(7.12) \quad \sum_k \dim V_{m,j,k}^{\text{ord}_q \geq h} \leq \min \left\{ \dim W_{m,j}, \sum_k \dim \bar{V}_{m,j,k}^{\text{ord}_q \geq h} \right\}.$$

The first bound follows by summing the actual row dimensions; the second is their filtered inclusion in the containing rows.

The actual rank density in the middle is $R(w) = (3 + 4w - 3w^2)/2$, by the volume calculation of Lemma 3.4. In the tail it is $2r^2 + O(r)$, where $r = 5m - 2j$. For the containing rows put

$$L = \max\{0, (d-1)/2\} + \max\{0, 2w - d\}, \quad U = \min\{1 + w, d\}.$$

The initial, middle and scaled terminal integrals of ℓ are 1/4, 223/96, 17/32, respectively.

For any nonnegative cutoff $c(w)$, use the first bound of (7.12) below order $mc(w)$ and the second above it. Relative to the uncapped middle moment, this saves

$$(7.13) \quad \int_0^1 \left(\iint_{L \leq \ell \leq U} \min\{c(w), \ell\} d\ell - c(w)R(w) \right) dw.$$

Choose the piecewise affine interpolation of $5w^2/4$ at 0, 1/4, 1/2, 3/4, 1. Its formulas and savings are

w interval	$c(w)$	saving
$[0, 1/4]$	$5w/16$	3305/25165824
$[1/4, 1/2]$	$15w/16 - 5/32$	148765/25165824
$[1/2, 3/4]$	$25w/16 - 15/32$	1104181/25165824
$[3/4, 1]$	$35w/16 - 15/16$	74328067/478150656

Their sum is 24549209/119537664. The complete cell integrals in Step 5 verify these values.

In the tail choose cutoff $5r/4$ and actual rank $2r^2$. At unit scale the four degree intervals $[1, 5/4]$, $[5/4, 2]$, $[2, 7/2]$, $[7/2, 5]$ contribute 25/512, 387/512, 63/32, 45/64 to the integral of $\min\{5/4, \ell\}$. The total is 445/128. Subtracting $(5/4) \cdot 2$ and multiplying by 1/8 saves 125/1024. The actual fixed divisor on the fiber has degree five, and its mean coefficient is 1/72. Hence

$$(7.14) \quad \begin{aligned} S(V; q) &\leq \frac{1}{3} \left(\frac{1}{4} + \frac{223}{96} + \frac{17}{32} - \frac{24549209}{119537664} - \frac{125}{1024} \right) + \frac{5}{72} \\ &= \frac{356827303}{358612992} < \frac{215}{216}. \end{aligned}$$

The final strict comparison has positive numerator $215 \cdot 358612992 - 216 \cdot 356827303 = 27095832$.

Step 4. We cover the rest of the fiber and control finite-degree errors. At a point different from q , the direct complete polynomial-row bound gives moving moment 29/36, as in (7.9) after dropping its lower exponent. The fixed contribution is at most 5/72, so the total is at most

7/8. All first-moment lattice and endpoint errors are $O(m^3)$. In (7.12), an $O(m)$ rank error is summed over $O(m)$ order levels and $O(m)$ grades, also giving $O(m^3)$. Thus the estimates hold uniformly after division by $mN_m = 3m^4 + O(m^3)$.

Step 5. We give the rational integration cells. In the following tables every polygon is listed by its vertices (w, d) . The first set integrates $d(U - L) - (U^2 - L^2)/2$; its cells are precisely the regions where the displayed max/min functions select the stated affine endpoints. Their interiors are disjoint. The tail entries already include the factor 1/8. Width four, no residual point.

L	U	Integral	Vertices
0	d	1/32	(0, 0), (1/4, 1), (0, 1)
$4w/3 - d/3$	d	1/32	(0, 0), (1, 1), (1/4, 1)
$d/2 - 1/2$	$w + 1$	3209/2592	(1, 4), (1, 5), (0, 3), (0, 1), (1/3, 4/3)
$d/2 - 1/2$	d	241/10368	(0, 1), (1/4, 1), (1/3, 4/3)
$4w/3 + d/6 - 1/2$	$w + 1$	185/324	(1, 2), (1, 4), (1/3, 4/3)
$4w/3 + d/6 - 1/2$	d	1253/10368	(1, 1), (1, 2), (1/3, 4/3), (1/4, 1)

Tail intervals: $[1, 2]$: $L = d/6 + 5/6$, $U = d$, contribution 25/1728; $[2, 4]$: $L = d/6 + 5/6$, $U = 2$, contribution 181/864; $[4, 5]$: $L = d/2 - 1/2$, $U = 2$, contribution 5/64.

Width three, a residual simple point.

L	U	Integral	Vertices
0	d	1/24	(0, 0), (1/3, 1), (0, 1)
$3w/2 - d/2$	d	1/36	(0, 0), (1, 1), (1/3, 1)
$d/2 - 1/2$	$w + 1$	821/768	(1, 3), (0, 3), (0, 1), (1/2, 3/2)
$d/2 - 1/2$	d	113/2304	(0, 1), (1/3, 1), (1/2, 3/2)
$3w/2 - 1/2$	$w + 1$	181/768	(1, 2), (1, 3), (1/2, 3/2)
$3w/2 - 1/2$	d	305/2304	(1, 1), (1, 2), (1/2, 3/2), (1/3, 1)
$d - 2$	$w + 1$	7/24	(1, 3), (1, 4), (0, 3)

Tail intervals: $[1, 2]$: $L = 1$, $U = d$, contribution 1/48; $[2, 3]$: $L = 1$, $U = 2$, contribution 1/8; $[3, 4]$: $L = d - 2$, $U = 2$, contribution 5/48.

Width two, a residual horizontal double.

L	U	Integral	Vertices
0	d	1/16	(0, 0), (1/2, 1), (0, 1)
$2w - d$	d	1/48	(0, 0), (1, 1), (1/2, 1)
$d/2 - 1/2$	$w + 1$	131/96	(1, 2), (1, 3), (0, 3), (0, 1)
$d/2 - 1/2$	d	11/64	(1, 2), (0, 1), (1/2, 1)
$2w - d/2 - 1/2$	d	19/192	(1, 1), (1, 2), (1/2, 1)

Tail intervals: $[1, 2]$: $L = 3/2 - d/2$, $U = d$, contribution 3/64; $[2, 3]$: $L = d/2 - 1/2$, $U = 2$, contribution 11/64.

Width two, a residual vertical double.

L	U	Integral	Vertices
0	d	1/16	(0, 0), (1/2, 1), (0, 1)
$2w - d$	d	1/48	(0, 0), (1, 1), (1/2, 1)
$d/2 - 1/2$	$w + 1$	79/162	(2/3, 5/3), (0, 3), (0, 1)
$d/2 - 1/2$	d	5525/41472	(3/4, 3/2), (2/3, 5/3), (0, 1), (1/2, 1)
$2w - d/2 - 1/2$	d	61/1536	(1, 1), (3/4, 3/2), (1/2, 1)
$w/3 + 2d/3 - 1$	$w + 1$	8/9	(1, 2), (1, 4), (0, 3), (2/3, 5/3)
$w/3 + 2d/3 - 1$	d	491/13824	(1, 2), (2/3, 5/3), (3/4, 3/2)
$7w/3 - d/3 - 1$	d	721/13824	(1, 1), (1, 2), (3/4, 3/2)

Tail intervals: $[1, 2]$: $L = 4/3 - d/3$, $U = d$, contribution $1/27$; $[2, 4]$: $L = 2d/3 - 2/3$, $U = 2$, contribution $5/27$.

The next table integrates $\min\{c(w), \ell\}$ over the secondary-row cells. The cutoff $c(w)$ is fixed on each of the four intervals displayed above. In every row the integration in ℓ runs from L to U ; the polygon is in (w, d) .

Cell	L	U	Integral	Vertices
1	0	d	$3235/786432$	$(0, 0), (1/4, 1/2), (1/4, 1), (0, 1)$
2	$2w - d$	d	$605/4194304$	$(0, 0), (1/4, 1/4), (1/4, 27/64)$
3	$2w - d$	d	$3925/25165824$	$(0, 0), (1/4, 27/64), (1/4, 1/2)$
4	$d/2 - 1/2$	$w + 1$	$2255/196608$	$(1/4, 5/4), (1/4, 7/2), (0, 3), (0, 1)$
5	$d/2 - 1/2$	d	$2755/4194304$	$(1/4, 37/32), (1/4, 5/4), (0, 1)$
6	$d/2 - 1/2$	d	$2175/2097152$	$(1/4, 1), (1/4, 37/32), (0, 1)$
7	0	d	$5885/786432$	$(1/2, 1), (1/4, 1), (1/4, 1/2)$
8	$2w - d$	d	$20195/12582912$	$(1/2, 1/2), (1/2, 11/16), (1/4, 27/64), (1/4, 1/4)$
9	$2w - d$	d	$151225/25165824$	$(1/2, 11/16), (1/2, 1), (1/4, 1/2), (1/4, 27/64)$
10	$d/2 - 1/2$	$w + 1$	$4550825/67436544$	$(1/2, 13/8), (1/2, 4), (1/4, 7/2), (1/4, 5/4), (5/14, 19/14)$
11	$d/2 - 1/2$	$w + 1$	$12149/4214784$	$(1/2, 3/2), (1/2, 13/8), (5/14, 19/14)$
12	$d/2 - 1/2$	d	$912105/1438646272$	$(5/14, 19/14), (1/4, 5/4), (1/4, 37/32)$
13	$d/2 - 1/2$	d	$43299287/2157969408$	$(1/2, 1), (1/2, 3/2), (5/14, 19/14), (1/4, 37/32), (1/4, 1)$
14	$2w - d$	d	$7065/4194304$	$(3/4, 3/4), (3/4, 51/64), (1/2, 11/16), (1/2, 1/2)$
15	$2w - d$	d	$366353/25165824$	$(3/4, 51/64), (3/4, 1), (1/2, 1), (1/2, 11/16)$
16	$d/2 - 1/2$	$w + 1$	$654745/4194304$	$(3/4, 77/32), (3/4, 9/2), (1/2, 4), (1/2, 13/8)$
17	$d/2 - 1/2$	$w + 1$	$67655/1048576$	$(3/4, 7/4), (3/4, 77/32), (1/2, 13/8), (1/2, 3/2)$
18	$d/2 - 1/2$	d	$81307/1572864$	$(3/4, 3/2), (3/4, 7/4), (1/2, 3/2), (1/2, 1)$
19	$2w - d/2 - 1/2$	d	$23053/786432$	$(3/4, 1), (3/4, 3/2), (1/2, 1)$
20	$2w - d$	d	$31725/1514143744$	$(3/4, 51/64), (3/4, 3/4), (15/19, 15/19)$
21	$2w - d$	d	$66570721309/20500709376000$	$(31/35, 1), (3/4, 1), (3/4, 51/64), (15/19, 15/19)$
22	$2w - d$	d	$37888/46433625$	$(1, 1), (31/35, 1), (15/19, 15/19)$
23	$d/2 - 1/2$	$w + 1$	$2425775/12582912$	$(1, 7/2), (1, 5), (3/4, 9/2), (3/4, 77/32)$
24	$d/2 - 1/2$	$w + 1$	$315695/1048576$	$(1, 2), (1, 7/2), (3/4, 77/32), (3/4, 7/4)$
25	$d/2 - 1/2$	d	$53525/1572864$	$(1, 2), (3/4, 7/4), (3/4, 3/2)$
26	$2w - d/2 - 1/2$	d	$4384610497/33718272000$	$(1, 5/4), (1, 2), (3/4, 3/2), (3/4, 1), (31/35, 1)$
27	$2w - d/2 - 1/2$	d	$186619/65856000$	$(1, 1), (1, 5/4), (31/35, 1)$

The sums of the cell integrals on the four cutoff intervals are respectively

$$\frac{442345}{25165824}, \frac{2671645}{25165824}, \frac{7999541}{25165824}, \frac{105964289}{159383552}.$$

The corresponding integrals of $c(w)R(w)$ are $1715/98304$, $3285/32768$, $26935/98304$, and $50075/98304$. Subtraction gives the four savings in (7.13).

Step 6. We apply adjunction with the retained coefficient. The first fiber bound $533/576$, the secondary bound (7.14), and $7/8$ away from q are all below $215/216$. Lemma 2.7 therefore gives that uniform surface bound along f_x . Both E and its fiber are smooth, with zero different.

Applying Lemma 2.6, first with a slightly smaller reciprocal constant and then taking a limit, gives

$$A_X(F) \geq \frac{216}{215} S_X(F) + \left(2 - \frac{216}{215} \frac{101}{72}\right) \text{ord}_F(E),$$

which is (7.10). If an off-section line was available, its surface bound $143/144 < 215/216$ gives the asserted conclusion with that choice of L . $\square$

8. THREE REDUCED LINES IN DEGREE EIGHTEEN

Setup 8.1. Let $Y = G_2/P_2 \subseteq \mathbb{P}^{13}$ be the adjoint fivefold, with hyperplane class H_Y . We use coordinates $(a, b, c, d, e, f, g, h, i, J, k, l, m, n)$ and the Pfaffian presentation

$$(8.1) \quad \begin{pmatrix} 0 & -f & e & g & h & i & a \\ f & 0 & -d & J & k & l & b \\ -e & d & 0 & m & n & -g-k & c \\ -g & -J & -m & 0 & c & -b & d \\ -h & -k & -n & -c & 0 & a & e \\ -i & -l & g+k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}$$

from [Kap11, §2]; its 4×4 Pfaffians vanish. The ambient data are

$$(8.2) \quad \dim Y = 5, \quad \text{Pic}(Y) = \mathbb{Z}[H_Y], \quad -K_Y = 3H_Y, \quad H_Y^5 = 18$$

as in [Lat23, Theorem 2.1 and its proof]. Every smooth prime Fano threefold of degree 18 is a transverse codimension-two linear section of this model, by [Vir26, Theorem 2.2].

On the chart $h = 1$ the independent coordinates are (i, a, e, n, g) , and the Pfaffians give

$$(8.3) \quad \begin{aligned} f &= ie - a^2, & k &= -g - ae - in, & c &= e^2 + an, \\ b &= -ag - ic, & d &= eg + ac, & J &= fc + gk, \\ l &= -af + ik, & m &= -ec + gn. \end{aligned}$$

The contact weights are 0 on h , 1 on i, a, e, n , 2 on f, g, k, c , 3 on l, b, d, m , and 4 on J .

Lemma 8.2. *Every pair of independent hyperplanes cuts Y in a pure three-dimensional Gorenstein scheme. The universal family of these sections is flat. Its degree- M section spaces commute with base change for $M \geq 0$, and its section ring is the quotient of the homogeneous section ring of Y by the two hyperplane equations.*

Proof. A hyperplane divisor is integral. Its prime components have positive integral classes in $\mathbb{Z}[H_Y]$, and their multiplicities and classes sum to H_Y . There exists therefore just one component of multiplicity one. The Cartier hypersurface is Cohen–Macaulay, so it is reduced. Independent hyperplanes give different divisors: otherwise their ratio would be constant on the projective variety Y . The second equation is a nonzero divisor on the first hypersurface. They form a regular sequence, proving the first assertion.

For the universal family over $\text{Gr}(2, H^0(Y, H_Y))$, let $\mathcal{K}$ be the kernel of evaluation to $\mathcal{O}_Y(H_Y)$. The incidence total space is $\text{Gr}_Y(2, \mathcal{K})$, hence smooth; all fibers have dimension three. The flatness criterion for an equidimensional Cohen–Macaulay morphism to a smooth base gives flatness. Equivalently, the fiberwise regular Koszul resolution is flat over the base. That resolution has twists $M, M-1, M-2$ on Y . Since $K_Y = -3H_Y$, Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem] applies even at $M-2 = -2$. It gives the asserted base change and section-ring quotient. $\square$

Proposition 8.3. *Let X be a smooth prime Fano threefold of degree 18, and suppose that $F_1(X, p)$ consists of three reduced points. Coordinates as in Setup 8.1 can be chosen with $p = [h]$ so that the leading forms of its two equations are*

$$(8.4) \quad n + i, \quad ug + vk + \lambda f + \mu c, \quad u - v = 1.$$

They satisfy

$$(8.5) \quad (u - \mu\zeta, v\zeta + \lambda) \neq (0, 0) \quad (\zeta^3 = 1).$$

On the exceptional plane E of $\mathrm{Bl}_p X$, put

$$(8.6) \quad \begin{aligned} f &= ie - a^2, & c &= e^2 - ai, & Q &= ae - i^2, & G &= vQ - \lambda f - \mu c, \\ g &= G, & k &= -G - Q, & b &= -aG - ic, & d &= eG + ac, \\ l &= -af - i(G + Q), & m &= -ec - iG, & J &= fc - G^2 - GQ. \end{aligned}$$

The entire associated graded anticanonical section ring is the quotient of the homogeneous ring of Y by $n + i$ and $ug + vk + \lambda f + \mu c$, embedded by (8.6) with $h = 1$.

Write $W_{M,r}$ for the ordinary-point refinement and put $c_N = 0$ for $N < 0$, $c_N = (\lfloor N/2 \rfloor + 1)(N + 1 - \lfloor N/2 \rfloor)$ for $N \geq 0$. Then

$$(8.7) \quad \dim W_{M,r} = \binom{r+2}{2} - 3c_{r-M-1} \quad (0 \leq r \leq 3M).$$

For $3M \leq r \leq 4M$, put $q = 4M - r$. Then

$$(8.8) \quad \begin{aligned} W_{M,r} &= J^{r-3M} W_{4M-r, 3(4M-r)}, \\ \dim W_{M,r} &= \frac{3q^2 + 3q + 2}{2}. \end{aligned}$$

There exist no terms for $r > 4M$. For $B_4 = (J = 0)$, the pair $(E, B_4/2)$ is log canonical, and

$$(8.9) \quad \mathrm{vol}(H - tE) = \begin{cases} 18 - t^3, & 0 \leq t \leq 1, \\ 18 - t^3 + \frac{3}{2}(t-1)^3, & 1 \leq t \leq 3, \\ 3(4-t)^3, & 3 \leq t \leq 4, \\ 0, & t \geq 4. \end{cases} \quad S_X(E) = 9/4.$$

Proof. We first establish the coefficient condition on the original smooth threefold, and then identify every degree of the limiting section ring.

Step 1. We normalize the contact plane. By [KP23, Lemmas 5.6 and A.5], the incident-line scheme is the intersection of the tangent direction plane with the twisted cubic in the contact $\mathbb{P}^3$. Three distinct points span a plane; a line cannot meet this quadric-cut cubic in length three. Thus the tangent plane is inside the contact space. The derived SL_2 of the Levi stabilizer acts on it as $\mathrm{Sym}^3 \mathbb{C}^2$, as described in [Vir26, Proposition 5.4]. It is triply transitive on the projective twisted cubic, parametrized by $(s^3, s^2t, st^2, -t^3)$ in coordinates (i, a, e, n) . The three-point plane can therefore be written $n + i = 0$. Row reduction gives the second leading equation above, since g and k have opposite differentials at p .

The contact action is an actual G_2 cocharacter. Its seven matrix-index weights are $(-1, 1, 0, 1, -1, 0, 0)$; adding two to their pairwise sums gives the displayed coordinate weights. It preserves the four-form

$$x_1x_2x_3x_7 + x_4x_5x_6x_7 + x_2x_3x_5x_6 + x_1x_3x_4x_6 + x_1x_2x_4x_5,$$

where products are exterior products. Every summand has weight zero. The shift by two is the normalization by the point coordinate h .

Step 2. Global smoothness forces (8.5). Write the actual equations as

$$F_1 = n + i + A, \quad F_2 = ug + vk + \lambda f + \mu c + \beta_b b + \beta_d d + \beta_l l + \beta_m m + \beta_J J,$$

where A is an arbitrary linear combination of coordinates of weight at least two. Suppose $u = \mu\zeta$, $\lambda = -v\zeta$ for some $\zeta^3 = 1$. The affine line

$$(i, a, e, n, g) = (t, \zeta t, \zeta^2 t, -t, 0)$$

lies on X , because every higher-weight coordinate in (8.3) vanishes there. Its far point is $q_\zeta = [h : i : a : e : n] = [0 : 1 : \zeta : \zeta^2 : -1]$. Along its finite part, put $Q = ae + in$. Then

$$\begin{aligned} df &= t(\zeta^2 di + de - 2\zeta da), & dc &= t(2\zeta^2 de - da + \zeta dn), \\ dQ &= -\zeta df + \zeta^2 dc, & \kappa_t &= dg + \zeta^2 dc \neq 0. \end{aligned}$$

The coefficient assumptions give

$$d(ug + vk + \lambda f + \mu c) = \kappa_t, \quad (db, dd, dl, dm, dJ) = (-\zeta t, \zeta^2 t, -t, -t, 0)\kappa_t.$$

Consequently $dF_2 = (1 + \gamma t)\kappa_t$, where $\gamma = -\zeta\beta_b + \zeta^2\beta_d - \beta_l - \beta_m$. If $\gamma \neq 0$, this ambient differential vanishes at $t = -1/\gamma$, contradicting smoothness of the codimension-two section.

If $\gamma = 0$, use the chart $i = 1$, with independent coordinates (a, f, g, h, l) . The Pfaffians give

$$(8.10) \quad \begin{aligned} e &= hf + a^2, & k &= af + hl, & b &= -f^2 + al, & c &= -ag - hb, \\ n &= -a^3 - 2ahf - hg - h^2l, & d &= gf - ab, & m &= eb - g(g + k), & J &= gl - fb. \end{aligned}$$

At q_ζ the differentials are $dk = \zeta df$, $dc = -\zeta dg$, $db = \zeta dl$, $dd = -\zeta^2 dl$, $dm = dl$, $dJ = 0$. Thus $dF_2 = -\gamma dl = 0$ there, contradicting smoothness. All higher terms of both original equations have been retained. This proves (8.5).

Step 3. The leading projective section is integral. Let Z be the section of Y by the two leading equations. Its $h = 1$ chart is $\mathbb{A}_{i,a,e}^3$, by (8.3) and $u - v = 1$. On the chart $i = 1, h = 0$, the first equation gives $a^3 = 1$, and the second becomes

$$(u - \mu\zeta)g + (v\zeta + \lambda)f = 0.$$

This is a nonzero equation in (f, g, l) by Step 2, so this boundary has dimension two. On $h = i = 0$, the Pfaffians first force $a = e = f = n = 0$, then $g = k = c = 0$, on the reduced support. The remaining coordinates satisfy

$$ld + b^2 = lm + bd = bm - d^2 = 0.$$

They form the projective cone, with vertex coordinate J , over a twisted cubic, of dimension two. Thus every component of Z meets its integral affine chart. Lemma 8.2 makes Z Cohen–Macaulay and pure-dimensional. It has one generically reduced component and no embedded associated primes, hence is integral.

Step 4. The degeneration identifies the full ring. Weight-homogenize the original equations by multiplying a term of weight w in F_1 by t^{w-1} and one in F_2 by t^{w-2} . Their independent leading terms give a flat family by Lemma 8.2, with central fiber Z and nonzero fibers translates of X . Lemma 8.2 also gives all-degree base change and the stated quotient ring. The ring of Y is generated in degree one: by Borel–Weil, as presented in [Ser54, Theorem 4], its degree- M space is the irreducible Cartan component of highest weight $M\theta$, and multiplication is a nonzero equivariant map onto it.

In each degree choose homogeneous lifts of a central eigenbasis. Their determinant is nonzero at zero and equivariant. Its zero set is invariant under the transitive action on the punctured base, and is therefore empty. They give a basis on X . The local implicit equations have $n = -i +$ terms of ordinary degree at least two and $g = G +$ terms of degree at least three. Thus a lifted eigenvector of weight r has initial ordinary polynomial its central weight- r polynomial in i, a, e . These polynomials are nonzero and independent: the h -chart is dense in integral Z . The basis is adapted to ordinary order, proving equality of the entire associated graded ring and its multiplication.

Step 5. We compute the exact ranks. The shifted weight character of $H^0(Y, MH_Y)$ is $P_M(q) = T_M(q)/((1 - q)^4(1 - q^2))$, where

$$\begin{aligned} T_M(q) &= 1 - (3M + 4)q^{M+1} + (3M + 5)q^{M+2} \\ &\quad - (3M + 5)q^{3M+4} + (3M + 4)q^{3M+5} - q^{4M+6}. \end{aligned}$$

Put $P_M = 0$ for $M < 0$. To verify this character, let α, β be the short and long simple roots. Then $\theta = 3\alpha + 2\beta$, $\rho = 5\alpha + 3\beta$, and the cocharacter has weights zero and one on α, β . The reflections are $s_\alpha(a, b) = (-a + 3b, b)$ and $s_\beta(a, b) = (a, a - b)$. Apply the Weyl character formula [Kna23, Theorem 5.75 and Corollary 5.76]. Pairing its twelve numerator terms under s_α , differentiating in the α variable and setting it to one gives

$$q^{2M+3} - (3M+4)q^{M+2} + (3M+5)q^{M+1} - (3M+5)q^{-M-1} + (3M+4)q^{-M-2} - q^{-2M-3}.$$

Its denominator is $q^3(1 - q^{-1})^4(1 - q^{-2})$. Multiplication by q^{2M} and reversal of weights gives the formula.

The equivariant Koszul resolution gives

$$\sum_r \dim W_{M,r} q^r = P_M(q) - (q + q^2)P_{M-1}(q) + q^3P_{M-2}(q).$$

Through degree $3M$ this equals

$$\frac{1}{(1-q)^3} - \frac{3q^{M+1}}{(1-q)^2(1-q^2)}.$$

The coefficient of q^N in the second denominator is $\sum_{b=0}^{\lfloor N/2 \rfloor} (N - 2b + 1) = c_N$, proving (8.7). At $M = 0, 1$ the characters are directly 1 and $1 + 3q + 3q^2 + 4q^3 + q^4$.

Every degree-one generator has weight at most three except J . A monomial of section degree M and weight $r \geq 3M$ contains at least $r - 3M$ copies of J . Generation and integrality therefore give (8.8), including injectivity and its exact dimension. They also give vanishing beyond $4M$. Uniformly on the support,

$$\dim W_{M,r} = M^2 \rho(r/M) + O(M), \quad \rho(t) = \begin{cases} t^2/2, & 0 \leq t \leq 1, \\ t^2/2 - 3(t-1)^2/4, & 1 \leq t \leq 3, \\ 3(4-t)^2/2, & 3 \leq t \leq 4. \end{cases}$$

The restriction sequences give volume $6 \int_t^4 \rho(s) ds$. Integration proves (8.9).

Step 6. We keep the entire quartic divisor. The conics (f, c, Q) form the complete conic system through the three noncollinear points $p_\zeta = (1 : \zeta : \zeta^2)$, $\zeta^3 = 1$. Their common zeros are precisely those reduced points, and the Vandermonde determinant proves noncollinearity. Let $p_0 : S_0 \rightarrow E$ and $q_0 : S_0 \rightarrow \mathbb{P}^2$ resolve the corresponding quadratic transformation. The conic $D_2 = (fc - G^2 - GQ = 0)$ is nonzero: its form is $fc + Q^2/4 - (G + Q/2)^2$, a rank-three form minus a square. At the level of divisors of sections,

$$p_0^* B_4 = q_0^* D_2 + 2 \sum e_\zeta, \quad K_{S_0} = p_0^* K_E + \sum e_\zeta.$$

The log pullback of $(E, B_4/2)$ is thus $(S_0, q_0^* D_2/2)$. A conic is smooth, two distinct lines, or a double line. Blowing up any subset of three distinct proper points gives total transform with simple normal-crossing support and multiplicities at most two: at a node the two branches meet the exceptional curve at different points. Hence this pair is log canonical, proving the last assertion. $\square$

Lemma 8.4. *In Proposition 8.3, the cubic system $W_{1,3} = \langle b, d, l, m \rangle$ has exactly six simple base points: the three points $p_\zeta = (1 : \zeta : \zeta^2)$ and one successive point over each. Their blowup $p_S : S \rightarrow E$ has base point free anticanonical class*

$$(8.11) \quad A = 3h - \sum_{\zeta} (e_\zeta + e'_\zeta) = -K_S, \quad A^2 = 3,$$

where $h = p_S^* \mathcal{O}_E(1)$ and the e_ζ, e'_ζ are total exceptional classes. Its anticanonical ring is the full diagonal $\bigoplus_{M \geq 0} W_{M,3M}$.

For every closed point $z \in E \setminus B_4$, a smooth plane cubic C through z satisfies

$$(8.12) \quad S(W; C) = 1/4, \quad S(W \text{ refined along } C; z) \leq 1.$$

Consequently $S(W; w) \leq A_E(w)$ for all divisors whose centers are not contained in B_4 .

Proof. We determine the cubic map and then its exact divisibility filtration.

Step 1. In this step, we determine the cubic map and its base points. Put $F_0 = ld + b^2$, $P_0 = bm - d^2$, $R_0 = lm + bd$. The polynomial identities (8.6) give

$$(8.13) \quad \begin{aligned} Jf &= F_0, & Jc &= P_0, & JQ &= R_0, \\ Jg &= vR_0 - \lambda F_0 - \mu P_0, & Jk &= -(v+1)R_0 + \lambda F_0 + \mu P_0, \\ Ji &= gl - fb, & Ja &= fd + gb, & Je &= -fm - gd, & Jh &= fc + gk. \end{aligned}$$

Multiplying these identities by further powers of J expresses J^2i, J^2a, J^2e cubically, and J^3h quartically, in b, d, l, m . In a monomial of section degree M and weight $3M$, the number of J factors equals the number of weight-two factors plus twice the number of weight-one factors plus three times the number of h factors. Pairing them as above proves that the entire diagonal ring is generated by b, d, l, m .

Its degree- M dimension is $1 + 3M(M+1)/2$ by (8.7). It is a domain, so its homogeneous prime kernel in $\mathbb{C}[b, d, l, m]$ has height one. The Hilbert function makes its generator cubic. One equation of the resulting integral cubic surface $\bar{T} \subseteq \mathbb{P}^3$ is

$$(vl - (v+1)m)R_0 + (\lambda(m-l) - b)F_0 + (\mu(m-l) - d)P_0 = 0.$$

The four cubics have no common curve. If an irreducible curve divided them, its square would divide Jf, Jc, JQ , hence divide J . Under the quadratic transformation $(f : c : Q)$, J is the pullback of the reduced conic $fc + Q^2/4 - (G + Q/2)^2$: its rank is at least two. A multiple component in the original plane must therefore be one of the three lines joining the proper base points. On the line $e = -i - a$ put $R = i^2 + ia + a^2$. Then $(f, c, Q) = (-R, R, -R)$ and $G = \gamma R$ for a constant γ , so $b = -(\gamma a + i)R$ is not identically zero on that line. The cyclic change $(i, a, e) \mapsto (i, \zeta a, \zeta^2 e)$ treats the other two lines.

At p_ζ , the linear parts F_1, C_1 of f, c are independent and the linear part of Q is $-\zeta F_1 + \zeta^2 C_1$. Set

$$T_\zeta = -(v\zeta + \lambda)F_1 + \zeta^2(u - \mu\zeta)C_1.$$

It is nonzero by (8.5). The linear parts of (b, d, l, m) are $(-\zeta T_\zeta, \zeta^2 T_\zeta, -T_\zeta, -T_\zeta)$. Thus each proper point and its successor direction $T_\zeta = 0$ are base points of multiplicity at least one. The degree formula for the resolved cubic map is

$$9 - \sum \text{mult}(\text{base point})^2 = 3 \deg(E \dashrightarrow \bar{T}).$$

The six exhibited points force degree one and equality in the sum. There exist no further base points and all six are simple. This proves (8.11). Riemann–Roch and the boundary-free case of Kawamata–Viehweg vanishing [DEL00, p. 142, Nadel Vanishing Theorem] give $h^0(S, qA) = 1 + 3q(q+1)/2$ because $(q+1)A$ is nef and big. Dimension equality identifies the full anticanonical ring. In particular $\bar{T}$ is normal, and $S \rightarrow \bar{T}$ is its birational anticanonical morphism.

Put $G_2 = vR_0 - \lambda F_0 - \mu P_0$. The inverse rational map is

$$[b : d : l : m] \longmapsto [G_2 l - F_0 b : F_0 d + G_2 b : -F_0 m - G_2 d].$$

Its composition with the cubic map is $J^2[i : a : e]$. Thus it is regular and inverse near a point outside B_4 . A positive-dimensional fiber there would map under p_S to that same point, outside the exceptional locus of p_S , which is impossible. Properness and normality imply that the cubic morphism is an isomorphism near the image of such a point.

Step 2. In this step, we construct a smooth cubic flag and prove exact divisibility. Fix $z \notin B_4$. A general hyperplane section through its image avoids the finitely many exceptional images and is smooth by Bertini. Its plane image C is a smooth cubic, including at the proper base points: its nonzero first derivative is a multiple of T_ζ . The hyperplane can be chosen so that, as a section on the integral threefold Z used in Proposition 8.3, it contains no component of $h = 0$.

To check this, on the $i = 1$ boundary we obtain $a = \zeta$, one nonzero relation on (f, g) , and l free; there $J = f^3 + (g - \zeta f)l$ is not identically zero on any component. On $h = i = 0$, a two-dimensional boundary component is the cone over the twisted cubic above, where J is generically nonzero. The inverse vector is $J^2[1 : \zeta : \zeta^2]$ in the first case and zero in the second. No boundary component therefore has constant cubic-map image equal to the image of z , whose inverse vector recovers $z \neq p_\zeta$. A general hyperplane through that image avoids all these components.

If s_C is the cubic equation, we obtain exact divisibility in every degree:

$$(8.14) \quad \{s \in W_{M,r} : s_C^k \mid s\} = s_C^k W_{M-k, r-3k}.$$

For the nontrivial inclusion, use the full ring of Z . Division is regular on $h = 1$ by polynomial divisibility, and where $s_C \neq 0$. The complement has codimension at least two, because the chosen hyperplane contains no component of $h = 0$. Since Z is Cohen–Macaulay, hence S_2 , the section of $\mathcal{O}_Z(M - k)$ extends uniquely and has weight $r - 3k$. The restriction quotients are therefore exactly the actual images in $H^0(C, \mathcal{O}_C(r))$. Summing (8.14) gives

$$S(W; C) = \lim_{M \rightarrow \infty} \frac{\sum_{k=1}^M N_{M-k}}{MN_M} = \frac{1}{4}.$$

Step 3. In this step, we bound the secondary degree. At each p_ζ choose regular parameters (t, y) with tangent $t = 0$ equal to the successor direction. The weight- $(2, 1)$ valuation has orders at least zero on weight-one generators, one on weight-two generators, and two on the cubics. It has order at least three on J : use $Jf = F_0$, $Jc = P_0$, and independence of the linear parts of f, c . Thus every element of $W_{M,r}$ has weighted order at least $r - M$. On the smooth cubic, $t = O(y^2)$, so restriction vanishes to order at least $\max\{r - M, 0\}$ at each p_ζ .

After removing those fixed zeros, and the exact J factor in the tail, the degree bound on the actual restriction image is

$$D_{M,r} = \begin{cases} 3r, & 0 \leq r \leq M, \\ 3M, & M \leq r \leq 3M, \\ 3(4M - r), & 3M \leq r \leq 4M. \end{cases}$$

All removed divisors avoid z . A degree- D line bundle on a smooth curve has $h^0(L(-kz)) \leq \max\{D - k + 1, 0\}$, so any subspace has total point weight at most $D(D + 1)/2$. Let $\eta(t) = 3t, 3, 3(4 - t)$ on the three intervals above. Since $\int_0^4 \eta(t)^2 dt = 24$, refining all original grades along C and using (8.14) gives total point weight

$$\sum_{M=0}^n \sum_{r=0}^{4M} \left(\frac{D_{M,r}^2}{2} + O(D_{M,r}) \right) \leq 3n^4 + O(n^3).$$

Division by $nN_n = 3n^4 + O(n^3)$ proves (8.12). The smooth surface flag theorem, [AZ22, Theorem 3.2], now gives the asserted bound for every valuation whose center contains z , and hence for every center not contained in B_4 . $\square$

Lemma 8.5. *On the surface of Lemma 8.4, put $R_\zeta = e_\zeta - e'_\zeta$, $F_\zeta = e'_\zeta$, and $B = 2h - \sum e_\zeta$. The full pullback series is a fixed divisor $\Delta_{M,r}$ times the following complete base point free series:*

	range	$\Delta_{M,r}$	residual divisor
(8.15)	$0 \leq r \leq M$	0	rh
	$M < r \leq 3M$	$\sum(aR_\zeta + sF_\zeta)$	$rh - \sum(ae_\zeta + be'_\zeta)$
	$3M < r \leq 4M$	$(r - 3M)p_S^*B_4 + q\sum(R_\zeta + 2F_\zeta)$	qA .

In the middle row $s = r - M$, $a = \lceil s/2 \rceil$, $b = \lfloor s/2 \rfloor$; in the last row $q = 4M - r$. If S_{mov} denotes the expected order of these residual complete spaces, with the original denominator MN_M , then

$$(8.16) \quad S(W; w) = S_{\text{mov}}(w) + \frac{41}{72} \sum w(R_\zeta) + \frac{41}{36} \sum w(F_\zeta) + \frac{1}{24} w(p_S^*B_4).$$

Thus $S(W; w) \leq A_E(w)$ is equivalent to

$$(8.17) \quad S_{\text{mov}}(w) + \frac{1}{24}w(p_S^*B_4) \leq A_S(w) + \frac{31}{72} \sum w(R_\zeta) + \frac{31}{36} \sum w(F_\zeta).$$

Proof. The classes h, A, B are base point free; B is the pulled-back conic system through the three proper points. The weighted generator estimate in Lemma 8.4 gives $\text{ord}_{F_\zeta} W_{M,r} \geq s$. A germ of ordinary multiplicity m_0 has weight- $(2, 1)$ order at most $2m_0$, since a nonzero monomial of ordinary degree m_0 has that bound. Therefore $\text{ord}_{R_\zeta} W_{M,r} \geq \lceil s/2 \rceil$. Removing the resulting effective divisor gives inclusion in the middle complete space.

If $s = 2n$, its residual divisor is $(r - 3n)h + nA$. If $s = 2n + 1$, it is $(r - 3n - 2)h + nA + B$. All coefficients are nonnegative; in the odd case r and $3M$ have opposite parity, so $r \leq 3M - 1$. The residual divisor is nef and base point free, and its sum with A is nef and big. Vanishing and Riemann–Roch give

$$h^0 = \binom{r+2}{2} - \frac{3}{2}(a(a+1) + b(b+1)) = \binom{r+2}{2} - 3c_{s-1}.$$

This equals the actual rank (8.7), so the inclusion is an equality. The first row follows from its complete rank, and the last from (8.8) and the middle row at $(q, 3q)$. Base point freeness proves that the fixed divisors are complete.

The limiting nef divisors are

$$(8.18) \quad P_t = \begin{cases} th, & 0 \leq t \leq 1, \\ \frac{3-t}{2}h + \frac{t-1}{2}A, & 1 \leq t \leq 3, \\ (4-t)A, & 3 \leq t \leq 4, \end{cases} \quad \rho(t) = P_t^2/2.$$

Multiplication of a rank- $b_{M,r}$ space by a fixed divisor shifts every jumping number by its valuation. Hence its normalized coefficients are

$$\frac{1}{3} \left(\int_1^3 \frac{t-1}{2} \rho(t) dt + \int_3^4 (4-t) \rho(t) dt \right) = \frac{41}{72}, \quad \frac{1}{3} \int_3^4 (t-3) \rho(t) dt = \frac{1}{24},$$

and twice the first value for F_ζ . Bounded rounding changes the weight sum by $O(M^3)$ for each fixed valuation. Convergence follows from [AZ22, Lemma 2.9]: support is $0 \leq r \leq 4M$, and $W_{2k,k} = H^0(E, \mathcal{O}_E(k))$ supplies an ample interior ray. For linear boundedness, realize $w = \text{ord}_G$ on a smooth surface over E and intersect the effective divisor $p^*D - w(D)G$ with an ample class; this gives $w(D) \leq C_w \deg D$. The finite-degree identity and convergence prove (8.16), including existence of S_{mov} . Finally $K_S = p_S^*K_E + \sum (R_\zeta + 2F_\zeta)$ gives (8.17). $\square$

Lemma 8.6. *Let $\mathcal{N}$ be the union of negative irreducible curves on S . For every $z \in S \setminus \mathcal{N}$ there exists a smooth rational fiber C through z of one of the three line pencils, not contained in $p_S^*B_4$, such that*

$$(8.19) \quad S(p_S^*W; C) \leq 67/72, \quad S(p_S^*W \text{ refined along } C; z) \leq 133/144.$$

Thus $S(W; w) \leq (67/72)A_E(w)$ for any center not contained in $p_S(\mathcal{N})$.

Proof. Adjunction and nefness of $A = -K_S$ show that a negative irreducible curve has $(C^2, A \cdot C) = (-1, 1)$ or $(-2, 0)$ and is smooth rational. There exist finitely many such lattice classes by negative definiteness of $A^\perp$. Distinct negative curves cannot share a numerical class.

The pencils $h - e_i$ are base point free with square zero, h -degree one and A -degree two. Every component of a reducible fiber is negative, being a strict plane line through an additional center or an exceptional curve. Their fibers through z are consequently smooth irreducible rational curves. The three plane lines are distinct and none passes through a second proper base point. At least one is not a component of B_4 : otherwise removing all three from the quartic leaves a line that must pass through all three noncollinear proper points, since the quartic has multiplicity at least two at each. Choose that fiber C . Its local intersection with $p_S^*B_4$ is at most four.

Write the Zariski decomposition $P_t - uC = P_{t,u} + N_{t,u}$ and put $d(t, u) = P_{t,u} \cdot C$. All negative parts are supported on $\mathcal{N}$ and avoid z . Nefness of C gives

$$0 \leq d(t, u) \leq d_0(t) := \begin{cases} t, & 0 \leq t \leq 1, \\ (t+1)/2, & 1 \leq t \leq 3, \\ 2(4-t), & 3 \leq t \leq 4. \end{cases}$$

The volume derivative is $-2d(t, u)$, since the positive part is orthogonal to the fixed negative support. At the pseudo-effective endpoint the volume is zero. Thus $\int d(t, u) du = \rho(t)$.

The threshold of C is at most t , $3/2$, and $3(4-t)/2$ on the three intervals, respectively: intersect with h on the first and A on the other two. The inequality $S_{P_t}(C) \leq 2\tau(P_t; C)/3$ follows from [AZ22, Lemma 4.1], first for $P_t + \epsilon L$ ample and then by continuity as $\epsilon \downarrow 0$. The thresholds are uniformly bounded, so dominated convergence applies. The original rank-weighted normalization gives

$$S(p_S^* W; C) \leq \frac{2}{9} \left(\frac{1}{8} + \frac{7}{2} + \frac{9}{16} \right) = \frac{67}{72}.$$

No fixed divisor contains C .

For the secondary moment, after Zariski fixed parts are removed the actual restriction rows have degree $Md(t, u) + O(1)$ on $\mathbb{P}^1$. Lemma 2.8 and the original denominator give a moving bound

$$\frac{1}{6} \int_0^4 \int d(t, u)^2 du dt \leq \frac{1}{6} \int_0^4 d_0(t) \rho(t) dt = \frac{1}{6} \left(\frac{1}{8} + \frac{11}{3} + \frac{3}{4} \right) = \frac{109}{144}.$$

This is also the smooth-flag formula of [Fuj26b, Corollary 9.1]. Slice convergence is [AZ22, Lemmas 2.9 and 2.10], or follows directly from (8.15). There exist finitely many negative supports, and bounded parity errors vanish in the Riemann sums. The exceptional fixed divisors avoid z . The full quartic adds at most $4/24 = 1/6$, including after refinement, because the row ranks sum to the original rank in each grade. This proves (8.19). The surface flag theorem gives the claimed $67/72$ bound. Finally p_S is an isomorphism near z , so $A_S(w) = A_E(w)$ there. $\square$

Lemma 8.7. *Number the three clusters by 1, 2, 3, and put $Q_4 = p_S^* B_4$. Then*

$$(8.20) \quad Q_4 = B^{\text{str}} + \sum_i b_i R_i + 3 \sum_i F_i, \quad b_i \in \{2, 3\},$$

where B^{str} is effective with no exceptional component and each of its prime components has multiplicity at most two. Moreover

$$(8.21) \quad B^{\text{str}} \cdot R_i = 2b_i - 3, \quad B^{\text{str}} \cdot F_i = 3 - b_i.$$

Besides R_i, F_i , the possible negative irreducible curves have classes

$$(8.22) \quad \begin{aligned} L_{ij} &= h - e_i - e_j, & T_i &= h - e_i - e'_i, & K_i &= 2h - \sum_j e_j - \sum_{j \neq i} e'_j, \\ D_{ij} &= h - e_i - e'_i - e_j, & Z_0 &= 2h - \sum_j (e_j + e'_j). \end{aligned}$$

The first row consists of (-1) candidates, the second of (-2) candidates. Draw $i \rightarrow j$ when the successor at i lies in the direction of the line joining the proper points i, j . Up to relabeling the configurations are: no arrows and no Z_0 , no arrows with Z_0 , one arrow $1 \rightarrow 2$, a star $2 \rightarrow 1, 3 \rightarrow 1$, a path $1 \rightarrow 2, 2 \rightarrow 3$, and the cycle $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$. In each configuration the irreducible (-1) curves are precisely the candidates having nonnegative intersection with every root present.

Proof. On the first blowup S_0 , the quadratic transformation maps the first exceptional curve indexed by ζ to $\ell_\zeta : Q = -\zeta f + \zeta^2 c$. On this line put

$$T_\zeta = -(v\zeta + \lambda)f + \zeta^2(u - \mu\zeta)c, \quad U_\zeta = \zeta f + \zeta^2 c.$$

Then $G = T_\zeta - \zeta^2 c$ and $D_2|_{\ell_\zeta} = T_\zeta(U_\zeta - T_\zeta)$. The successor maps to $T_\zeta = 0$, a nonzero linear form by (8.5). If U_ζ is nonzero there, D_2 is smooth and transverse to ℓ_ζ . If $U_\zeta = 0$, then $G + Q/2 = 0$, so the gradient of $D_2 = fc + Q^2/4 - (G + Q/2)^2$ equals the nonzero gradient of the smooth conic $fc + Q^2/4$ there. This includes the case when ℓ_ζ is a component, since then $T_\zeta = U_\zeta$.

The three ℓ_ζ are distinct tangent lines to that smooth conic, with distinct tangency points. Their pairwise intersection vertices are off it. If the marked image is a Cremona vertex, transversality makes the strict conic transform avoid the strict ℓ_ζ there; the total transform has multiplicity one, from the exceptional curve of q_0 . Elsewhere q_0 is an isomorphism at the marked point, again with multiplicity one. Its generic multiplicity along a first exceptional curve is zero or one. Pulling $p_0^*B_4 = q_0^*D_2 + 2\sum e_i$ through the second blowups gives (8.20). Intersection with R_i, F_i gives (8.21); multiplicity at most two for the other components follows from log canonicity of $(E, B_4/2)$.

For the curve list, write a nonexceptional negative curve as $dh - \sum(m_i e_i + n_i e'_i)$, where $m_i \geq n_i \geq 0$. For a (-1) curve the sums of multiplicities and their squares are $3d - 1$ and $d^2 + 1$; for a (-2) curve they are $3d$ and $d^2 + 2$. Cauchy's inequality gives $d \leq 2$ in both cases. Solving these integer conditions gives (8.22); the line through all three proper points is excluded by noncollinearity.

An arrow exists exactly when the corresponding D_{ij} is effective. There exists at most one outgoing arrow per vertex. Opposite arrows would give a strict line of square -3 and negative intersection with A . These facts give the five arrow types listed in the statement. An irreducible Z_0 is impossible with an arrow: its three conditions on the joining line force that line to be a component. With no arrows, a conic satisfying all six conditions is irreducible, since a reducible one would force an arrow. This gives the six configurations.

Each (-1) candidate is effective: the conic imposes only five linear conditions, and the line cases are explicit. Negative intersection with a root forces reducibility. Conversely, if an effective candidate $D = C + V$ is reducible with $A \cdot C = 1$ and $A \cdot V = 0$, all components of V are roots. Hodge index [Mil24, Corollaries 11.51–11.52] and adjunction give $C^2 = -1$. Since $D^2 = -1$, we obtain $2C \cdot V + V^2 = 0$. Nonnegative intersection of D with every root gives $0 \leq D \cdot V = -C \cdot V \leq 0$. Thus $V^2 = 0$; negative definiteness of $A^\perp$ forces $V = 0$. This proves the irreducibility test. $\square$

Lemma 8.8. *For a negative curve C on S , write $P_t - uC = P_{t,u} + N_{t,u}$ for its Zariski decomposition up to the big threshold, with P_t as in (8.18), and define*

$$(8.23) \quad \begin{aligned} s_C &= \frac{1}{6} \iint P_{t,u}^2 du dt, \\ b_C &= \frac{1}{6} \iint (P_{t,u} \cdot C)^2 du dt, \\ k_{C,D} &= \frac{1}{3} \iint (P_{t,u} \cdot C) \text{coeff}_D(N_{t,u}) du dt. \end{aligned}$$

Then $S_{\text{mov}}(C) = s_C$, and at $z \in C$ the secondary moving moment is at most

$$(8.24) \quad b_C + \sum_{D \neq C} k_{C,D} I_z(C, D).$$

The following bounds hold in every configuration:

$$(8.25) \quad \begin{array}{c|cc} C & s_C \text{ bound} & b_C \text{ bound} \\ \hline F_i & 17/18 & 5/12 \\ R_i & 65/72 & 19/36 \\ L_{ij} & 17/24 & 5/12 \\ T_i & 17/24 & 11/18 \\ K_i & 17/24 & 47/72 \\ D_{ij} & 17/24 & 4/9 \\ Z_0 & 17/24 & 2/3. \end{array}$$

For F_i , every neighbor coefficient other than that of R_i is at most $121/288$, and $b_{F_i} + k_{F_i, R_i} \leq 59/72$, except at the star hub F_1 . For R_i , every neighbor coefficient other than that of F_i is at most $7/24$. At the star hub $b_{R_1} = 5/12$ and $k_{R_1, F_1} = 7/18$. The sums of nonexceptional neighbor coefficients for L, T, K, D, Z_0 are at most $1/18, 1/72 + 5/27, 1/216, 5/72$, and zero, respectively. For either exceptional flag, its neighbors other than its paired exceptional curve are mutually disjoint and disjoint from that paired curve.

Proof. We compute the moments from the Zariski chambers and record the complete six-configuration calculation.

Step 1. In this step, we verify the intersection and nefness tests for the chamber decompositions.

The intersection matrix in the ordered total-transform basis $(h, e_1, e_2, e_3, e'_1, e'_2, e'_3)$ is

$$\text{diag}(1, -1, -1, -1, -1, -1, -1).$$

The complete residual series in (8.15) give the factor $1/6$ in the volume integral, because the denominator is $3M^4 + O(M^3)$. After removing the Zariski fixed part at a flag level, the actual restriction image has degree $M(P_{t,u} \cdot C) + O(1)$ on $C \simeq \mathbb{P}^1$. Its free point weight is at most $d(d+1)/2$, and the fixed part contributes its actual rank times its local intersection. This proves (8.24) and the factors in (8.23). It agrees with the complete surface refinement formula in [AZ22, Lemma 4.5] and its slice formulas [AZ22, Lemmas 2.9 and 2.10].

We record the full finite chamber calculation below. In each cell set $P = P_t - uC - N$. The stated negative support has negative-definite intersection matrix, nonnegative coefficients, and P is orthogonal to it. At the polygon vertices, substitution into the displayed intersection matrix gives $P^2 \geq 0$, $A \cdot P \geq 0$, and nonnegative intersection with every listed negative curve. These conditions prove nefness: a nonnegative-square irreducible curve lies in the closed positive cone containing A , and pairs nonnegatively with P by Hodge index [Mil24, Corollaries 11.51–11.52]. Convexity extends the check over each polygon. The vertical sections of these polygons concatenate without gaps from $u = 0$ to an upper envelope where $P^2 = 0$. Volume remains zero thereafter by monotonicity. Hence the cells cover the entire big range and give the stated Zariski decompositions.

Step 2. In this step, we integrate over the chambers and control the finite-degree errors uniformly. For a triangle of area V with vertices v_0, v_1, v_2 , the product of affine functions ℓ_1, ℓ_2 has integral

$$(8.26) \quad \frac{V}{12} \left[\left(\sum_{i=0}^2 \ell_1(v_i) \right) \left(\sum_{i=0}^2 \ell_2(v_i) \right) + \sum_{i=0}^2 \ell_1(v_i) \ell_2(v_i) \right].$$

Triangulating each polygon from its first vertex and using (8.26) gives the values $s_C, b_C, k_{C,D}$ printed with each flag. Thus the certificate consists of exact rational intersection and integration calculations.

There exist finitely many rational chambers. After clearing their denominators, Riemann–Roch and vanishing for nef divisors give uniform $O(1)$ errors in restriction ranks and fixed coefficients in the interiors. Wall strips of width $O(1/M)$ are bounded by these linear degree estimates and contribute $o(1)$ after normalization. The bounded parity changes in (8.15) have

zero limiting contribution. The estimates are therefore uniform in the point on the chosen flag. Every nonzero intersection of two distinct listed curves is one, so these intersections are transverse. The neighbor assertions follow from the displayed class lists.

Step 3. In this step, we give the complete chamber and moment tables for all six configurations. For readability in the certificate, a polygon is specified by its vertices (t, u) , and $d_C = P \cdot C$. Only flag types up to the symmetries of each configuration are listed. The printed $k_{C,D}$ are restricted to neighbors, meaning $C \cdot D > 0$; an unlisted neighbor coefficient is zero. Coefficients of disjoint curves may be nonzero, but their local intersection contributions vanish.

Generic configuration.

The negative curves are $F_1, F_2, F_3, L_{12}, L_{13}, L_{23}, T_1, T_2, T_3, K_1, K_2, K_3, R_1, R_2, R_3$.

Flag $C = F_1$. $s_C = 3/4$, $b_C = 5/12$, and

$$k_{C,T_1} = 1/12, k_{C,K_2} = 1/72, k_{C,K_3} = 1/72, k_{C,R_1} = 3/8.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)T_1 + (u/2)R_1$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(3, 1), (3, 4/3), (1, 2)$	$(u - 1)T_1 + (t + 2u - 5)K_2 + (t + 2u - 5)K_3 +$ $(u/2)R_1 + (t/2 + u - 5/2)R_2 + (t/2 + u - 5/2)R_3$	$-3t/2$ $9u/2 + 21/2$
$(3, 1), (1, 2), (1, 1)$	$(u - 1)T_1 + (u/2)R_1$	$t/2 - u/2 + 1/2$
$(1, 0), (3, 0), (3, 1), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(4, 0), (3, 4/3), (3, 1)$	$(t + u - 4)T_1 + (2t + 2u - 8)K_2 + (2t + 2u -$ $8)K_3 + (u/2)R_1 + (t + u - 4)R_2 + (t + u - 4)R_3$	$-6t - 9u/2 +$ 24
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1$	$-t + u/2 + 4$

Flag $C = R_1$. $s_C = 3/4$, $b_C = 19/36$, and

$$k_{C,F_1} = 19/72, k_{C,L_{12}} = 1/12, k_{C,L_{13}} = 1/12, k_{C,K_1} = 1/72.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(-t/2 + u + 1/2)F_1 + (2u - 2)L_{12} + (2u - 2)L_{13} +$ $(t/2 + u - 5/2)K_1 + (u - 1)R_2 + (u - 1)R_3$	$6 - 4u$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + u + 1/2)F_1 + (2u - 2)L_{12} + (2u - 2)L_{13} +$ $(u - 1)R_2 + (u - 1)R_3$	$t/2 - 3u + 7/2$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(t + u - 4)F_1 + (2t + 2u - 8)L_{12} + (2t + 2u -$ $8)L_{13} + (t + u - 4)K_1 + (t + u - 4)R_2 + (t + u -$ $4)R_3$	$-6t - 4u + 24$

Flag $C = L_{12}$. $s_C = 11/18$, $b_C = 5/12$, and

$$k_{C,T_3} = 1/18, k_{C,R_1} = 1/3, k_{C,R_2} = 1/3.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1 + (u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_2 + (u - 1)T_3 +$ $(-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_2 + (-t/2 + u +$ $1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(u - 1)T_3 + (u/2)R_1 + (u/2)R_2$	$2 - u$
$(1, 0), (3, 0), (3, 1), (2, 1)$	$(u/2)R_1 + (u/2)R_2$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_3 + (u/2)R_1 + (u/2)R_2$	$-2t - u + 8$

Generic configuration, $C = L_{12}$ (continued)

Polygon vertices	N	d_C
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1 + (u/2)R_2$	$4 - t$

Flag $C = T_1$. $s_C = 17/36$, $b_C = 11/18$, and

$$k_{C,F_1} = 7/36.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)R_1$	$t - u$
$(1, 0), (3, 0), (3, 1)$	0	$u + 1$
$(1, 0), (3, 1), (1, 1)$	$(-t + 2u + 1)F_1 + (-t/2 + u + 1/2)R_1$	$t - u$
$(3, 0), (4, 0), (3, 1)$	0	$-t + u + 4$

Flag $C = K_1$. $s_C = 91/216$, $b_C = 47/72$, and

$$k_{C,F_2} = 23/216, k_{C,F_3} = 23/216, k_{C,T_1} = 1/216, k_{C,R_1} = 47/216.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1/2)$	$(u)F_1 + (2u)F_2 + (2u)F_3 + (u)R_1 + (u)R_2 + (u)R_3$	$2t - 4u$
$(1, 0), (2, 1), (1, 1/2)$	$(-t + u + 1)F_1 + (-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$2t - 4u$
$(3, 1), (3, 4/3), (2, 1)$	$(-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (u - 1)T_1 + (u/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$3t/2 - 9u/2 + 3/2$
$(1, 0), (3, 1), (2, 1)$	$(-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (u/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$3t/2 - 7u/2 + 1/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_1$	$-t/2 + u/2 + 5/2$
$(4, 0), (3, 4/3), (3, 1)$	$(2t + 2u - 8)F_2 + (2t + 2u - 8)F_3 + (t + u - 4)T_1 + (u/2)R_1 + (t + u - 4)R_2 + (t + u - 4)R_3$	$-6t - 9u/2 + 24$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1$	$-t + u/2 + 4$

Conic configuration.

The negative curves are $F_1, F_2, F_3, L_{12}, L_{13}, L_{23}, T_1, T_2, T_3, R_1, R_2, R_3, Z_0$.

Flag $C = F_1$. $s_C = 29/36$, $b_C = 25/72$, and

$$k_{C,T_1} = 1/8, k_{C,R_1} = 29/72, k_{C,Z_0} = 11/72.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)T_1 + (u/2)R_1$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(2, 1), (1, 2), (1, 1)$	$(u - 1)T_1 + (u/2)R_1$	$t/2 - u/2 + 1/2$
$(3, 1), (3, 2), (1, 2), (2, 1)$	$(u - 1)T_1 + (u/2)R_1 + (t/2 + u/2 - 3/2)Z_0$	$2 - u$
$(1, 0), (3, 0), (2, 1), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 1), (2, 1)$	$(u/2)R_1 + (t/2 + u/2 - 3/2)Z_0$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_1 + (u/2)R_1 + (u/2)Z_0$	$-2t - u + 8$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1 + (u/2)Z_0$	$4 - t$

Flag $C = R_1$. $s_C = 3/4$, $b_C = 19/36$, and
 $k_{C,F_1} = 5/18, k_{C,L_{12}} = 1/12, k_{C,L_{13}} = 1/12$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + u + 1/2)F_1 + (2u - 2)L_{12} + (2u - 2)L_{13} + (u - 1)R_2 + (u - 1)R_3$	$t/2 - 3u + 7/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(2u - 2)F_1 + (2u - 2)L_{12} + (2u - 2)L_{13} + (u - 1)R_2 + (u - 1)R_3 + (t/2 + u - 5/2)Z_0$	$6 - 4u$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(2t + 2u - 8)F_1 + (2t + 2u - 8)L_{12} + (2t + 2u - 8)L_{13} + (t + u - 4)R_2 + (t + u - 4)R_3 + (t + u - 4)Z_0$	$-6t - 4u + 24$

Flag $C = L_{12}$. $s_C = 11/18$, $b_C = 5/12$, and
 $k_{C,T_3} = 1/18, k_{C,R_1} = 1/3, k_{C,R_2} = 1/3$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1 + (u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_2 + (u - 1)T_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_2 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(u - 1)T_3 + (u/2)R_1 + (u/2)R_2$	$2 - u$
$(1, 0), (3, 0), (3, 1), (2, 1)$	$(u/2)R_1 + (u/2)R_2$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_3 + (u/2)R_1 + (u/2)R_2$	$-2t - u + 8$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1 + (u/2)R_2$	$4 - t$

Flag $C = T_1$. $s_C = 17/36$, $b_C = 11/18$, and
 $k_{C,F_1} = 7/36$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)R_1$	$t - u$
$(1, 0), (3, 0), (3, 1)$	0	$u + 1$
$(1, 0), (3, 1), (1, 1)$	$(-t + 2u + 1)F_1 + (-t/2 + u + 1/2)R_1$	$t - u$
$(3, 0), (4, 0), (3, 1)$	0	$-t + u + 4$

Flag $C = Z_0$. $s_C = 1/2$, $b_C = 2/3$, and
 $k_{C,F_1} = 5/36, k_{C,F_2} = 5/36, k_{C,F_3} = 5/36$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1/2)$	$(2u)F_1 + (2u)F_2 + (2u)F_3 + (u)R_1 + (u)R_2 + (u)R_3$	$2t - 4u$
$(1, 0), (3, 0), (3, 1)$	0	$-t + 2u + 3$
$(1, 0), (3, 1), (3, 3/2), (1, 1/2)$	$(-t + 2u + 1)F_1 + (-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$2t - 4u$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(2t + 2u - 8)F_1 + (2t + 2u - 8)F_2 + (2t + 2u - 8)F_3 + (t + u - 4)R_1 + (t + u - 4)R_2 + (t + u - 4)R_3$	$-6t - 4u + 24$

Edge configuration.

The negative curves are $F_1, F_2, F_3, L_{13}, L_{23}, T_2, T_3, K_1, R_1, R_2, R_3, D_{12}$.

Flag $C = F_1$. $s_C = 7/8$, $b_C = 5/16$, and

$$k_{C,R_1} = 7/16, k_{C,D_{12}} = 55/144.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t+u)F_2 + (u/2)R_1 + (-t+u)R_2 + (-t+u)D_{12}$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(2, 2), (1, 2), (1, 1)$	$(-t+u)F_2 + (u/2)R_1 + (-t/2 + u - 1/2)R_2 + (u-1)D_{12}$	$t/2 - u/2 + 1/2$
$(3, 3/2), (3, 2), (2, 2)$	$(t+2u-6)T_3 + (u/2)R_1 + (t/2 + u - 5/2)R_2 + (t+2u-5)D_{12}$	$-t/2 - 3u/2 + 9/2$
$(1, 0), (3, 0), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 3/2), (2, 2), (1, 1)$	$(u/2)R_1 + (t/6 + u/3 - 1/2)R_2 + (t/3 + 2u/3 - 1)D_{12}$	$t/6 - u/6 + 1/2$
$(4, 0), (3, 2), (3, 3/2)$	$(3t+2u-12)T_3 + (u/2)R_1 + (t+u-4)R_2 + (2t+2u-8)D_{12}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 3/2)$	$(u/2)R_1 + (u/3)R_2 + (2u/3)D_{12}$	$-t - u/6 + 4$

Flag $C = F_2$. $s_C = 55/72$, $b_C = 85/216$, and

$$k_{C,T_2} = 5/54, k_{C,K_1} = 1/36, k_{C,R_2} = 23/54.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t+u)T_2 + (u/2)R_2$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_2$	$u/2$
$(3, 1), (3, 3/2), (1, 2)$	$(u-1)T_2 + (t+2u-5)K_1 + (t/2 + u - 5/2)R_1 + (t/6 + 2u/3 - 1/2)R_2 + (t/3 + u/3 - 1)D_{12}$	$-2t/3 - 8u/3 + 6$
$(2, 1), (1, 2), (1, 1)$	$(u-1)T_2 + (u/2)R_2$	$t/2 - u/2 + 1/2$
$(3, 1), (1, 2), (2, 1)$	$(u-1)T_2 + (t/6 + 2u/3 - 1/2)R_2 + (t/3 + u/3 - 1)D_{12}$	$t/3 - 2u/3 + 1$
$(1, 0), (3, 0), (2, 1), (1, 1)$	$(u/2)R_2$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 1), (2, 1)$	$(t/6 + 2u/3 - 1/2)R_2 + (t/3 + u/3 - 1)D_{12}$	$t/3 + u/3$
$(4, 0), (3, 3/2), (3, 1)$	$(t+u-4)T_2 + (2t+2u-8)K_1 + (t+u-4)R_1 + (2u/3)R_2 + (u/3)D_{12}$	$-4t - 8u/3 + 16$
$(3, 0), (4, 0), (3, 1)$	$(2u/3)R_2 + (u/3)D_{12}$	$-t + u/3 + 4$

Flag $C = F_3$. $s_C = 3/4$, $b_C = 5/12$, and

$$k_{C,T_3} = 7/72, k_{C,K_1} = 1/72, k_{C,R_3} = 3/8.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t+u)T_3 + (u/2)R_3$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_3$	$u/2$
$(3, 1), (3, 4/3), (1, 2)$	$(t+3u-6)T_3 + (t+2u-5)K_1 + (t/2 + u - 5/2)R_1 + (t/2 + u - 5/2)R_2 + (u/2)R_3 + (t+2u-5)D_{12}$	$-3t/2 - 9u/2 + 21/2$
$(3, 1), (1, 2), (1, 1)$	$(u-1)T_3 + (u/2)R_3$	$t/2 - u/2 + 1/2$
$(1, 0), (3, 0), (3, 1), (1, 1)$	$(u/2)R_3$	$t/2 + u/2 - 1/2$
$(4, 0), (3, 4/3), (3, 1)$	$(3t+3u-12)T_3 + (2t+2u-8)K_1 + (t+u-4)R_1 + (t+u-4)R_2 + (u/2)R_3 + (2t+2u-8)D_{12}$	$-6t - 9u/2 + 24$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_3$	$-t + u/2 + 4$

Flag C = L_{13} . $s_C = 11/18$, $b_C = 5/12$, and
 $k_{C,T_2} = 1/18$, $k_{C,R_1} = 1/3$, $k_{C,R_3} = 1/3$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1 + (u)F_3 + (u)R_1 + (u)R_3$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_3 + (u - 1)T_2 +$ $(-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_3$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_3 + (-t/2 + u +$ $1/2)R_1 + (-t/2 + u + 1/2)R_3$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(u - 1)T_2 + (u/2)R_1 + (u/2)R_3$	$2 - u$
$(1, 0), (3, 0), (3, 1), (2, 1)$	$(u/2)R_1 + (u/2)R_3$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_2 + (u/2)R_1 + (u/2)R_3$	$-2t - u + 8$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1 + (u/2)R_3$	$4 - t$

Flag C = L_{23} . $s_C = 5/8$, $b_C = 59/144$, and
 $k_{C,R_2} = 59/144$, $k_{C,R_3} = 49/144$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_2 + (u)F_3 + (u)R_2 + (u)R_3$	$t - u$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_2 + (-t + u + 1)F_3 + (-t/2 + u +$ $1/2)R_2 + (-t/2 + u + 1/2)R_3$	$t - u$
$(3, 2), (1, 1), (2, 1)$	$(-t + 2u)F_2 + (-t + u + 1)F_3 + (-t/2 + 2u -$ $1/2)R_2 + (-t/2 + u + 1/2)R_3 + (u - 1)D_{12}$	$t - 2u + 1$
$(3, 3/2), (3, 2), (2, 1)$	$(-t + 2u)F_2 + (-t/2 + 2u - 1/2)R_2 + (u/2)R_3 +$ $(u - 1)D_{12}$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 0), (2, 1)$	$(u/2)R_2 + (u/2)R_3$	1
$(3, 0), (3, 3/2), (2, 1)$	$(t/6 + 2u/3 - 1/2)R_2 + (u/2)R_3 + (t/3 + u/3 -$ $1)D_{12}$	$-t/6 - u/6 + 3/2$
$(4, 0), (3, 2), (3, 3/2)$	$(3t + 2u - 12)F_2 + (2t + 2u - 8)R_2 + (u/2)R_3 +$ $(t + u - 4)D_{12}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 3/2)$	$(2u/3)R_2 + (u/2)R_3 + (u/3)D_{12}$	$-t - u/6 + 4$

Flag C = T_2 . $s_C = 17/36$, $b_C = 11/18$, and
 $k_{C,F_2} = 7/36$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_2 + (u)R_2$	$t - u$
$(1, 0), (3, 0), (3, 1)$	0	$u + 1$
$(1, 0), (3, 1), (1, 1)$	$(-t + 2u + 1)F_2 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 0), (4, 0), (3, 1)$	0	$-t + u + 4$

Flag C = T_3 . $s_C = 37/72$, $b_C = 55/108$, and
 $k_{C,F_3} = 13/54$, $k_{C,T_2} = 1/72$, $k_{C,D_{12}} = 5/27$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_3 + (u)R_3$	$t - u$
$(1, 0), (3, 0), (2, 1/2)$	0	$u + 1$
$(3, 1), (3, 3/2), (1, 1)$	$(-t + 2u + 1)F_3 + (u - 1)T_2 + (t/6 + u/3 -$ $1/2)R_2 + (-t/2 + u + 1/2)R_3 + (t/3 + 2u/3 -$ $1)D_{12}$	$2t/3 - 8u/3 + 2$
$(2, 1/2), (3, 1), (1, 1)$	$(-t + 2u + 1)F_3 + (t/6 + u/3 - 1/2)R_2 + (-t/2 +$ $u + 1/2)R_3 + (t/3 + 2u/3 - 1)D_{12}$	$2t/3 - 5u/3 + 1$
$(1, 0), (2, 1/2), (1, 1)$	$(-t + 2u + 1)F_3 + (-t/2 + u + 1/2)R_3$	$t - u$

Edge configuration, $C = T_3$ (continued)

Polygon vertices	N	d_C
$(3, 0), (3, 1), (2, 1/2)$	$(t/6 + u/3 - 1/2)R_2 + (t/3 + 2u/3 - 1)D_{12}$	$-t/3 + u/3 + 2$
$(4, 0), (3, 3/2), (3, 1)$	$(2t + 2u - 8)F_3 + (t + u - 4)T_2 + (u/3)R_2 + (t + u - 4)R_3 + (2u/3)D_{12}$	$-4t - 8u/3 + 16$
$(3, 0), (4, 0), (3, 1)$	$(u/3)R_2 + (2u/3)D_{12}$	$-t + u/3 + 4$

Flag $C = K_1$. $s_C = 91/216$, $b_C = 47/72$, and
 $k_{C,F_2} = 1/9$, $k_{C,F_3} = 23/216$, $k_{C,R_1} = 47/216$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1/2)$	$(u)F_1 + (2u)F_2 + (2u)F_3 + (u)R_1 + (u)R_2 + (u)R_3$	$2t - 4u$
$(1, 0), (2, 1), (1, 1/2)$	$(-t + u + 1)F_1 + (-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$2t - 4u$
$(1, 0), (3, 1), (2, 1)$	$(-t + 2u + 1)F_2 + (-t + 2u + 1)F_3 + (u/2)R_1 + (-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$3t/2 - 7u/2 + 1/2$
$(3, 1), (3, 4/3), (2, 1)$	$(-t + 3u)F_2 + (-t + 2u + 1)F_3 + (u/2)R_1 + (-t/2 + 2u - 1/2)R_2 + (-t/2 + u + 1/2)R_3 + (u - 1)D_{12}$	$3t/2 - 9u/2 + 3/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_1$	$-t/2 + u/2 + 5/2$
$(4, 0), (3, 4/3), (3, 1)$	$(3t + 3u - 12)F_2 + (2t + 2u - 8)F_3 + (u/2)R_1 + (2t + 2u - 8)R_2 + (t + u - 4)R_3 + (t + u - 4)D_{12}$	$-6t - 9u/2 + 24$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_1$	$-t + u/2 + 4$

Flag $C = R_1$. $s_C = 3/4$, $b_C = 19/36$, and
 $k_{C,F_1} = 25/72$, $k_{C,L_{13}} = 1/12$, $k_{C,K_1} = 1/72$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(-t/2 + 3u - 3/2)F_1 + (2u - 2)L_{13} + (t/2 + u - 5/2)K_1 + (u - 1)R_2 + (u - 1)R_3 + (2u - 2)D_{12}$	$6 - 4u$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + 3u - 3/2)F_1 + (2u - 2)L_{13} + (u - 1)R_2 + (u - 1)R_3 + (2u - 2)D_{12}$	$t/2 - 3u + 7/2$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(3t + 3u - 12)F_1 + (2t + 2u - 8)L_{13} + (t + u - 4)K_1 + (t + u - 4)R_2 + (t + u - 4)R_3 + (2t + 2u - 8)D_{12}$	$-6t - 4u + 24$

Flag $C = R_2$. $s_C = 59/72$, $b_C = 4/9$, and
 $k_{C,F_2} = 23/72$, $k_{C,L_{23}} = 13/72$, $k_{C,D_{12}} = 1/4$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_2$	u
$(1, 0), (3, 0), (2, 1/2)$	0	$2u$
$(1, 0), (2, 1/2), (1, 1)$	$(-t/2 + u + 1/2)F_2$	$t/2 + u - 1/2$
$(3, 1), (3, 2), (1, 1)$	$(-t/2 + u + 1/2)F_2 + (2u - 2)L_{23} + (u - 1)R_3 + (t/4 + u/2 - 3/4)D_{12}$	$t/4 - 3u/2 + 9/4$
$(2, 1/2), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_2 + (t/4 + u/2 - 3/4)D_{12}$	$t/4 + u/2 + 1/4$

Edge configuration, $C = R_2$ (continued)

Polygon vertices	N	d_C
$(3, 0), (3, 1), (2, 1/2)$	$(t/4 + u/2 - 3/4)D_{12}$	$-t/4 + 3u/2 + 3/4$
$(4, 0), (3, 2), (3, 1)$	$(t+u-4)F_2 + (2t+2u-8)L_{23} + (t+u-4)R_3 + (u/2)D_{12}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)D_{12}$	$3u/2$

Flag $C = R_3$. $s_C = 3/4$, $b_C = 19/36$, and
 $k_{C,F_3} = 19/72$, $k_{C,L_{13}} = 1/12$, $k_{C,L_{23}} = 7/72$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_3$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_3$	$t/2 + u - 1/2$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + u + 1/2)F_3 + (2u-2)L_{13} + (2u-2)L_{23} + (u-1)R_1 + (u-1)R_2$	$t/2 - 3u + 7/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(-t/2 + u + 1/2)F_3 + (2u-2)L_{13} + (t/2 + 3u - 9/2)L_{23} + (u-1)R_1 + (t/2 + 2u - 7/2)R_2 + (t/2 + u - 5/2)D_{12}$	$6 - 4u$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(t+u-4)F_3 + (2t+2u-8)L_{13} + (3t+3u-12)L_{23} + (t+u-4)R_1 + (2t+2u-8)R_2 + (t+u-4)D_{12}$	$-6t - 4u + 24$

Flag $C = D_{12}$. $s_C = 25/36$, $b_C = 4/9$, and
 $k_{C,F_1} = 31/72$, $k_{C,T_3} = 5/72$, $k_{C,R_2} = 3/8$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t+2u+1)F_1 + (-t+u+1)F_2 + (u-1)T_3 + (-t/2+u+1/2)R_1 + (-t/2+u+1/2)R_2$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t+2u+1)F_1 + (-t+u+1)F_2 + (-t/2+u+1/2)R_1 + (-t/2+u+1/2)R_2$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(-t+2u+1)F_1 + (u-1)T_3 + (-t/2+u+1/2)R_1 + (u/2)R_2$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 1), (2, 1)$	$(-t+2u+1)F_1 + (-t/2+u+1/2)R_1 + (u/2)R_2$	$t/2 - u/2 + 1/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_2$	$-t/2 + 3u/2 + 3/2$
$(4, 0), (3, 2), (3, 1)$	$(2t+2u-8)F_1 + (t+u-4)T_3 + (t+u-4)R_1 + (u/2)R_2$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_2$	$3u/2$

Star configuration.

The negative curves are $F_1, F_2, F_3, L_{23}, T_1, R_1, R_2, R_3, D_{21}, D_{31}$.

Flag $C = F_1$. $s_C = 29/36$, $b_C = 25/72$, and

$$k_{C,T_1} = 1/8, k_{C,R_1} = 5/9.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)T_1 + (u/2)R_1$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(2, 1), (1, 2), (1, 1)$	$(u - 1)T_1 + (u/2)R_1$	$t/2 - u/2 + 1/2$
$(3, 1), (3, 2), (1, 2), (2, 1)$	$(u - 1)T_1 + (t/2 + u - 3/2)R_1 + (t/2 + u/2 - 3/2)D_{21} + (t/2 + u/2 - 3/2)D_{31}$	$2 - u$
$(1, 0), (3, 0), (2, 1), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 1), (2, 1)$	$(t/2 + u - 3/2)R_1 + (t/2 + u/2 - 3/2)D_{21} + (t/2 + u/2 - 3/2)D_{31}$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_1 + (u)R_1 + (u/2)D_{21} + (u/2)D_{31}$	$-2t - u + 8$
$(3, 0), (4, 0), (3, 1)$	$(u)R_1 + (u/2)D_{21} + (u/2)D_{31}$	$4 - t$

Flag $C = F_2$. $s_C = 8/9$, $b_C = 11/36$, and

$$k_{C,R_2} = 4/9, k_{C,D_{21}} = 29/72.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)F_1 + (-t + u)R_1 + (u/2)R_2 + (-t + u)D_{21}$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_2$	$u/2$
$(2, 2), (1, 2), (1, 1)$	$(-t + u)F_1 + (-t/2 + u - 1/2)R_1 + (u/2)R_2 + (u - 1)D_{21}$	$t/2 - u/2 + 1/2$
$(3, 0), (2, 2), (1, 1)$	$(t/6 + u/3 - 1/2)R_1 + (u/2)R_2 + (t/3 + 2u/3 - 1)D_{21}$	$t/6 - u/6 + 1/2$
$(3, 0), (3, 2), (2, 2)$	$(t/2 + u/2 - 3/2)R_1 + (u/2)R_2 + (t/2 + 3u/4 - 3/2)D_{21} + (t/2 + u/4 - 3/2)D_{31}$	$1 - u/4$
$(1, 0), (3, 0), (1, 1)$	$(u/2)R_2$	$t/2 + u/2 - 1/2$
$(3, 0), (4, 0), (3, 2)$	$(u/2)R_1 + (u/2)R_2 + (3u/4)D_{21} + (u/4)D_{31}$	$-t - u/4 + 4$

Flag $C = R_1$. $s_C = 65/72$, $b_C = 5/12$, and

$$k_{C,F_1} = 7/18, k_{C,D_{21}} = 7/24, k_{C,D_{31}} = 7/24.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (2, 1/2)$	0	$2u$
$(1, 0), (2, 1/2), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(2, 1/2), (3, 1), (3, 2), (1, 1)$	$(-t/2 + u + 1/2)F_1 + (t/4 + u/2 - 3/4)D_{21} + (t/4 + u/2 - 3/4)D_{31}$	1
$(3, 0), (3, 1), (2, 1/2)$	$(t/4 + u/2 - 3/4)D_{21} + (t/4 + u/2 - 3/4)D_{31}$	$-t/2 + u + 3/2$
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)F_1 + (u/2)D_{21} + (u/2)D_{31}$	$4 - t$
$(3, 0), (4, 0), (3, 1)$	$(u/2)D_{21} + (u/2)D_{31}$	u

Flag $C = R_2$. $s_C = 3/4$, $b_C = 19/36$, and
 $k_{C,F_2} = 13/36$, $k_{C,L_{23}} = 1/12$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_2$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_2$	$t/2 + u - 1/2$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + 3u - 3/2)F_2 + (2u - 2)L_{23} + (u - 1)R_1 +$ $(u - 1)R_3 + (2u - 2)D_{21}$	$t/2 - 3u + 7/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(4u - 4)F_2 + (2u - 2)L_{23} + (t/2 + 2u - 7/2)R_1 +$ $(u - 1)R_3 + (t/2 + 3u - 9/2)D_{21} + (t/2 + u -$ $5/2)D_{31}$	$6 - 4u$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(4t + 4u - 16)F_2 + (2t + 2u - 8)L_{23} + (2t +$ $2u - 8)R_1 + (t + u - 4)R_3 + (3t + 3u - 12)D_{21} +$ $(t + u - 4)D_{31}$	$-6t - 4u + 24$

Flag $C = T_1$. $s_C = 17/36$, $b_C = 11/18$, and
 $k_{C,F_1} = 7/36$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)R_1$	$t - u$
$(1, 0), (3, 0), (3, 1)$	0	$u + 1$
$(1, 0), (3, 1), (1, 1)$	$(-t + 2u + 1)F_1 + (-t/2 + u + 1/2)R_1$	$t - u$
$(3, 0), (4, 0), (3, 1)$	0	$-t + u + 4$

Flag $C = L_{23}$. $s_C = 11/18$, $b_C = 5/12$, and
 $k_{C,T_1} = 1/18$, $k_{C,R_2} = 1/3$, $k_{C,R_3} = 1/3$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_2 + (u)F_3 + (u)R_2 + (u)R_3$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + u + 1)F_2 + (-t + u + 1)F_3 + (u - 1)T_1 +$ $(-t/2 + u + 1/2)R_2 + (-t/2 + u + 1/2)R_3$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_2 + (-t + u + 1)F_3 + (-t/2 + u +$ $1/2)R_2 + (-t/2 + u + 1/2)R_3$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(u - 1)T_1 + (u/2)R_2 + (u/2)R_3$	$2 - u$
$(1, 0), (3, 0), (3, 1), (2, 1)$	$(u/2)R_2 + (u/2)R_3$	1
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)T_1 + (u/2)R_2 + (u/2)R_3$	$-2t - u + 8$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_2 + (u/2)R_3$	$4 - t$

Flag $C = D_{21}$. $s_C = 17/24$, $b_C = 71/162$, and
 $k_{C,F_2} = 73/162$, $k_{C,R_1} = 301/648$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1 + (2u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_1 + (-t + 2u + 1)F_2 + (-t/2 + u +$ $1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 3/2), (3, 2), (1, 1), (2, 1)$	$(-t + 2u)F_1 + (-t + 2u + 1)F_2 + (-t/2 + 2u -$ $1/2)R_1 + (-t/2 + u + 1/2)R_2 + (u - 1)D_{31}$	$t - 2u + 1$
$(1, 0), (7/3, 2/3), (2, 1)$	$(-t + 2u + 1)F_2 + (u/2)R_1 + (-t/2 + u + 1/2)R_2$	$t/2 - u/2 + 1/2$
$(7/3, 2/3), (3, 1), (3, 3/2), (2, 1)$	$(-t + 2u + 1)F_2 + (t/6 + 2u/3 - 1/2)R_1 + (-t/2 +$ $u + 1/2)R_2 + (t/3 + u/3 - 1)D_{31}$	$t/3 - 2u/3 + 1$

Star configuration, $C = D_{21}$ (continued)

Polygon vertices	N	d_C
$(1, 0), (3, 0), (7/3, 2/3)$	$(u/2)R_1$	$-t/2 + 3u/2 + 3/2$
$(3, 0), (3, 1), (7/3, 2/3)$	$(t/6 + 2u/3 - 1/2)R_1 + (t/3 + u/3 - 1)D_{31}$	$-2t/3 + 4u/3 + 2$
$(4, 0), (3, 2), (3, 3/2)$	$(3t + 2u - 12)F_1 + (2t + 2u - 8)F_2 + (2t + 2u - 8)R_1 + (t + u - 4)R_2 + (t + u - 4)D_{31}$	$-4t - 2u + 16$
$(4, 0), (3, 3/2), (3, 1)$	$(2t + 2u - 8)F_2 + (2u/3)R_1 + (t + u - 4)R_2 + (u/3)D_{31}$	$-2t - 2u/3 + 8$
$(3, 0), (4, 0), (3, 1)$	$(2u/3)R_1 + (u/3)D_{31}$	$4u/3$

Path configuration.

The negative curves are $F_1, F_2, F_3, L_{13}, T_3, R_1, R_2, R_3, D_{12}, D_{23}$.

Flag $C = F_1$. $s_C = 7/8$, $b_C = 5/16$, and

$$k_{C,R_1} = 7/16, k_{C,D_{12}} = 55/144.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t+u)F_2 + (u/2)R_1 + (-t+u)R_2 + (-t+u)D_{12}$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(2, 2), (1, 2), (1, 1)$	$(-t+u)F_2 + (u/2)R_1 + (-t/2 + u - 1/2)R_2 + (u-1)D_{12}$	$t/2 - u/2 + 1/2$
$(3, 3/2), (3, 2), (2, 2)$	$(t+2u-6)T_3 + (u/2)R_1 + (t/2 + u - 5/2)R_2 + (t+2u-5)D_{12}$	$-t/2 - 3u/2 + 9/2$
$(1, 0), (3, 0), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 3/2), (2, 2), (1, 1)$	$(u/2)R_1 + (t/6 + u/3 - 1/2)R_2 + (t/3 + 2u/3 - 1)D_{12}$	$t/6 - u/6 + 1/2$
$(4, 0), (3, 2), (3, 3/2)$	$(3t + 2u - 12)T_3 + (u/2)R_1 + (t + u - 4)R_2 + (2t + 2u - 8)D_{12}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 3/2)$	$(u/2)R_1 + (u/3)R_2 + (2u/3)D_{12}$	$-t - u/6 + 4$

Flag $C = F_2$. $s_C = 17/18$, $b_C = 79/288$, and

$$k_{C,R_2} = 157/288, k_{C,D_{23}} = 121/288.$$

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t+u)F_3 + (u/2)R_2 + (-t+u)R_3 + (-t+u)D_{23}$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_2$	$u/2$
$(3, 3), (1, 2), (3/2, 3/2)$	$(-t+u)F_3 + (t/6 + 2u/3 - 1/2)R_2 + (-t/2 + u - 1/2)R_3 + (t/3 + u/3 - 1)D_{12} + (u-1)D_{23}$	$t/3 - 2u/3 + 1$
$(3/2, 3/2), (1, 2), (1, 1)$	$(-t+u)F_3 + (u/2)R_2 + (-t/2 + u - 1/2)R_3 + (u-1)D_{23}$	$t/2 - u/2 + 1/2$
$(1, 0), (3, 0), (1, 1)$	$(u/2)R_2$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 3), (3/2, 3/2)$	$(t/6 + 2u/3 - 1/2)R_2 + (t/6 + u/3 - 1/2)R_3 + (t/3 + u/3 - 1)D_{12} + (t/3 + 2u/3 - 1)D_{23}$	$1 - u/3$
$(3, 0), (3/2, 3/2), (1, 1)$	$(u/2)R_2 + (t/6 + u/3 - 1/2)R_3 + (t/3 + 2u/3 - 1)D_{23}$	$t/6 - u/6 + 1/2$
$(3, 0), (4, 0), (3, 3)$	$(2u/3)R_2 + (u/3)R_3 + (u/3)D_{12} + (2u/3)D_{23}$	$-t - u/3 + 4$

Flag $C = F_3$. $s_C = 55/72$, $b_C = 85/216$, and
 $k_{C,T_3} = 13/108$, $k_{C,R_3} = 23/54$.

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)T_3 + (u/2)R_3$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_3$	$u/2$
$(3, 1), (3, 3/2), (1, 2)$	$(t+3u-6)T_3 + (t/2+u-5/2)R_2 + (t/6+2u/3-1/2)R_3 + (t+2u-5)D_{12} + (t/3+u/3-1)D_{23}$	$-2t/3 - 8u/3 + 6$
$(2, 1), (1, 2), (1, 1)$	$(u-1)T_3 + (u/2)R_3$	$t/2 - u/2 + 1/2$
$(3, 1), (1, 2), (2, 1)$	$(u-1)T_3 + (t/6+2u/3-1/2)R_3 + (t/3+u/3-1)D_{23}$	$t/3 - 2u/3 + 1$
$(1, 0), (3, 0), (2, 1), (1, 1)$	$(u/2)R_3$	$t/2 + u/2 - 1/2$
$(3, 0), (3, 1), (2, 1)$	$(t/6+2u/3-1/2)R_3 + (t/3+u/3-1)D_{23}$	$t/3 + u/3$
$(4, 0), (3, 3/2), (3, 1)$	$(3t+3u-12)T_3 + (t+u-4)R_2 + (2u/3)R_3 + (2t+2u-8)D_{12} + (u/3)D_{23}$	$-4t - 8u/3 + 16$
$(3, 0), (4, 0), (3, 1)$	$(2u/3)R_3 + (u/3)D_{23}$	$-t + u/3 + 4$

Flag $C = L_{13}$. $s_C = 5/8$, $b_C = 59/144$, and
 $k_{C,R_1} = 49/144$, $k_{C,R_3} = 59/144$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1 + (u)F_3 + (u)R_1 + (u)R_3$	$t - u$
$(1, 0), (2, 1), (1, 1)$	$(-t + u + 1)F_1 + (-t + u + 1)F_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_3$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + u + 1)F_1 + (-t + 2u)F_3 + (-t/2 + u + 1/2)R_1 + (-t/2 + 2u - 1/2)R_3 + (u-1)D_{23}$	$t - 2u + 1$
$(3, 3/2), (3, 2), (2, 1)$	$(-t + 2u)F_3 + (u/2)R_1 + (-t/2 + 2u - 1/2)R_3 + (u-1)D_{23}$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 0), (2, 1)$	$(u/2)R_1 + (u/2)R_3$	1
$(3, 0), (3, 3/2), (2, 1)$	$(u/2)R_1 + (t/6+2u/3-1/2)R_3 + (t/3+u/3-1)D_{23}$	$-t/6 - u/6 + 3/2$
$(4, 0), (3, 2), (3, 3/2)$	$(3t+2u-12)F_3 + (u/2)R_1 + (2t+2u-8)R_3 + (t+u-4)D_{23}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 3/2)$	$(u/2)R_1 + (2u/3)R_3 + (u/3)D_{23}$	$-t - u/6 + 4$

Flag $C = T_3$. $s_C = 37/72$, $b_C = 55/108$, and
 $k_{C,F_3} = 55/216$, $k_{C,D_{12}} = 5/27$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_3 + (u)R_3$	$t - u$
$(1, 0), (3, 0), (2, 1/2)$	0	$u + 1$
$(2, 1/2), (3, 1), (1, 1)$	$(-t+2u+1)F_3 + (t/6+u/3-1/2)R_2 + (-t/2+u+1/2)R_3 + (t/3+2u/3-1)D_{12}$	$2t/3 - 5u/3 + 1$
$(3, 1), (3, 3/2), (1, 1)$	$(-t+3u)F_3 + (t/6+u/3-1/2)R_2 + (-t/2+2u-1/2)R_3 + (t/3+2u/3-1)D_{12} + (u-1)D_{23}$	$2t/3 - 8u/3 + 2$
$(1, 0), (2, 1/2), (1, 1)$	$(-t+2u+1)F_3 + (-t/2+u+1/2)R_3$	$t - u$
$(3, 0), (3, 1), (2, 1/2)$	$(t/6+u/3-1/2)R_2 + (t/3+2u/3-1)D_{12}$	$-t/3 + u/3 + 2$
$(4, 0), (3, 3/2), (3, 1)$	$(3t+3u-12)F_3 + (u/3)R_2 + (2t+2u-8)R_3 + (2u/3)D_{12} + (t+u-4)D_{23}$	$-4t - 8u/3 + 16$
$(3, 0), (4, 0), (3, 1)$	$(u/3)R_2 + (2u/3)D_{12}$	$-t + u/3 + 4$

Flag C = R_1 . $s_C = 3/4$, $b_C = 19/36$, and
 $k_{C,F_1} = 25/72$, $k_{C,L_{13}} = 7/72$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (3, 1)$	0	$2u$
$(1, 0), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(3, 1), (2, 3/2), (1, 1)$	$(-t/2 + 3u - 3/2)F_1 + (2u - 2)L_{13} + (u - 1)R_2 + (u - 1)R_3 + (2u - 2)D_{12}$	$t/2 - 3u + 7/2$
$(3, 1), (3, 3/2), (2, 3/2)$	$(-t/2 + 3u - 3/2)F_1 + (t/2 + 3u - 9/2)L_{13} + (u - 1)R_2 + (t/2 + 2u - 7/2)R_3 + (2u - 2)D_{12} + (t/2 + u - 5/2)D_{23}$	$6 - 4u$
$(3, 0), (4, 0), (3, 1)$	0	$2u$
$(4, 0), (3, 3/2), (3, 1)$	$(3t + 3u - 12)F_1 + (3t + 3u - 12)L_{13} + (t + u - 4)R_2 + (2t + 2u - 8)R_3 + (2t + 2u - 8)D_{12} + (t + u - 4)D_{23}$	$-6t - 4u + 24$

Flag C = R_2 . $s_C = 59/72$, $b_C = 4/9$, and
 $k_{C,F_2} = 1/2$, $k_{C,D_{12}} = 1/4$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_2$	u
$(1, 0), (3, 0), (2, 1/2)$	0	$2u$
$(1, 0), (2, 1/2), (1, 1)$	$(-t/2 + u + 1/2)F_2$	$t/2 + u - 1/2$
$(3, 1), (3, 2), (1, 1)$	$(-t/2 + 3u - 3/2)F_2 + (u - 1)R_3 + (t/4 + u/2 - 3/4)D_{12} + (2u - 2)D_{23}$	$t/4 - 3u/2 + 9/4$
$(2, 1/2), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_2 + (t/4 + u/2 - 3/4)D_{12}$	$t/4 + u/2 + 1/4$
$(3, 0), (3, 1), (2, 1/2)$	$(t/4 + u/2 - 3/4)D_{12}$	$-t/4 + 3u/2 + 3/4$
$(4, 0), (3, 2), (3, 1)$	$(3t + 3u - 12)F_2 + (t + u - 4)R_3 + (u/2)D_{12} + (2t + 2u - 8)D_{23}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)D_{12}$	$3u/2$

Flag C = R_3 . $s_C = 59/72$, $b_C = 4/9$, and
 $k_{C,F_3} = 23/72$, $k_{C,L_{13}} = 13/72$, $k_{C,D_{23}} = 1/4$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_3$	u
$(1, 0), (3, 0), (2, 1/2)$	0	$2u$
$(1, 0), (2, 1/2), (1, 1)$	$(-t/2 + u + 1/2)F_3$	$t/2 + u - 1/2$
$(3, 1), (3, 2), (1, 1)$	$(-t/2 + u + 1/2)F_3 + (2u - 2)L_{13} + (u - 1)R_1 + (t/4 + u/2 - 3/4)D_{23}$	$t/4 - 3u/2 + 9/4$
$(2, 1/2), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_3 + (t/4 + u/2 - 3/4)D_{23}$	$t/4 + u/2 + 1/4$
$(3, 0), (3, 1), (2, 1/2)$	$(t/4 + u/2 - 3/4)D_{23}$	$-t/4 + 3u/2 + 3/4$
$(4, 0), (3, 2), (3, 1)$	$(t + u - 4)F_3 + (2t + 2u - 8)L_{13} + (t + u - 4)R_1 + (u/2)D_{23}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)D_{23}$	$3u/2$

Flag $C = D_{12}$. $s_C = 25/36$, $b_C = 4/9$, and
 $k_{C,F_1} = 31/72$, $k_{C,T_3} = 5/72$, $k_{C,R_2} = 3/8$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(2, 1), (3, 2), (1, 1)$	$(-t + 2u + 1)F_1 + (-t + u + 1)F_2 + (u - 1)T_3 +$ $(-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - 2u + 1$
$(1, 0), (2, 1), (1, 1)$	$(-t + 2u + 1)F_1 + (-t + u + 1)F_2 + (-t/2 + u +$ $1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 1), (3, 2), (2, 1)$	$(-t + 2u + 1)F_1 + (u - 1)T_3 + (-t/2 + u + 1/2)R_1 +$ $(u/2)R_2$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 1), (2, 1)$	$(-t + 2u + 1)F_1 + (-t/2 + u + 1/2)R_1 + (u/2)R_2$	$t/2 - u/2 + 1/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_2$	$-t/2 + 3u/2 + 3/2$
$(4, 0), (3, 2), (3, 1)$	$(2t + 2u - 8)F_1 + (t + u - 4)T_3 + (t + u - 4)R_1 +$ $(u/2)R_2$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_2$	$3u/2$

Flag $C = D_{23}$. $s_C = 25/36$, $b_C = 4/9$, and
 $k_{C,F_2} = 1/2$, $k_{C,R_3} = 3/8$.

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_2 + (u)F_3 + (u)R_2 + (u)R_3$	$t - u$
$(1, 0), (2, 1), (1, 1)$	$(-t + 2u + 1)F_2 + (-t + u + 1)F_3 + (-t/2 + u +$ $1/2)R_2 + (-t/2 + u + 1/2)R_3$	$t - u$
$(3, 2), (1, 1), (2, 1)$	$(-t + 3u)F_2 + (-t + u + 1)F_3 + (-t/2 + 2u -$ $1/2)R_2 + (-t/2 + u + 1/2)R_3 + (u - 1)D_{12}$	$t - 2u + 1$
$(1, 0), (3, 1), (2, 1)$	$(-t + 2u + 1)F_2 + (-t/2 + u + 1/2)R_2 + (u/2)R_3$	$t/2 - u/2 + 1/2$
$(3, 1), (3, 2), (2, 1)$	$(-t + 3u)F_2 + (-t/2 + 2u - 1/2)R_2 + (u/2)R_3 +$ $(u - 1)D_{12}$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_3$	$-t/2 + 3u/2 + 3/2$
$(4, 0), (3, 2), (3, 1)$	$(3t + 3u - 12)F_2 + (2t + 2u - 8)R_2 + (u/2)R_3 +$ $(t + u - 4)D_{12}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_3$	$3u/2$

Cycle configuration.

The negative curves are $F_1, F_2, F_3, R_1, R_2, R_3, D_{12}, D_{23}, D_{31}$.

Flag $C = F_1$. $s_C = 17/18$, $b_C = 79/288$, and

$k_{C,R_1} = 157/288$, $k_{C,D_{12}} = 121/288$.

Polygon vertices	N	d_C
$(0, 0), (1, 1), (1, 2)$	$(-t + u)F_2 + (u/2)R_1 + (-t + u)R_2 + (-t + u)D_{12}$	$t - u/2$
$(0, 0), (1, 0), (1, 1)$	$(u/2)R_1$	$u/2$
$(3/2, 3/2), (1, 2), (1, 1)$	$(-t + u)F_2 + (u/2)R_1 + (-t/2 + u - 1/2)R_2 +$ $(u - 1)D_{12}$	$t/2 - u/2 + 1/2$
$(3, 3), (1, 2), (3/2, 3/2)$	$(-t + u)F_2 + (t/6 + 2u/3 - 1/2)R_1 + (-t/2 +$ $u - 1/2)R_2 + (u - 1)D_{12} + (t/3 + u/3 - 1)D_{31}$	$t/3 - 2u/3 + 1$
$(1, 0), (3, 0), (1, 1)$	$(u/2)R_1$	$t/2 + u/2 - 1/2$
$(3, 0), (3/2, 3/2), (1, 1)$	$(u/2)R_1 + (t/6 + u/3 - 1/2)R_2 + (t/3 + 2u/3 -$ $1)D_{12}$	$t/6 - u/6 + 1/2$
$(3, 0), (3, 3), (3/2, 3/2)$	$(t/6 + 2u/3 - 1/2)R_1 + (t/6 + u/3 - 1/2)R_2 +$ $(t/3 + 2u/3 - 1)D_{12} + (t/3 + u/3 - 1)D_{31}$	$1 - u/3$

Cycle configuration, $C = F_1$ (continued)

Polygon vertices	N	d_C
$(3, 0), (4, 0), (3, 3)$	$(2u/3)R_1 + (u/3)R_2 + (2u/3)D_{12} + (u/3)D_{31}$	$-t - u/3 + 4$

Flag $C = R_1$. $s_C = 59/72$, $b_C = 4/9$, and

$$k_{C,F_1} = 1/2, k_{C,D_{31}} = 1/4.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(u)F_1$	u
$(1, 0), (3, 0), (2, 1/2)$	0	$2u$
$(1, 0), (2, 1/2), (1, 1)$	$(-t/2 + u + 1/2)F_1$	$t/2 + u - 1/2$
$(3, 1), (3, 2), (1, 1)$	$(-t/2 + 3u - 3/2)F_1 + (u-1)R_2 + (2u-2)D_{12} + (t/4 + u/2 - 3/4)D_{31}$	$t/4 - 3u/2 + 9/4$
$(2, 1/2), (3, 1), (1, 1)$	$(-t/2 + u + 1/2)F_1 + (t/4 + u/2 - 3/4)D_{31}$	$t/4 + u/2 + 1/4$
$(3, 0), (3, 1), (2, 1/2)$	$(t/4 + u/2 - 3/4)D_{31}$	$-t/4 + 3u/2 + 3/4$
$(4, 0), (3, 2), (3, 1)$	$(3t + 3u - 12)F_1 + (t + u - 4)R_2 + (2t + 2u - 8)D_{12} + (u/2)D_{31}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)D_{31}$	$3u/2$

Flag $C = D_{12}$. $s_C = 25/36$, $b_C = 4/9$, and

$$k_{C,F_1} = 1/2, k_{C,R_2} = 3/8.$$

Polygon vertices	N	d_C
$(0, 0), (1, 0), (1, 1)$	$(2u)F_1 + (u)F_2 + (u)R_1 + (u)R_2$	$t - u$
$(1, 0), (2, 1), (1, 1)$	$(-t + 2u + 1)F_1 + (-t + u + 1)F_2 + (-t/2 + u + 1/2)R_1 + (-t/2 + u + 1/2)R_2$	$t - u$
$(3, 2), (1, 1), (2, 1)$	$(-t + 3u)F_1 + (-t + u + 1)F_2 + (-t/2 + 2u - 1/2)R_1 + (-t/2 + u + 1/2)R_2 + (u-1)D_{31}$	$t - 2u + 1$
$(1, 0), (3, 1), (2, 1)$	$(-t + 2u + 1)F_1 + (-t/2 + u + 1/2)R_1 + (u/2)R_2$	$t/2 - u/2 + 1/2$
$(3, 1), (3, 2), (2, 1)$	$(-t + 3u)F_1 + (-t/2 + 2u - 1/2)R_1 + (u/2)R_2 + (u-1)D_{31}$	$t/2 - 3u/2 + 3/2$
$(1, 0), (3, 0), (3, 1)$	$(u/2)R_2$	$-t/2 + 3u/2 + 3/2$
$(4, 0), (3, 2), (3, 1)$	$(3t + 3u - 12)F_1 + (2t + 2u - 8)R_1 + (u/2)R_2 + (t + u - 4)D_{31}$	$-3t - 3u/2 + 12$
$(3, 0), (4, 0), (3, 1)$	$(u/2)R_2$	$3u/2$ $\square$

Proposition 8.9. *Let X be a smooth prime Fano threefold of degree 18, and suppose that $F_1(X, p)$ consists of three reduced points. For the exceptional plane E of $\text{Bl}_p X$ and every prime divisor G with $p \in c_X(G)$,*

$$(8.27) \quad A_X(G) \geq S_X(G) + \frac{3}{4} \text{ord}_G(E).$$

In particular, $\delta_p(X) \geq 1$, and $\beta_X(G) > 0$ if $c_X(G) = \{p\}$.

Proof. By Proposition 8.3, every such smooth threefold satisfies the coefficient condition, so all preceding complete-series calculations apply. We prove (8.17) on the negative curves; their complement is covered by Lemma 8.6.

Step 1. We state the effective-boundary criterion. For a negative curve C set $a_C = 31/72$ if $C = R_i$, $a_C = 31/36$ if $C = F_i$, and zero otherwise; let $m_C = \text{coeff}_C(Q_4)$. At $z \in C$ it suffices

that

$$(8.28) \quad s_C + m_C/24 - a_C < 1, \\ b_C + \sum_{D \neq C} k_{C,D} I_z(C, D) + \frac{1}{24} I_z(C, Q_4 - m_C C) < 1.$$

To see this, choose in every residual complete space a basis compatible with order along C and a given valuation w . Let D_M be their normalized effective divisor. Its C -coefficient tends to s_C , its value tends to $S_{\text{mov}}(w)$, and its residual restriction has the uniform point bound (8.24). Write $D_M + Q_4/24 = \alpha_M C + \Theta_M$, where Θ_M is effective and does not contain C . The strict bounds give $\alpha_M - a_C < 1$ and $\text{mult}_z(\Theta_M|_C) < 1$ for large M . Inversion of adjunction makes $(S, C + \Theta_M)$ log canonical near z . Subtracting the effective multiple $(1 - \alpha_M + a_C)C$ gives

$$w(D_M) + w(Q_4)/24 \leq A_S(w) + a_C w(C).$$

Taking limits and adding the other nonnegative credits proves (8.17). In particular, the residual boundary used for adjunction is effective; only the coefficient of the chosen flag is removed.

The first inequality of (8.28) follows for every flag from

$$\frac{17}{18} + \frac{3}{24} - \frac{31}{36} = \frac{5}{24}, \quad \frac{65}{72} + \frac{3}{24} - \frac{31}{72} = \frac{43}{72}, \quad \frac{17}{24} + \frac{2}{24} = \frac{19}{24}.$$

Step 2. We cover the exceptional curves. At $z \in F_i \setminus R_i$ choose $C = F_i$. There exists at most one other negative neighbor there. By (8.21) the strict quartic intersects F_i in at most one point counted with multiplicity. The second quantity of (8.28) is at most $5/12 + 121/288 + 1/24 = 253/288 < 1$. At $z \in R_i \setminus F_i$ choose R_i ; the corresponding bound is $19/36 + 7/24 + 3/24 = 17/18$.

At $R_i \cap F_i$, use F_i except at the star hub. No other negative neighbor meets this point, and $(Q_4 - 3F_i) \cdot F_i = 3$. The bound is $59/72 + 3/24 = 17/18$. At the star hub use R_1 . If $b_1 = 2$, the strict quartic has total intersection one with R_1 . If $b_1 = 3$, it is disjoint from F_1 and misses the crossing. Thus the residual quartic intersection there is at most four, and the bound is $5/12 + 7/18 + 4/24 = 35/36$. This includes every possible strict-quartic contact at these points.

Step 3. We cover the other negative curves. Let z lie on a nonexceptional negative curve C , outside all R_i, F_i , and let $m \leq 2$ be its coefficient in B^{str} . Put $B_0 = 4h - 2 \sum R_i - 3 \sum F_i$. Because $b_i \geq 2$, we obtain $B^{\text{str}} \cdot C \leq B_0 \cdot C$. The effective residual quartic therefore satisfies

$$I_z(C, B^{\text{str}} - mC) \leq B_0 \cdot C - mC^2.$$

The intersections $B_0 \cdot L = B_0 \cdot K = 0$, $B_0 \cdot T = 1$, and $B_0 \cdot D = B_0 \cdot Z_0 = -1$ bound this by two for L, K and by three for T, D, Z_0 . Only nonexceptional neighbors contribute at z . Lemma 8.8 gives the point bounds

$$\begin{array}{l|l} L & 5/12 + 1/18 + 2/24 = 5/9 \\ T & 11/18 + 1/72 + 5/27 + 3/24 = 101/108 \\ K & 47/72 + 1/216 + 2/24 = 20/27 \\ D & 4/9 + 5/72 + 3/24 = 23/36 \\ Z_0 & 2/3 + 3/24 = 19/24. \end{array}$$

All are strictly below one, with no transversality assumption on the quartic. The criterion proves (8.17) over the whole negative locus. Together with Lemma 8.6, it proves $S(W; w) \leq A_E(w)$ for every divisor over E .

Finally the ordinary point extraction has discrepancy three and zero different, while (8.9) gives $S_X(E) = 9/4$. The retained-coefficient adjunction of [AZ22, Theorem 3.2 and equation (3.6)], applied to this complete refinement, gives (8.27). For a center equal to p , the order of E is positive. $\square$

9. A SINGLE DOUBLED INCIDENCE POINT IN DEGREE EIGHTEEN

Lemma 9.1. *Let X be a smooth prime Fano threefold of degree 18 or 22 and L a special line. On $Y = \text{Bl}_L X$, let s be the negative section of the exceptional $\mathbb{F}_3$. The normal sequence is*

$$(9.1) \quad 0 \rightarrow \mathcal{O}(-3) \rightarrow N_{s/Y} \rightarrow \mathcal{O}(1) \rightarrow 0.$$

Let $e \in H^1(\mathcal{O}(-4))$ be its class. A nonzero first-order deformation of L fixing $x \in L$ extends over $\mathbb{C}[t]/(t^3)$ if and only if

$$(9.2) \quad \delta_e(\ell_x) = 0 \in H^1(\mathcal{O}(-3)), \quad \ell_x \in H^0(\mathcal{O}(1)(-x)) \setminus \{0\}.$$

If $N_{s/Y} = \mathcal{O} \oplus \mathcal{O}(-2)$, there exists exactly one such point x_0 , and $F_1(X, x_0)$ is the entire curvilinear triple supported at L . On the second exceptional $\mathbb{F}_2$, its negative section σ meets the strict first exceptional divisor in one transverse point over x_0 . If $N_{s/Y} = \mathcal{O}(-1)^{\oplus 2}$, there exists no such point.

Proof. The normal types of s follow from the line links used above. The tangent-normal sequence of L splits, since $H^1(\mathcal{O}(1) \oplus \mathcal{O}(4)) = 0$. Choose formal normal coordinates u, v in the positive and negative normal directions. There exist no linear normal terms in the tangent coordinate changes. The relevant negative quadratic transition is

$$v_j = h_{ij}v_i + a_{ij}u_i^2 + \text{terms divisible by } u_i v_i, v_i^2 + \text{terms of normal degree at least three.}$$

Here $u_j = g_{ij}u_i + \text{terms of degree at least two}$, and the class of a_{ij} lies in $H^1(\mathcal{O}(-4))$. On the positive blowup chart $v_i = u_i w_i$ the linear transition is

$$u_j = g_{ij}u_i, \quad w_j = (h_{ij}/g_{ij})w_i + (a_{ij}/g_{ij})u_i.$$

It identifies the quadratic class with the extension class in (9.1).

A second-order graph through the fixed point has $u_i = t\ell_i + t^2 r_i$, $v_i = t^2 b_i$, with $r_i(x) = b_i(x) = 0$. The negative gluing obstruction is $a\ell^2$ in $H^1(\mathcal{O}(-2)(-x))$. Multiplication by ℓ_x identifies $\mathcal{O}(-x)$ with $\mathcal{O}(-1)$, so this is precisely $e\ell_x \in H^1(\mathcal{O}(-3))$. If it vanishes, the positive corrections and parameter changes fixing x can be solved because $H^1(\mathcal{O}(1)(-x)) = H^1(T_L(-x)) = 0$. The glued graphs are flat and have the Hilbert polynomial of a line. This proves necessity and sufficiency in (9.2).

The kernel of δ_e is $H^0(N_{s/Y})$. It is zero in the balanced normal type and one-dimensional in the other. In the unbalanced normal type the nowhere-vanishing section of the trivial summand maps to a nonzero section of $\mathcal{O}(1)$, with a unique zero x_0 . A nonzero tangent lifting to $\mathbb{C}[t]/(t^3)$ forces local incidence length at least three. Lemma 3.2 bounds the entire length by three, proving the assertion. On $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$, the negative section is the trivial subline, whereas the strict first exceptional divisor is the subline $\mathcal{O}(-3)$. They meet exactly where its projection to $\mathcal{O}(1)$ vanishes. Their intersection number is $\sigma \cdot (\sigma + 3f) = 1$, so the meeting is transverse. $\square$

Lemma 9.2. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, with ruling classes σ, f , and let $B \in |\sigma + 2f|$ be smooth. Let ι be the involution of the degree-two map from B to the ruling base with fiber σ . Let Δ be a degree-three divisor on B whose maps to both ruling bases are closed immersions. Set*

$$(9.3) \quad R_{m,k} = H^0(S, k\sigma + mf \otimes \mathcal{I}_\Delta^{\max\{k-m, 0\}}), \quad 0 \leq k \leq 2m.$$

Then $\sum_k \dim R_{m,k} = \frac{3}{2}m^3 + O(m^2)$ and

$$(9.4) \quad S_R(v) \leq A_{S,B/2}(v)$$

for every divisor whose center is not contained in $\text{Supp } \Delta$. The zeros imposed by the ideal in (9.3) are included in S_R .

Proof. We resolve the base conditions, keep the conjugate fixed part on B , and change the flag at a repeated conjugate contact.

Step 1. We identify nef classes and ranks. Resolve the three successive points along B on $b: T \rightarrow S$ and write $G = \sum e_i$ in total-transform notation. A length- r local ideal is (u, t^r) , with

$B = (u = 0)$. Its c th power consists of monomials with $r \operatorname{ord}_u + \operatorname{ord}_t \geq rc$, and resolves with multiplicity c at every successive center. Thus the ideal powers in the statement give exactly the complete spaces on T .

The classes $U = 2\sigma + f - G$ and $V = \sigma + 2f - G$ are globally generated. For U , push the fiber-linear system down the ruling with fiber σ . The evaluation kernel for Δ has rank two, degree one, and embeds in $\mathcal{O}(2)^{\oplus 2}$. Its splitting is $\mathcal{O}(a) \oplus \mathcal{O}(1-a)$, $1 \leq a \leq 2$. If $a = 2$, its constant direction puts Δ in a fiber of the other ruling, contradicting the second closed-immersion assumption. Hence the kernel is $\mathcal{O}(1) \oplus \mathcal{O}$. Locally its sections generate $(u - P(t), Q(t))$, proving global generation after blowup. Interchanging the rulings proves it for V . Thus a class $p\sigma + qf - cG$ is nef when

$$p \geq c, \quad q \geq c, \quad p + q \geq 3c, \quad c \geq 0 :$$

subtract a suitable convex combination of cU and cV . Also $-K_T = U + f$ is nef and big. Vanishing and Riemann–Roch give uniform section counts for these nef classes. With $a = k/m$ the rank density is a on $[0, 1]$ and $a - \frac{3}{2}(a-1)^2$ on $[1, 2]$. Its integral is $3/2$.

Step 2. We compute the full B refinement. For B order $u = \ell/m$, the support is $0 \leq u \leq \min\{a, 1/2\}$. Put $v = a - 1 - u$. When $v \leq 0$ the residual nef class is $(a-u)\sigma + (1-2u)f$. For $v \geq 0$ subtract vG ; this is nef up to $u = 2 - a$. For $a \geq 3/2$ and $2 - a \leq u \leq 1/2$ its negative and positive parts are

$$N = (a + u - 2)(3\sigma - G), \quad P = (1 - 2u)U + (4 - 2a)\sigma.$$

For a cluster of length r , its part in $3\sigma - G$ is $rC_0 + (r-1)E_1 + \cdots + E_{r-1}$ along the $(-1), (-2), \dots, (-2)$ chain of its fiber and preceding exceptional curves. These disjoint chains have negative-definite matrices, and P is orthogonal to them. The formulas therefore give the Zariski decomposition, also at a simple branch point.

On the original B , the fixed divisor is $mv_+\Delta + mw_+\iota(\Delta)$, where $v_+ = \max\{v, 0\}$ and $w_+ = a + u - 2$ only in the last chamber. The fiber through a cluster intersects B in that cluster and its conjugate; the original-point coefficients combine to $v_+\Delta$, and the other coefficients give $w_+\iota(\Delta)$. At a branch point they add. The remaining restriction degree is $mn + O(1)$:

$$(9.5) \quad \begin{array}{c|c|c|c|c} a & u & n & v_+ & w_+ \\ \hline [0, 1/2] & [0, a] & 2a + 1 - 4u & 0 & 0 \\ [1/2, 1] & [0, 1/2] & 2a + 1 - 4u & 0 & 0 \\ [1, 3/2] & [0, a-1] & 4 - a - u & a - 1 - u & 0 \\ [1, 3/2] & [a-1, 1/2] & 2a + 1 - 4u & 0 & 0 \\ [3/2, 2] & [0, 2-a] & 4 - a - u & a - 1 - u & 0 \\ [3/2, 2] & [2-a, 1/2] & 10 - 4a - 4u & a - 1 - u & a + u - 2. \end{array}$$

Inside each chamber, differences of the surface dimensions give row ranks $mn + O(1)$. Rows in bounded-width strips along chamber walls contribute $O(m^3)$ to the total order numerators. Integrating over the six regions gives respectively

$$\iint n = \frac{3}{2}, \quad \iint un = \frac{5}{16}, \quad \iint \frac{n^2}{2} = \frac{71}{48}, \quad \iint v_+n = \frac{13}{48}, \quad \iint w_+n = \frac{1}{48}.$$

Hence $S_R(B) = 5/24$, and at $q \in B \setminus \operatorname{Supp} \Delta$ the secondary moment is at most $(71 + \operatorname{mult}_q \iota \Delta)/72$. All these estimates retain the fixed divisor and use the original normalization $3m^4/2 + O(m^3)$.

Step 3. We handle repeated conjugate points. If $q \notin \operatorname{Supp} \Delta$ has conjugate multiplicity $r \geq 2$, let $q' = \iota(q)$ and let C be their common σ fiber. The points are distinct and the intersections with B are transverse, since multiple branch points are excluded. Forgetting base conditions bounds $S_R(C) \leq 8/9$. After removing order ℓ along C , the condition $(u, t^r)^{k-m}$ at q' forces a residual zero of order

$$h = \max\{k - m - \lfloor \ell/r \rfloor, 0\} \geq \max\{k - m - \lfloor \ell/2 \rfloor, 0\}.$$

The remaining degree on C is at most $m - h$, so the total order at $q \neq q'$ is at most $(m - h)(m - h + 1)/2$. Writing $z = \ell/m$, the normalized residual degree is $d(a, z) = 1 - \max\{a - 1 - z/2, 0\}$ for $1 \leq a \leq 2$. We obtain

$$\int_0^a d(a, z)^2 dz = a - 2(a - 1)^2 + \frac{2}{3}(a - 1)^3.$$

Its integral on $[1, 2]$ is one; the early integral is $1/2$. Division by two and then $3/2$ gives secondary moment at most $1/2$, exactly the available discrepancy at $C \cap B$.

Step 4. We apply local adjunction. For $q \in B \setminus (\text{Supp } \Delta \cup \text{Supp } \iota\Delta)$ use the B flag: $5/24 < 1/2$ and $71/72 < 1$ give local constant $72/71$. At conjugate multiplicity one the secondary bound is one, giving the limiting constant one. At multiplicity at least two use Step 3, retaining the half different at q . For $q \notin B$, its f fiber has first moment at most $2/3$ and secondary moment at most $8/9$, by inclusion in the complete bidegree- (k, m) spaces; this gives local constant $9/8$.

For each application choose only two compatible filtrations: the tested valuation and the chosen flag. The residual divisor after removing the flag coefficient is effective. The curve bounds apply to every such basis by maximality of filtration weights, and their errors are $O(1/m)$, uniformly in the point. For a constant strictly below the stated endpoint, inversion of adjunction applies at finite degree; then take the degree and constant limits. This proves (9.4), including generic curve centers. $\square$

Lemma 9.3. *Let $S = \mathbb{F}_2$ with negative section σ , ruling f , and smooth $B \in |\sigma + 3f|$. Put $q_0 = B \cap \sigma$. For a length-three cluster Δ , define*

$$(9.6) \quad R_{m,k} = \begin{cases} H^0(S, k\sigma + (m+k)f), & 0 \leq k \leq m, \\ s_\sigma^{k-m} H^0(S, m\sigma + (m+k)f \otimes \mathcal{I}_\Delta^{k-m}), & m \leq k \leq 2m. \end{cases}$$

Consider the following configurations.

- (1) $\Delta = Z_B \sqcup \{q_3\}$, where Z_B is a degree-two divisor on $B \setminus \sigma$ mapping as a closed immersion to the ruling base, and $q_3 \in \sigma \setminus B$ is on a different fiber.
- (2) Δ consists of a length-two cluster at q_0 , transverse to both σ and its ruling fiber, and a point $q_3 \in B$ on a different fiber.
- (3) Δ is a length-three free curvilinear cluster at q_0 , transverse to σ and to the ruling fiber.

In each case $\sum_k \dim R_{m,k} = \frac{3}{2}m^3 + O(m^2)$. For every divisor whose center meets a fiber C with the restrictions below, the indicated inequality holds:

configuration	excluded fibers	$A_{S,B/2}(v) \geq c S_R(v)$
(i)	fibers through Z_B and q_0	$c = 36/31$
(ii)	fibers through q_0, q_3	$c = 8/7$
(iii)	fiber through q_0	$c = 108/95$.

The fiber through q_3 in case (i) is included.

Proof. We describe the effective cones, list the full fiber chambers and then apply the surface flag lemma. Throughout, put $a = k/m$, $v = \max\{a - 1, 0\}$, and let u be the scaled fiber order. All exceptional classes below are total transforms.

Step 1. In this step, we determine the effective cones.

In case (i), blow up Z_B and q_3 , with classes e_1, e_2, e_3 . For distinct points the negative generators are $s = \sigma - e_3$, $c_i = f - e_i$, and e_i . Here $s^2 = -3$, $c_i^2 = e_i^2 = -1$, $s \cdot c_1 = s \cdot c_2 = s \cdot e_3 = 1$, and $c_i \cdot e_i = 1$; the other pairings vanish. A nonvertical curve different from σ has class $\alpha\sigma + \beta f - \sum m_i e_i$, with $\beta \geq 2\alpha$ and $0 \leq m_i \leq \alpha$, and is the nonnegative combination

$$\alpha s + m_1 c_1 + m_2 c_2 + (\beta - m_1 - m_2)f + (\alpha - m_3)e_3.$$

Thus these curves generate the effective cone. For a double Z_B , replace c_1, c_2 by $C_1 = f - e_1$ and $D = e_1 - e_2$, with squares $-1, -2$. Their equal-coefficient sum becomes $2C_1 + D$. All subsequent positive parts have equal e_1, e_2 coefficients and pair to zero with D . The cone expression and

negative support tests apply after this replacement. The boundary $s/2$ makes the negative log canonical divisor nef and big.

In case (ii), the generators are

$$s = \sigma - e_1, \quad C_1 = f - e_1, \quad C_3 = f - e_3, \quad D = e_1 - e_2, \quad e_2, e_3.$$

Their squares are $-3, -1, -1, -2, -1, -1$, and the edges with intersection one are sC_3, sD, C_1D, C_3e_3 , and De_2 . The two transversality assumptions ensure that s, C_1 avoid the second center. The cone expression is

$$\alpha s + m_2(f - e_2) + m_3C_3 + (\beta - m_2 - m_3)f + (\alpha - m_1)e_1,$$

where $f - e_2 = C_1 + D$, $f = C_3 + e_3$, and $e_1 = D + e_2$. The boundary $s/2 + D/4$ is klt and its negative log canonical divisor is nef and big; its intersections with s, D are $1/4, 0$.

In case (iii), the generators are

$$s = \sigma - e_1, \quad C_1 = f - e_1, \quad D_1 = e_1 - e_2, \quad D_2 = e_2 - e_3, \quad e_3$$

with squares $-3, -1, -2, -2, -1$ and edges sD_1, C_1D_1, D_1D_2 , and D_2e_3 . The cone expression is

$$\alpha s + m_2(f - e_2) + m_3(f - e_3) + (\beta - m_2 - m_3)f + (\alpha - m_1)e_1,$$

using $m_1 \geq m_2 \geq m_3$, $f - e_2 = C_1 + D_1$, $f - e_3 = C_1 + D_1 + D_2$, $f = C_1 + D_1 + D_2 + e_3$, and $e_1 = D_1 + D_2 + e_3$. The klt boundary $s/2 + D_1/3 + D_2/6$ has nef and big negative log canonical divisor.

These cone descriptions give complete nefness tests. The displayed boundaries have simple normal crossings on the smooth surfaces and coefficients below one, so their multiplier ideals are trivial. The smooth-pair Kawamata–Viehweg consequence of [DEL00, Definition 1.1 and p. 142, Nadel Vanishing Theorem], followed by Riemann–Roch, gives $h^0 = m^2P^2/2 + O(m)$ uniformly in every rational chamber below. The local ideals $(x, y^r)^c$ resolve with equal multiplicity c at their r simple free centers, so these counts apply to the specified ideal powers. The early class $a\sigma + (a+1)f$ has half-square a . The middle free class

$$L_a = \sigma + (a+1)f - (a-1)(e_1 + e_2 + e_3)$$

is nef and has half-square $a - \frac{3}{2}(a-1)^2$. Their integrated rank is $3/2$.

Step 2. In this step, we compute the full fiber chambers. For a fiber disjoint from the base cluster, subtract uf . Let w be the coefficient of s in the negative part and n the fiber degree of the positive part. Restriction has fixed coefficient $m(v+w)$ at the σ point and free degree $mn + O(1)$. The following tables give all positive-volume chambers, with polygon vertices in (a, u) . The two early rows are common to the three cases:

vertices	support	n	w
$(0, 0), (1, 0), (0, 1)$	$\emptyset$	a	0
$(1, 0), (1, 2), (0, 1)$	total transform of σ	$(a - u + 1)/2$	$(a + u - 1)/2$

In the second row the support is s, e_3 in (i), s, D, e_2 in (ii), and s, D_1, D_2, e_3 in (iii). The middle rows are as follows:

Configuration (i), fiber away from the cluster			
vertices	support	n	w
$(1, 0), (2, 0), (3/2, 3/2)$	s	$1 - u/3$	$u/3$
$(1, 0), (3/2, 3/2), (1, 2)$	s, e_3	$(a - u + 1)/2$	$(-a + u + 1)/2$
$(2, 0), (2, 1), (3/2, 3/2)$	s, c_1, c_2	$5 - 2a - u$	$2a + u - 4$

(9.8)

Configuration (ii)				
	vertices	support	n	w
(9.9)	$(1, 0), (2, 0), (4/3, 5/3)$	s, D	$1 - 2u/5$	$2u/5$
	$(2, 0), (3/2, 2), (4/3, 5/3)$	s, C_3, D	$(7 - 2a - 2u)/3$	$(2a + 2u - 4)/3$
	$(1, 0), (4/3, 5/3), (1, 2)$	s, D, e_2	$(a - u + 1)/2$	$(-a + u + 1)/2$
	$(2, 0), (2, 1), (3/2, 2)$	s, C_1, C_3, D	$5 - 2a - u$	$2a + u - 4$
	$(1, 2), (4/3, 5/3), (3/2, 2)$	s, C_3, D, e_2	$2 - u$	$u - 1$

Configuration (iii)				
	vertices	support	n	w
(9.10)	$(1, 0), (2, 0), (4/3, 7/3)$	s, D_1, D_2	$1 - 3u/7$	$3u/7$
	$(2, 0), (2, 1), (4/3, 7/3)$	s, C_1, D_1, D_2	$5 - 2a - u$	$2a + u - 4$
	$(1, 0), (4/3, 7/3), (1, 2)$	s, D_1, D_2, e_3	$(a - u + 1)/2$	$(-a + u + 1)/2$

Every remaining negative coefficient is specified by $M_I^{-1}(L \cdot C_i)_{i \in I}$, using the displayed support intersection matrix M_I and the relevant residual divisor L . These matrices are negative definite. Substitution gives nonnegative coefficients exactly on the listed polygons and nonnegative intersection of the positive part with every cone generator. For example the first middle negative parts are $(u/3)s$, $(2u/5)s + (u/5)D$, and $(3u/7)s + (2u/7)D_1 + (u/7)D_2$, respectively. In the last separated row the c_1, c_2 coefficients are $w - (2 - a)$. On the upper edges the positive square is zero, and the cone inequalities leave no further positive-volume region. This verifies the entire chamber range.

The integrals, with measure $da du$, are

	$\iint n$	$\iint un$	$\iint n^2/2$	$\iint n(v + w)$
(i)	3/2	11/12	1/2	19/24
(ii)	3/2	17/18	23/48	5/6
(iii)	3/2	23/24	17/36	61/72

They follow by integrating the displayed affine functions on the triangles, using (8.26). The common normalization is $2/3$. Thus the first moments are $11/18, 17/27, 23/36$; secondary moments at σ are $31/36, 7/8, 95/108$, and elsewhere are $1/3, 23/72, 17/54$.

Step 3. In this step, we treat the remaining base fiber and apply adjunction. It remains to include the q_3 fiber in (i). After removal of its order, the middle base condition at q_3 has exponent $h_3 = \max\{a - 1 - u, 0\}$, by the colon ideal with its fiber equation. Its free divisor is $\sigma + (a + 1 - u)f - (a - 1)(e_1 + e_2) - h_3e_3$. The early rows above remain valid; the two middle rows are

vertices	support	n	w	h_3
$(1, 0), (2, 0), (2, 1)$	$\emptyset$	$2 - a + u$	0	$a - 1 - u$
$(1, 0), (2, 1), (1, 2)$	s, e_3	$(a - u + 1)/2$	$(-a + u + 1)/2$	0

When present, the e_3 coefficient is w . The only fixed point on the fiber is q_3 , with coefficient $m(v + w + h_3)$. The four integrals are $3/2, 1, 1/2, 19/24$ in the order of (9.11), replacing its last integrand by $n(v + w + h_3)$. Hence its first moment is $2/3$, its secondary moment at q_3 is at most $31/36$, and elsewhere at most $1/3$.

Every permitted fiber avoids q_0 , so its intersections with B and σ are distinct. The B point has discrepancy $1/2$ on the fiber and every other point has discrepancy one. The listed first and secondary bounds therefore allow constants $36/31, 8/7, 108/95$ in the three cases. Actual row inclusion in the complete rational-curve systems gives uniform $O(1/m)$ errors; the bounded rational rounding also contributes $O(1/m)$. Apply Lemma 2.7 at any smaller constant and take the limit. This proves (9.7). $\square$

Proposition 9.4. *Let X be a smooth prime Fano threefold of degree 18, and suppose that $F_1(X, p)$ has length two and is supported at one line L . Let F be the second exceptional divisor*

obtained by blowing up L and then the negative section. For every prime divisor G centered at p ,

$$(9.12) \quad \beta_X(G) \geq \frac{27}{95} \operatorname{ord}_G(F) > 0.$$

If the quartic surface in the line link has normalization $\mathbb{F}_0$, the coefficient may be replaced by $11/18$. In the $\mathbb{F}_2$ normalization case we obtain the stronger inequality

$$(9.13) \quad A_X(G) \geq \frac{108}{95} S_X(G) + \frac{27}{95} \operatorname{ord}_G(F).$$

Proof. We identify the actual half-boundary refinement with the surface series just treated, and verify which fibers can contain p .

Step 1. We keep the plt extraction and its different. The nonzero pointed tangent forces L to be special by Lemma 3.2. Its doubled conic has ideal (y, x^2) , and the blowup $U \rightarrow X$ has charts $y = x^2 t$ and $x^2 = yt$. The latter is the transverse quotient $(\alpha, \beta) \mapsto (-\alpha, -\beta)$ with $x = \alpha\beta$, $y = \alpha^2$, $t = \beta^2$. Thus (U, F) is plt, $2F$ is Cartier and $-F$ is relatively ample. Its different is $B/2$, where B is the smooth intersection of F with the strict first exceptional divisor E_1 on the resolution. The curve B maps isomorphically to L . Proposition 4.4 gives

$$(9.14) \quad A_X(F) = 3, \quad S_X(F) = 43/18.$$

Step 2. We compare the normal bundles on the two sides. Use the reversed model of Proposition 4.4: $V = \operatorname{Bl}_\Lambda Q$, with exceptional $G_0 = \mathbb{F}_1$, negative section g and fiber η , and put $l = h - G_0$. Projection from Λ makes $V \rightarrow \mathbb{P}^2$ a $\mathbb{P}^1$ -bundle. On G_0 ,

$$h|_{G_0} = \eta, \quad G_0|_{G_0} = -g, \quad l|_{G_0} = g + \eta.$$

The normalized quartic T is the inverse image of a smooth plane conic K . Its lifted smooth genus-two curve Γ_T meets G_0 in a length-three scheme Z_0 whose map to Λ is a closed immersion, as established in that proposition.

If K avoids the image of g , then $T = \mathbb{F}_0$ and its conductor C_0 is smooth of class $2g + 2\eta$, disjoint from g . If K passes through that point, then $T = \mathbb{F}_2$ and

$$h|_T = \sigma_T + 3f_T, \quad G_0|_T = \sigma_T + f_0, \quad \Gamma_T = 2\sigma_T + 5f_T.$$

On G_0 the conductor is $g + C_1$, with $C_1 \in |g + 2\eta|$ meeting g transversely once; these are f_0, σ_T , respectively. These are the two normalization descriptions in [KPZ14, Proposition 2.2].

The modified inverse normal bundle, twisted by $\mathcal{O}(2)$, is

$$K_0 = r_* \mathcal{I}_{Z_0/G_0}(g + 3\eta) \subseteq \mathcal{O}(3) \oplus \mathcal{O}(2), \quad \deg K_0 = 2.$$

The splitting $\mathcal{O}(3) \oplus \mathcal{O}(-1)$ would put Z_0 on g , which is impossible. In the balanced normalization, $\mathcal{O}(2) \oplus \mathcal{O}$ would give a section of $\mathcal{I}_{Z_0}(g + \eta)$, hence a line containing a length-three subscheme of the smooth conic K , again impossible. Thus the inverse flop curve has normal bundle $\mathcal{O}(-1)^{\oplus 2}$. In the unbalanced normalization $Z_0 \cap g$ has length two. Multiplying g by the fiber through the residual length-one quotient gives a section of $\mathcal{I}_{Z_0}(g + \eta)$, including when the contacts coalesce. Hence $K_0 = \mathcal{O}(2) \oplus \mathcal{O}$, and the inverse normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$. The modification formula holds for repeated contacts: locally $\Lambda = (x = y = 0)$ and $\Gamma = (x = 0, y = t^r)$, and the blowup equation $x = (y - t^r)w$ modifies one normal direction by t^r .

The first exceptional surfaces of the common Atiyah resolution are therefore $\mathbb{F}_0$ in the first case. In the second case, the pagoda identifies the two first $\mathbb{F}_2$ surfaces, extending across their negative sections by the common minimal resolution of the cone. Its base-surface identification and restriction reflection are [Rei83, Theorem (6.3) and Claim (6.5)], after the first blowup in [Rei83, (5.4)–(5.8)]. This applies at every finite width, including when the remaining flop is Atiyah. The other global flopping curves are disjoint from s .

Step 3. We transport every grade and both parities. Let $W = \text{Bl}_{\Gamma_T} V$, with exceptional D , strict transforms G, T_W , and classes

$$A = -K_W = 3h - G - D, \quad T_W = 2h - 2G - D, \quad M = 5h - 2D - G.$$

Resolve the strict G surface by the three successive blowups of Z_0 . On the resulting P , put $J = e_1 + e_2 + e_3$. Then

$$A|_P = g + 3\eta - J, \quad l|_P = g + \eta, \quad T_W|_P = 2g + 2\eta - J, \quad G|_P = -g.$$

Elementary transformation gives $b : P \rightarrow S$, where S is the actual second exceptional surface. Its total exceptional divisor is $J_b = 3\eta - J$.

In the balanced case choose f as the fiber over L ; this ruling is opposite to the inverse ruling over Λ . Then

$$b^*\sigma = \eta, \quad b^*f = g + 2\eta - J, \quad A|_P = b^*(\sigma + f), \quad l|_P = b^*(2\sigma + f) - J_b.$$

The inverse length-three cluster Δ lies on $B \in |\sigma + 2f|$ and maps as a closed immersion to both ruling bases. To verify the cluster, a fiber through a point of Z_0 meets the conductor at that point and its conjugate; blowing up the former and contracting the fiber marks the latter. Repeating the elementary transformation preserves the multiplicity on the conjugate branch. A multiple contact cannot be a ramification point, because $Z_0 \rightarrow \Lambda$ is a closed immersion; a simple ramification point is allowed. The other ruling immersion follows from $B \rightarrow L$ being an isomorphism. On S we obtain

$$H|_S = f, \quad F|_S = -\sigma - f, \quad E_1|_S = B = \sigma + 2f.$$

In the unbalanced case the rulings agree, and

$$b^*f = \eta, \quad b^*\sigma = g + \eta - J, \quad A|_P = b^*(\sigma + 2f), \quad l|_P = b^*(\sigma + 3f) - J_b.$$

On S the corresponding classes are

$$H|_S = f, \quad F|_S = -\sigma - 2f, \quad E_1|_S = B = \sigma + 3f.$$

A section of F order at least j has E_1 order at least $\lceil j/2 \rceil$, because the transverse valuation (1, 2) is bounded by twice ordinary order. After removing these mandatory powers, the even grade $j = 2k$ has containing class $k\sigma + mf$ in the balanced case and $k\sigma + (m+k)f$ in the unbalanced case. The odd class subtracts one fiber in either case. Its intrinsic fractional fixed part is $B/2$, retained from restriction on U .

For $k \geq m$ the even divided divisor on W is

$$mM - 2kG - kT_W = (2m - k)A + (k - m)l.$$

In the balanced case its restriction is $b^*(k\sigma + mf) - (k - m)J_b$, imposing exactly $\mathcal{I}_\Delta^{k-m}$. The odd change is $-G - T_W$, whose restriction is $-b^*f$. Thus these actual images embed in (9.3), with the stated odd fiber shift.

In the unbalanced case the even restriction is $b^*(m\sigma + (m+k)f) - (k - m)J_b$. The restriction reflection in [Rei83, Theorem (6.3)] is

$$D_S \mapsto D_S + (D_S \cdot \sigma)\sigma.$$

The original class $k\sigma + (m+k)f$ has σ intersection $m-k$, so its reflection is precisely $m\sigma + (m+k)f$. The rational leading sections agree away from σ , and their class difference is $(k - m)\sigma$. Hence the original image has fixed factor s_σ^{k-m} before the indicated cluster conditions. For odd j , the restriction change is $-b^*(\sigma + f)$; reflection gives fixed power $k + 1 - m$. Multiplying its free part by s_σ embeds it in

$$s_\sigma^{k-m} H^0(S, m\sigma + (m+k-1)f \otimes \mathcal{I}_\Delta^{k-m}).$$

This proves the comparison with (9.6) in both parities.

These are comparisons of actual associated graded images: if a divided section restricts to zero on P , it has strictly larger F order. The ideals $(u, t^r)^{k-m}$ retain all repeated base conditions.

There exist no grades $j > 4m$, since $mM - jG$ has degree $4m - j$ on the covering general fibers of $V \rightarrow \mathbb{P}^2$.

Each containing surface family has total dimension $\frac{3}{2}m^3 + O(m^2)$, whereas $N_m = 3m^3 + O(m^2)$. There exist two parity copies. An odd section is included in the even containing system by multiplication by a fixed fiber. The fiber shift and $B/2$ are bounded divisors, with $O(1/m)$ effects on the normalized flag bounds. The total dimension deficit is $O(m^2)$, and the relevant orders are $O(m)$; thus the original denominator is retained with only $O(m^3)$ numerator error. The surface lemmas consequently apply to every actual refined basis.

Step 4. We identify the unbalanced contact clusters. The two conductor components on T have intersection degrees one and two with Γ_T . Away from their crossing, elementary transformation conjugates the branches. The two contacts on $g = f_0$ give a degree-two divisor Z_B on $B \setminus \sigma$, and the contact on $C_1 = \sigma_T$ gives $q_3 \in \sigma \setminus B$. Their ruling images are disjoint; a repeated contact remains a closed immersion to the ruling base. This is configuration (i) of Lemma 9.3.

At the crossing, use the normalization chart

$$(b, t) \mapsto [1 : t - b : bt : bt^2 : bt^3], \quad XV - YW - Z^2 = 0.$$

The conductor is $bt = 0$. Since $\Gamma_T \cdot \sigma_T = 1$, write $t = \phi(b)$. Its image is smooth only if $\phi'(0) \neq 1$. In G_0 coordinates $y = t - b$, $r = t$, the conductor branches are $r = 0, r = y$. If $\phi'(0) \neq 0, 1$, the contact graph is $r = dy \bmod y^2$, $d \neq 0, 1$, plus one additional contact on g . The two elementary transformations $r_1 = r/y$, $r_2 = (r_1 - d)/y$ give inverse coordinate $u = 1/r_2$, cluster (u, y^2) and branch images $u = -y/d$, $u = y/(1 - d)$ up to higher terms. This is configuration (ii).

If $\phi'(0) = 0$, its first nonzero term has degree two because $\Gamma_T \cdot f_0 = 2$. All contacts coalesce, with $r = dy^2 \bmod y^3$, $d \neq 0$. Using $r_1 = r/y$, $r_2 = r_1/y$, $r_3 = (r_2 - d)/y$ gives the inverse cluster (u, y^3) , $u = 1/r_3$, with branch images $u = -y/d$ and $u = y^2/(1 - dy)$. This is configuration (iii), transverse to σ and the fiber and tangent to B to order two. The degree-one and degree-two intersections exhaust all cases. The transformed conductor has image $\sigma + B$; applying this reflection to E_1 leaves the class and support of B . Thus this is the actual different curve, rather than a replacement in its linear system.

Step 5. We locate all excluded fibers. Each balanced base point of Δ gives another line through its image on L . Let z be that point and z' its conjugate contact of Γ_T on the conductor. If $z \neq z'$, the ruling through z meets Γ_T in length two away from the conductor; embeddedness of Γ prevents it from containing both conjugate points. Downstairs z' contributes one more intersection on Λ . At a branch point use the pinch normalization $(u, v) \mapsto (x = uv, y = u, z = v^2)$ and ruling $v = 0$. Smoothness of Γ gives its lift as $v = g(u)$, with $d = \text{ord}_u g \geq 1$. The lifted intersection is d , whereas the downstairs ideal (x, z) restricts to (ug, g^2) , of length $\min\{d + 1, 2d\} = d + 1$. It again adds one to the total ruling intersection two. On $Z = \text{Bl}_\Gamma Q$, let D_Z be the exceptional divisor. The nef anticanonical divisor $3h - D_Z$ bounds that intersection by three. Hence the image ruling is a trisecant distinct from Λ , and [KPZ14, Proposition 2.12] identifies its flop with another line meeting L .

In the unbalanced case every base point on $B \setminus \{q_0\}$ also gives another line through its image on L . Its conjugate lies on $\sigma_T \setminus f_0$. The ruling there has two contacts with Γ_T away from the conductor and one further downstairs contact on Λ , hence is a trisecant. This excludes the two Z_B fibers in (i) and the extra q_3 fiber in (ii), under the sole-line hypothesis. It does not exclude the σ base fiber in (i), which the surface lemma includes.

The meeting parameter is the actual point on L : $A|_{T_W} = 3f_T$, while on E_1 the anticanonical contraction has class three times its ruling over L . The common contraction identifies these bases. In the balanced case this agrees with the opposite-ruling formulas in Step 3; in the unbalanced case C_1 maps isomorphically both to Λ and to the ruling base of T . Thus the trisecants occur on exactly the stated fibers over L .

Finally, in the unbalanced case Lemma 9.1 shows that $q_0 = \sigma \cap B$ lies over the unique point with entire incidence length three. Our point has length two, so its fiber avoids q_0 . All restrictions of Lemma 9.3 are now verified.

Step 6. We retain the discrepancy surplus. The balanced surface lemma gives constant one above the fiber of p . The unbalanced configurations give $36/31$, $8/7$, and $108/95$, each at least $108/95$. For a basis of $H^0(X, mH)$ compatible with G, F , write its pullback to U as $S_m(F)F + \Theta_m$, with Θ_m effective and no F component. Its restriction is the actual refinement bounded in Step 3. For a constant c strictly below the corresponding endpoint, plt inversion of adjunction gives

$$A_X(G) \geq cS_m(G) + (3 - cS_m(F)) \text{ord}_G(F).$$

Take $m \rightarrow \infty$ and then c to the endpoint, using (9.14). The balanced surplus is $3 - 43/18 = 11/18$. The unbalanced surplus is $3 - (108/95)(43/18) = 27/95$. This proves both assertions. Since G is centered on the doubled-line ideal at p and $2F$ is its Cartier exceptional divisor, $\text{ord}_G(F) > 0$. $\square$

10. MIXED INCIDENCE POINTS IN DEGREE EIGHTEEN

Let the incidence scheme at p consist of an ordinary line L and a double point at another line. We use the ordinary-line link from L . Its exceptional surface is $E = \mathbb{F}_1$, with negative section σ and fiber f_p . Lemma 7.1 identifies the double point with a cluster of width $n \in \{2, 3, 4\}$. The distinction between these three widths is essential in this section.

Lemma 10.1. *Let Z be the length-four base scheme on E , let $p_T : T \rightarrow E$ resolve its four simple free base points, and denote the total exceptional classes by $e_1, \dots, e_4$. Set $P = p_T^*(2\sigma + 3f) - \sum e_i$. Then $P = -K_T$ is nef, $P^2 = 4$, and the actual divisor $D_0 = (2H - 5E)|_E$ satisfies*

$$(10.1) \quad \Delta = p_T^*D_0 - 3 \sum e_i = C + N, \quad P \cdot C = 7, \quad C^2 = \Delta^2 = 9,$$

where C is isomorphic to the smooth inverse curve of degree seven and genus two, and N is effective and supported on the P -trivial curves. At most one base point lies on the successive strict transforms of σ .

Writing $W_{m,j}$ for the actual refinement along E , we obtain

$$(10.2) \quad \begin{aligned} W_{m,j} &\subseteq H^0(E, j\sigma + (m+j)f), & 0 \leq j \leq m, \\ W_{m,j} &\subseteq H^0(E, (j\sigma + (m+j)f) \otimes I_Z^{j-m}), & m \leq j \leq 2m, \\ W_{m,j} &= D_0^{j-2m} U_{5m-2j}, & 2m \leq j \leq \lfloor 5m/2 \rfloor. \end{aligned}$$

Here U_r is the actual restriction image of $H^0(Y, r(H - 2E))$, contained in $H^0(E, r(2\sigma + 3f) \otimes I_Z^r)$. There exist no later pieces. If $w = j/m - 1 \in [0, 1]$, then

$$(10.3) \quad \dim W_{m,j} = \frac{3 + 4w - 3w^2}{2} m^2 + O(m), \quad \dim U_r = 2r^2 + O(r), \quad S_X(E) = \frac{101}{72}, \quad J = \frac{1}{72},$$

where J is the normalized total exponent of D_0 in the tail.

Proof. The ordinary link maps E birationally onto the surface $S \in |\mathcal{O}_Q(2)|$ containing the smooth inverse curve Γ . The exact base ideals of Lemma 7.1 give $P^2 = 8 - 4 = 4$ and $P = -K_T$. Its moving system is base point free. If k_σ successive centers are on σ , then $P \cdot \sigma_T = 1 - k_\sigma \geq 0$.

Put $A = H - E$ and $B = H - 2E$. On a common resolution, the pullbacks of A, A^+ agree and $B = B^+ + G_b$, with G_b the fixed base divisor. Since $D = 2H - 5E = 3B - A$, we obtain $D = D^+ + 3G_b$. Its restriction to T proves effectivity of Δ in (10.1). The surface S contains Γ with generic multiplicity one, so Δ has a component mapping onto Γ . Its P -degree is $19 - 12 = 7$. The degree of Γ forces this component to have coefficient one and map degree one, and every other component to have degree zero. A nonconstant morphism of integral projective curves is finite. The resulting finite birational map to smooth Γ is an isomorphism. Adjunction gives $C^2 = 9$, while $\Delta^2 = (5\sigma + 7f)^2 - 36 = 9$.

Full section spaces on the two sides of the flop agree by their divisorial inequalities in the common function field. The pullback identities above impose multiplicity $j - m$ at every successive base point in the middle range. For a free chain this is exactly the indicated ordinary ideal power. In the terminal range the fixed divisor identity of Lemma 3.4 gives the equality in (10.2); multiplication by D_0 is injective because E is not a component of D . This argument uses no reducedness assumption on Z . The complete section differences and Riemann–Roch give the ranks in (10.3), with uniform linear errors. The volume calculation in that lemma gives $S_X(E)$, and

$$J = \frac{1}{3} \int_2^{5/2} 2(t-2)(5-2t)^2 dt = \frac{1}{72}.$$

□

Lemma 10.2. *If $n \geq 3$, then Z is disjoint from σ . If $n = 2$, write $D_0 = \mu f_p + D_{\text{res}}$, where f_p is not a component of D_{res} . Then $1 \leq \mu \leq 2$, and $\mu = 1$ if the marked root is on σ . For $n = 3$, one also has $1 \leq \mu \leq 2$. In each case $D_{\text{res}}|_{f_p}$ has degree five.*

Proof. We classify the P -trivial curves and use their intersection matrices to control the marked fiber and the negative section.

Step 1. In this step, we determine the root configurations.

Index the marked free chain first. Its first two centers lie on the successive strict fiber; for $n > 2$ its ruling order is exactly two. Thus $F = f - e_1 - e_2$ is the strict fiber, a P -trivial (-2) -curve. The vector of intersections of Δ with F starts with -1 , which forces F to occur in N .

We first classify the other P -trivial curves. A nonexceptional negative curve has class $a\sigma + bf - \sum m_i e_i$ and square -2 , by Hodge index [Mil24, Corollaries 11.51–11.52] and adjunction. Therefore

$$\sum m_i = a + 2b, \quad \sum m_i^2 = -a^2 + 2ab + 2, \quad 5a^2 - 4ab + 4b^2 \leq 8.$$

Here a, b, m_i are nonnegative integers. The last inequality first gives $a \leq 1$, then permits precisely $(a, b) = (0, 1), (1, 0), (1, 1)$. These are a fiber through two centers, σ through one center, and a member of $|\sigma + f|$ through three centers, respectively, all with multiplicities one. The exceptional roots are consecutive differences in the free chains.

For $n = 2$, a $|\sigma + f|$ curve cannot pass through the second marked center, since its intersection with the original fiber is one. Its strict transform is consequently disjoint from F . The component of the zero-curve graph containing F is one of F , $F - S$, or $F - S - R$, where S is the strict negative section and R is the exceptional root of a double residual cluster on σ . There exists no further vertex: a $|\sigma + f|$ curve is disjoint from σ , and the other zero fibers become disjoint from S after its base point is blown up. If the marked root lies on σ , F is isolated. The respective Δ -intersection vectors are (-1) , $(-1, -1)$, and $(-1, -1, 0)$.

Step 2. In this step, we bound the fixed fiber coefficient using the intersection form. Let A be the positive Cartan matrix of this component, $c_i = C \cdot R_i$, and n_i its coefficients in N . Then $An = c - d$ and $\mu = n_1$. Orthogonal projection of $C - \frac{7}{4}P$, whose square is $-13/4$, to this negative definite root space gives $c^t A^{-1} c \leq 13/4$. Thus

$$\mu \leq (A^{-1}(-d))_1 + \sqrt{(A^{-1})_{11} 13/4}.$$

For the three graphs these bounds are $\frac{1}{2} + \sqrt{13/8} < 2$, $1 + \sqrt{13/6} < 3$, and $\frac{5}{4} + \sqrt{39/16} < 3$. Integrality proves the asserted width-two bounds.

For $n \geq 3$, no $|\sigma + f|$ root exists: it can contain at most the first marked center and the one possible residual center. There exists no other zero fiber. Suppose a connected collection $R_1, \dots, R_k$ is contracted to o . Since $C \rightarrow \Gamma$ is an isomorphism, the intersections satisfy

$$(10.4) \quad c_i \in \mathbb{Z}_{\geq 0}, \quad \sum_i c_i \leq 1.$$

To prove this, choose a regular function at the smooth embedded curve $\Gamma \subseteq Q$ vanishing to order exactly one on Γ at o . Its pullback contains every contracted R_i with multiplicity at least one. Restrict to C . The resulting order one bounds the sum of all local intersection multiplicities, including tangencies and crossings. If C misses the collection, the inequality is immediate.

Step 3. In this step, we prove that the base scheme avoids the negative section when $n \geq 3$. There exist three possible ways Z could meet σ for $n \geq 3$. For $n = 3$ with the marked root off σ and the residual root on σ , the entire zero graph, in order, is

$$e_1 - e_2, e_2 - e_3, F, \sigma - e_4; \quad d = (0, 0, -1, -1).$$

It is A_4 . For $n = 3$ with the marked root on σ , it is again A_4 , ordered as $F, e_2 - e_3, e_1 - e_2, \sigma - e_1$, with $d = (-1, 0, 0, -1)$. For $n = 4$ with the root on σ , it is D_5 , ordered as $\sigma - e_1, e_1 - e_2, e_2 - e_3, e_3 - e_4, F$, with edges 12, 23, 34, 35 and $d = (-1, 0, 0, 0, -1)$. These lists are complete by the root classification.

The equality $C^2 = \Delta^2$ and $c = d + An$ give

$$c^\dagger A^{-1} c = d^\dagger A^{-1} d.$$

By (10.4), the left side is zero or a diagonal entry of A^{-1} . The relevant inverses are

$$A_4^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad D_5^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3/2 & 3/2 \\ 1/2 & 1 & 3/2 & 5/4 & 3/4 \\ 1/2 & 1 & 3/2 & 3/4 & 5/4 \end{pmatrix}.$$

The right sides in the three cases are $16/5, 2, 13/4$, whereas the respective maximal diagonal entries are $6/5, 6/5, 3$. This contradiction proves disjointness from σ . In the remaining width-three case the component of F is A_3 , ordered $F, e_2 - e_3, e_1 - e_2$, with $d = (-1, 0, 0)$. The Hodge inequality [Mil24, Corollaries 11.51–11.52] for this A_3 configuration gives $\mu \leq \frac{3}{4} + \sqrt{39/16} < 3$. Finally D_{res} is effective with no fiber component, and $D_{\text{res}} \cdot f_p = D_0 \cdot f_p = 5$. $\square$

Proposition 10.3. *If $n = 2$ or 3, then*

$$(10.5) \quad A_X(G) \geq \frac{3360}{3323} S_X(G) + \frac{5798}{9969} \text{ord}_G(E) \quad \text{for every } G \text{ with } p \in c_X(G).$$

In particular, $\delta_p(X) \geq 3360/3323$.

Proof. We estimate the full refinement along $E \supseteq f_p$ with the original denominator $mN_m = 3m^4 + O(m^3)$.

Step 1. We bound fiber orders for width two. Choose z on the ruling base with $f_p = (z = 0)$ and infinity avoiding the base supports. In the positive chart the marked ideal is (z, u^2) ; in the negative chart it is (z, v^2) , where $v = 1/u$. Act by $z \mapsto \lambda^{-1}z, u \mapsto \lambda^B u$ for a fixed positive integer B . Both marked ideals are invariant. At infinity, $(\zeta, u_\infty) = (1/z, u/z)$ have positive weights $1, B+1$. The residual length two has precisely four limiting forms: $h = (u_\infty, \zeta^2)$, $v = (\zeta, u_\infty^2)$, one off-section and one on-section simple point, denoted $r + t$, or a transverse double root on the section, denoted t_v . When the marked root is on the section, only h, v occur.

These are flat families for every ideal power. Locally write a graph ideal as $(a - P_\lambda(b), Q_\lambda(b))$, with Q_λ monic of degree ℓ . After replacing a by $a - P_\lambda(b)$, the quotient by its q th power has free basis $a^i b^j$, $0 \leq i < q$, $0 \leq j < \ell(q - i)$. The marked and limiting residual supports are disjoint. Upper semicontinuity therefore bounds kernels with any imposed fiber order, and hence the first fiber moment. Only this first filtration is compared by degeneration.

Scale the middle degrees as $a = 1 + w$, $b = 2 + w$, $q = w$, $0 \leq w \leq 1$, and put $y = \ell/m$. The marked inequality is $2i + \ell \geq 2q$. Thus the lower i -bound is $L = \max\{0, w - y/2\}$. The

upper bound U in each limiting model is the minimum in the following table; the domain also has $0 \leq y \leq 1 + w$.

model	U	middle integral	tail integral
off/ h	$\min\{b - y, b + y - 2w\}$	$43/24$	$1/4$
off/ v	$\min\{b - y, b - y/2 - w\}$	$43/24$	$1/4$
off/ $(r + t)$	$\min\{b - y, b - w, a + b - 2y - w\}$	$23/12$	$7/24$
off/ t_v	$\min\{b - y, b - y + (a - y)/2 - w\}$	$49/24$	$1/3$
on/ h	$\min\{b - a + y, b + a - y - 2w\}$	$43/24$	$1/4$
on/ v	$\min\{b - a + y, b - a + y + (a - y)/2 - w\}$	$43/24$	$1/4$

The integrand is $(U^2 - L^2)/2$ where $L \leq U$. The tail is the corresponding unit integral at $(a, b, q) = (2, 3, 1)$ times $\int_2^{5/2} (5 - 2t)^3 dt = 1/8$. The initial contribution is $11/24$, and normalization divides by three. The free fiber moments are respectively $5/6, 5/6, 8/9, 17/18, 5/6, 5/6$. Adding $\mu J \leq 2/72$ gives

$$(10.6) \quad S(W; f_p) \leq 35/36.$$

When the marked root is on the section, $\mu = 1$ improves this to $61/72$.

For completeness, the following cells give every middle integral. Each row lists (L, U) , the vertices (w, y) of its domain, and its integral of $(U^2 - L^2)/2$. The max/min inequalities select these cells with disjoint interiors and cover the full domain.

Model	(L, U)	Polygon	Integral
off/ h	$0, w - y + 2$	$(0, 0), (1, 2), (0, 1)$	$11/24$
off/ h	$w - y/2, w - y + 2$	$(0, 0), (1, 1), (1, 2)$	$67/96$
off/ h	$w - y/2, -w + y + 2$	$(0, 0), (1, 0), (1, 1)$	$61/96$
off/ v	$0, w - y + 2$	$(0, 0), (1, 2), (0, 1)$	$11/24$
off/ v	$w - y/2, 2 - y/2$	$(0, 0), (1, 0), (1, 2)$	$4/3$
off/ $r + t$	$0, w - y + 2$	$(0, 0), (1/2, 1), (0, 1)$	$55/192$
off/ $r + t$	$0, w - 2y + 3$	$(1/2, 1), (1, 2), (0, 1)$	$19/192$
off/ $r + t$	$w - y/2, w - y + 2$	$(0, 0), (1, 1), (1/2, 1)$	$5/12$
off/ $r + t$	$w - y/2, 2$	$(0, 0), (1, 0), (1, 1)$	$89/96$
off/ $r + t$	$w - y/2, w - 2y + 3$	$(1, 1), (1, 2), (1/2, 1)$	$3/16$
off/ t_v	$0, w - y + 2$	$(0, 0), (1/3, 2/3), (0, 1)$	$133/648$
off/ t_v	$0, w/2 - 3y/2 + 5/2$	$(1/3, 2/3), (1, 2), (0, 1)$	$49/324$
off/ t_v	$w - y/2, w - y + 2$	$(0, 0), (1, 0), (1/3, 2/3)$	$131/162$
off/ t_v	$w - y/2, w/2 - 3y/2 + 5/2$	$(1, 0), (1, 2), (1/3, 2/3)$	$71/81$
on/ h	$0, y + 1$	$(0, 0), (1/2, 1), (0, 1)$	$17/48$
on/ h	$0, 3 - y$	$(1/2, 1), (1, 2), (0, 1)$	$17/48$
on/ h	$w - y/2, y + 1$	$(0, 0), (1, 0), (1, 1), (1/2, 1)$	$47/64$
on/ h	$w - y/2, 3 - y$	$(1, 1), (1, 2), (1/2, 1)$	$67/192$
on/ v	$0, y + 1$	$(0, 0), (1/3, 2/3), (0, 1)$	$133/648$
on/ v	$0, -w/2 + y/2 + 3/2$	$(1/3, 2/3), (1, 2), (0, 1)$	$193/324$
on/ v	$w - y/2, y + 1$	$(0, 0), (1, 0), (1/3, 2/3)$	$73/324$
on/ v	$w - y/2, -w/2 + y/2 + 3/2$	$(1, 0), (1, 2), (1/3, 2/3)$	$62/81$

Step 2. We bound all secondary orders on the actual rows for width two. After removing the fixed fiber factor, the tail contributes at most $5J = 5/72$ at every point, since the remaining divisor is $D_{\text{res}}|_{f_p}$.

For a marked root off σ , keep only its ideal $(z, u^2)^q$. The order- k row injects into the span of u^ℓ with

$$\max\{0, 2(q - k)\} \leq \ell \leq \min\{a, b - k\}.$$

At $q_0 = \sigma \cap f_p$ the orders are $a - \ell$. Their initial, middle, and free tail integrals are $1/4, 2, 5/12$, giving moving moment $8/9$ and full moment $23/24$. At every finite point, discard the marked condition as well. The containing polynomials have degree $\min\{a, b - k\}$. Lemma 2.8 gives the triangular-number bound, with integrals $5/24, 43/24, 5/12$ and moving moment $29/36$. The full moment is at most $7/8$.

If the marked root is q_0 , the complete negative-chart row has exponents $\max\{0, k - b + a\} \leq \ell \leq a$. At every point other than $v = 0$, its possible vanishing is described by the triangular number, so the full bound remains $7/8$. At q_0 itself the exact containing rows satisfy

$$0 \leq \ell \leq a, \quad \max\{0, q - \ell/2\} \leq k \leq b - a + \ell.$$

Their uncapped integrals are $1/4, 9/4, 1/2$, with normalized moment one. The actual total rank gives a strict saving. For middle degrees the density in $y = \ell/m$ is

$$h_w(y) = 1 + y - \max\{0, w - y/2\}, \quad 0 \leq y \leq 1 + w.$$

Its integral exceeds the actual rank $(3 + 4w - 3w^2)/2$ by w^2 . For any collection of subspaces of total rank r and any cutoff c , the total order is bounded by

$$\sum_{\text{ambient}} \ell - \sum_{\text{ambient}} \min\{cm, \ell\} + cmr.$$

This follows by retaining cm for each chosen vector and bounding the excess above that order by the containing filtered rows. Choose $c = w^2$. On $[0, c]$ the density is $1 - w + 3y/2$. The resulting saving, before normalization by three, is

$$\int_0^1 \left[cw^2 - \int_0^c (c - y)(1 - w + 3y/2) dy \right] dw = \int_0^1 \left(\frac{w^4}{2} + \frac{w^5}{2} - \frac{w^6}{4} \right) dw = \frac{31}{210}.$$

In the tail the ambient unit rank is three and the actual rank is two. The unit density is $3y/2$, and cutoff one saves $3/4$. The tail scaling multiplies this by $1/8$. Consequently every secondary moment is at most

$$(10.7) \quad 1 - \frac{1}{3} \left(\frac{31}{210} + \frac{3}{32} \right) + \frac{5}{72} = \frac{3323}{3360}.$$

All these comparisons are injections of the actual rows, with their actual total ranks.

Step 3. We treat width three. Lemma 10.2 places both supports off σ . The marked cluster has ideal $(z - u^2, u^3)$ after scaling z , and the residual simple point lies on a different fiber. The action $z \mapsto \lambda^{-2}z, u \mapsto \lambda^{-1}u$ preserves this ideal and f_p . At infinity its coordinates have weights $2, 1$, so the residual point tends to the off-section point. The graph-ideal argument of Step 1 gives flatness of all powers, again only for comparison of first fiber orders.

Group monomials $z^i u^\ell$ by $d = 2i + \ell$ and write a group as $u^d P(z/u^2)$. Membership in $(z - u^2, u^3)^q$ is equivalent to divisibility of $P(T)$ by $(T - 1)^h$, where $h = \max\{0, 3q - d\}$. The residual point imposes $i \leq b - q$. After scaling the middle range,

$$L_0 = \max\{0, (d - 1 - w)/2, d - 2 - w\}, \quad U_0 = \min\{d/2, 2\}.$$

Multiplication by $(T - 1)^h$ has nonzero constant term and is triangular in increasing T -degree. Thus the leading fiber exponents run from $L = L_0$ to $U = U_0 - \max\{0, 3w - d\}$. The complete

middle certificate for $(U^2 - L^2)/2$ is

L	U	vertices (w, d)	integral
0	$d/2$	$(0, 0), (1/2, 3/2), (0, 1)$	$19/768$
0	$-3w + 3d/2$	$(0, 0), (1, 2), (1/2, 3/2)$	$3/256$
$(d - w - 1)/2$	$d/2$	$(1, 3), (1, 4), (0, 3), (0, 1), (1/2, 3/2)$	$913/768$
$(d - w - 1)/2$	$-3w + 3d/2$	$(1, 2), (1, 3), (1/2, 3/2)$	$59/768$
$d - w - 2$	$d/2$	$(1, 4), (0, 4), (0, 3)$	$37/96$
$d - w - 2$	2	$(1, 4), (1, 5), (0, 4)$	$7/24$

The sum is $95/48$; the scaled tail integral is $29/96$. Including $11/24$ from the initial range and dividing by three gives free moment $263/288$. The fixed contribution is at most $2/72$, so $S(W; f_p) \leq 271/288$.

The actual ideal $(z - u^2, u^3)$ is contained in (z, u^2) . Every actual row therefore satisfies the off-section vertical-double condition used in Step 2. The ranks and the degree-five residual fixed contribution are unchanged. This gives secondary bounds $23/24$ at $\sigma \cap f_p$ and $7/8$ at every other point. In particular (10.7) holds.

Step 4. We pass to divisors over X . The rational polygons and fixed degree bounds give total lattice error $O(m^3)$ in numerators normalized by $3m^4 + O(m^3)$. The row bounds are uniform in their point. Both first fiber bounds are less than $3323/3360$. First apply Lemma 2.7 with bases compatible with the tested surface valuation and the fiber. This gives the uniform finite-degree assertion for every basis divisor of the surface series. Then choose a basis compatible with G and E and apply Lemma 2.6, or the retained form of [AZ22, Theorem 3.2 and (3.6)], give

$$A_X(G) \geq cS_X(G) + (2 - cS_X(E)) \operatorname{ord}_G(E), \quad c = \frac{3360}{3323}.$$

Since $S_X(E) = 101/72$, this is (10.5). A center not contained in L has exceptional order zero; otherwise its center on E meets f_p . Thus the argument applies to every center containing p . $\square$

Lemma 10.4. *Let $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ be generated by finitely many vectors of positive first coordinate, and suppose its group is $\mathbb{Z}^{d+1}$. Let P be the section of its rational cone at first coordinate one. If P is bounded and has dimension d , then*

$$(10.8) \quad \#\mathcal{S}_r = \operatorname{vol}(P)r^d + O(r^{d-1}).$$

The leading normalized sums of linear functions are their integrals over P .

Proof. Triangulate the rational cone into simplicial cones whose ray generators belong to $\mathcal{S}$. Subtract nonnegative multiples of those generators from each lattice point to place the remainder in one of finitely many bounded fundamental parallelepipeds. Thus the saturated lattice semigroup is a finite module over $\mathcal{S}$. Every remainder is a difference of two elements of $\mathcal{S}$, because its group is the full lattice. Adding the sum of all these negative parts gives one fixed translate taking the saturated semigroup into $\mathcal{S}$. The missing points therefore lie within bounded distance of the boundary of the cone. There exist $O(r^{d-1})$ such points in degree r . Comparing unit lattice cubes with the dilate rP proves (10.8); the boundary cubes also contribute $O(r^{d-1})$. For a linear function, its size is $O(r)$ on that slice, and the corresponding weighted boundary error is $O(r^d)$. Riemann sums give the claimed integral. $\square$

Proposition 10.5. *Suppose $n = 4$. There exists a plt weighted extraction at p with exceptional divisor E_p such that*

$$(10.9) \quad A_X(E_p) = 7, \quad S_X(E_p) = 67/12, \quad (E_p, \Delta) = \left(\mathbb{P}(1, 1, 3)_{[x:z:w]}, \frac{1}{2}(w = 0) \right).$$

For every prime divisor G centered at p ,

$$(10.10) \quad S_X(G) \leq \frac{71}{72}A_X(G) - \frac{95}{72} \operatorname{ord}_G(E_p).$$

In particular, $\beta_X(G) > 0$.

Proof. We identify the complete weighted section ring, estimate its surface refinement, and retain the extraction coefficient in adjunction.

Step 1. We determine the actual local section equations. Use the adjoint model of Setup 8.1. The length-three incidence scheme is noncollinear, so its tangent plane lies in the contact $\mathbb{P}^3$. Otherwise its intersection with that space would be a line containing the incidence scheme; the quadrics of the twisted cubic would all contain this line, a contradiction. The SL_2 contact action normalizes the double direction to $[i]$ and the simple direction to $[n]$. Their scheme-theoretic span is the plane with coordinates (i, a, n) . Thus $p = [h]$, the double-support line is $[h : i]$, and $L = [h : n]$.

In the chart $h = 1$, the exact Pfaffian identities are those of (8.3). The two actual linear equations, after row operations, take the form

$$(10.11) \quad \begin{aligned} F_1 &= e + Af + B(g + k) + Cc + Dl + Eb + Fd + Gm + Jj, \\ F_2 &= ug + vk + \lambda f + \mu c + Pl + Qb + Rd + \kappa m + Tj, \quad u - v = 1. \end{aligned}$$

Here m in the equations is the homogeneous coordinate of the adjoint model. All higher linear-section coefficients are retained.

A line near $[h : i]$ meeting L at $n = t, h = 1$ has, with parameter z_0 ,

$$i = z_0, \quad a = rz_0, \quad e = r^2 z_0, \quad n = t - r^3 z_0, \quad g = -tz_0.$$

To obtain this scheme-theoretically, start with $a = rz_0, e = sz_0, n = t + qz_0, g = g_1 z_0$. The quadratic coefficients in f, k, l force, successively, $s = r^2, q = -r^3, g_1 = -t$. Then the other coordinates are $f = k = l = b = d = j = 0, c = rtz_0, m = -t^2 z_0$. These formulas also prove the converse. The exact local incidence equations are therefore

$$(10.12) \quad r^2 - Bt + Crt - Gt^2 = 0, \quad t(-u + \mu r - \kappa t) = 0.$$

If $u \neq 0$, the local length is two. If $u = 0$, curvilinearity forces $B \neq 0$, since otherwise both equations have zero linear term. Solving gives $t = r^2/B + O(r^3)$. The length is three when $\mu \neq 0$, four when $\mu = 0, \kappa \neq 0$, and infinite when both vanish. Hence our actual width-four point has

$$u = \mu = 0, \quad v = -1, \quad B\kappa \neq 0.$$

Moreover $\lambda \neq 0$. On the chart $i = 1$, with free coordinates (a, f, g, h, l) , the Pfaffian identities give

$$\begin{aligned} e &= hf + a^2, \quad k = af + hl, \quad b = al - f^2, \quad c = hf^2 - ag - ahl, \\ n &= -a^3 - 2ahf - hg - h^2 l, \quad d = gf - ab, \quad m = eb - g(g + k), \quad j = gl - fb. \end{aligned}$$

Along the entire line $[h : i]$, $dF_2 = \lambda df + (P - h)dl$. If $\lambda = 0$, this differential vanishes at its actual point $h = P$, contradicting smoothness.

Step 2. We identify the whole weighted ring. Give the regular parameters (i, a, n) weights $(2, 3, 2)$. Recursive substitution in (10.11) and the exact Pfaffian identities gives

$$\begin{aligned} \mathrm{in}(e) &= Bin, \quad \mathrm{in}(g) = -in, \quad \mathrm{in}(c) = an, \quad \mathrm{in}(f) = Bi^2 n - a^2 =: F_0, \quad \mathrm{in}(m) = -in^2, \\ \mathrm{in}(k) &= \lambda F_0 - \kappa in^2 =: K_0, \quad \mathrm{in}(l) = iK_0, \quad \mathrm{in}(d) = -nF_0, \quad \mathrm{in}(b) = aK_0, \quad \mathrm{in}(j) = -inK_0. \end{aligned}$$

For the order check, the first equation starts as $e - Bin$ in weight four and the second as $-k + \lambda f + \kappa m$ in weight six. The remaining summands in these equations have weights at least five and eight, respectively. The exact identities then give errors of weights at least six in g , seven in f , and eight in c, m . Repeated substitution preserves these strict bounds.

Put $\nu = \kappa/\lambda$ and set $i = \nu x, n = Bz, a = B\nu y$. Rescale the sections independently. Their twelve symbols become

$$(10.13) \quad 1, x, y, z, xz, yz, F, xz^2, xK, zF, yK, xzK, \quad F = x^2 z - y^2, \quad K = F - xz^2.$$

Their weights are 0, 2, 3, 2, 4, 5, 6, 6, 8, 8, 9, 10. If T_0 is the initial symbol of the nonvanishing section h , let $\mathcal{A}$ be generated by T_0 times this list in $\mathbb{C}[T_0, x, y, z]$.

We prove completeness in every section degree. Filter further by lowest x exponent. The resulting exponent triples of the twelve generators are

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), \\ (0, 2, 0), (1, 0, 2), (1, 2, 0), (0, 2, 1), (0, 3, 0), (1, 2, 1).$$

Fix degree r , x exponent j , and $s = r - j$. For $s \geq 1$, every y exponent $0 \leq b \leq 3r - j$ occurs, and the z exponents are exactly $0 \leq d \leq D$, where

$$D = \begin{cases} r + j, & 0 \leq b \leq 2s, \\ 2r - \lceil b/2 \rceil, & 2s \leq b \leq 2r, \\ 3r - b, & 2r \leq b \leq 3r - j. \end{cases}$$

To check that there exist no gaps, the j factors with x exponent one can have y exponent $2k$, $0 \leq k \leq j$, and every z exponent from zero to $2j - k$. The other s factors give every $b' \in [0, 3s]$ and every z exponent from zero to $\min\{s, 3s - b'\}$. At fixed b , maximize $2j - k + \min\{s, 3s - b + 2k\}$ on

$$\max\{0, \lceil (b - 3s)/2 \rceil\} \leq k \leq \min\{j, \lfloor b/2 \rfloor\}.$$

The interval is nonempty and its maximum is the displayed D . Every contributing interval starts at zero. For $s = 0$, only even $b \in [0, 2r]$ occur and $0 \leq d \leq 2r - b/2$.

For fixed $s \geq 1$, summing $D + 1$ gives $3r^2 + 3r + 1 + rs + (s - 3s^2)/2$. Summing this over $s = 1, \dots, r$ gives $3r^3 + 3r^2 + r$; the $s = 0$ slice has $(r + 1)(3r + 2)/2$ points. Thus the number of independent symbols is exactly

$$N_r = 3r^3 + \frac{9}{2}r^2 + \frac{7}{2}r + 1 = h^0(X, rH).$$

The last equality follows from Riemann–Roch and Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem]. It proves that $\mathcal{A}$ is the entire associated graded ring.

The weighted blowup with primitive weights $(2, 3, 2)$ is plt and has discrepancy seven. Its second Veronese is $\mathbb{C}[x, z, w]$, $w = y^2$, of weights $(1, 1, 3)$. The quotient charts give $\Delta = \frac{1}{2}D$, $D = (w = 0)$, from the generic stabilizer of order two. The actual divisor of y is $D/2$; a weight- q polynomial has class $(q/2)\mathcal{O}_{E_p}(1)$. We keep these fractional divisors.

Step 3. We construct an asymptotically complete surface series. Filter $\mathcal{A}$ by auxiliary weights $(0, 1, 2)$ on (x, y, z) , retaining the original exceptional degree. Let $\mathcal{A}'$ be the full associated graded algebra. It contains the algebra $\mathcal{B}$ generated in degree one by

$$T_0(1, x, y, z, xz, yz, F, xz^2, xF, zF, yF, xzF)$$

and in degree two by $T_0^2 F^3$. The additional symbol follows from

$$(xK)(xzK) - (yK)^2 + (xzK)(zF) = F^2 K.$$

The auxiliary initial symbol of K is F .

Use characters $\text{char}(x) = (1, 0)$, $\text{char}(y) = (1, 1)$, $\text{char}(z) = (0, 2)$, and $\text{char}(F) = (2, 2)$. Write them as (u, v) ; the original degree is $2u + v$. Filter also by F order c and use $x = X_0, z = Z_0^2, y = X_0 Z_0$ on $F = 0$. The degree-one triples (a, b, c) are

$$(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 2, 0), (1, 2, 0), (1, 3, 0), \\ (0, 0, 1), (1, 4, 0), (1, 0, 1), (0, 2, 1), (1, 1, 1), (1, 2, 1),$$

and the degree-two triple is $(0, 0, 3)$. Under the determinant-one change $u = a + 2c, v = b + 2c$, the normalized convex hull is

(10.14)

$$\mathcal{P} = \{g(u) \leq c \leq h(u, v), \ u \leq 3, \ v \leq 4\}, \quad \begin{cases} g(u) = \max\{0, (u - 1)/2\}, \\ h(u, v) = \min\{u/2, v/2, 1 + u - v/2, 3 - v/2\}. \end{cases}$$

Its vertices are

$$(0, 0, 0), (1, 0, 0), (0, 2, 0), (1, 4, 0), (3, 2, 1), (3, 4, 1), (2, 4, 1), (3, 3, 3/2).$$

Each belongs to the normalized generating set, and every generator satisfies the inequalities. The generated group is the full lattice, using the degree-one zero vector and differences of the displayed triples. Lemma 10.4 applies.

For explicit integration, partition $\mathcal{P}$ as follows:

c	u	v
$[0, 1]$	$[2c, 1 + c]$	$[2c, 2 + 2u - 2c]$
$[0, 1]$	$[1 + c, 1 + 2c]$	$[2c, 4]$
$[1, 3/2]$	$[2c, 3]$	$[2c, 6 - 2c]$

These iterated intervals give

$$(10.15) \quad \int_{\mathcal{P}} (1, u, v, c, g) = \left(3, \frac{115}{24}, \frac{43}{6}, \frac{13}{8}, 1 \right).$$

For the last entry, write $g = (u - 1)/2 + (1 - u)_+/2$. The first part integrates to $43/48$. For $0 \leq u \leq 1$, the slice area is $u + u^2/2$, so the correction is $\frac{1}{2} \int_0^1 (1 - u)(u + u^2/2) du = 5/48$.

Define a containing polynomial space $\mathcal{C}_{r,u,v}$ as the span of monomials $x^{u-b}y^bz^{(v-b)/2}$ with $b \equiv v \pmod{2}$,

$$0 \leq b \leq \min\{u, v, 2r + 2u - v, 6r - v\}, \quad u \leq 3r, \quad v \leq 4r,$$

subject to divisibility by F^{g_r} , where $g_r = \max\{0, \lceil (u - r)/2 \rceil\}$. Every generator of $\mathcal{B}$ satisfies these additive inequalities and this divisibility, so $\mathcal{B}_r \subseteq \mathcal{C}_r$. With $Q_0 = y^2/(x^2z)$, a character space before divisibility is a fixed monomial times polynomials in Q_0 of degree at most h_r , where h_r is half the difference between the greatest and least allowed parity exponents. Divisibility means $(Q_0 - 1)^{g_r}$ divides that polynomial. Hence its dimension is $\max\{0, h_r - g_r + 1\}$. The limiting count is precisely (10.14). Thus

$$\dim \mathcal{B}_r = \dim \mathcal{C}_r = 3r^3 + O(r^2), \quad \dim \mathcal{A}'_r = N_r.$$

Both quotients by $\mathcal{B}_r$ have dimension $O(r^2)$.

All original degrees are at most $10r$, and all polynomial degrees and orders for the curves and the fixed weighted extraction below are $O(r)$, uniformly in their points and nonzero pencil parameters. The ambient valuation filtrations induce one common filtration on the actual subspace, so both inclusions are strict. Lemma 2.9 therefore identifies their limiting moments, including all fixed parts, with errors $O(1/r)$. This comparison concerns the complete $\mathcal{A}'_r$.

Step 4. We compute the surface flags. In a character space the basis $(Q_0 - 1)^{g_r} Q_0^j$, $0 \leq j \leq h_r - g_r$, is adapted to highest and lowest y exponent. After normalization, their exponents are $2c$ and $2(c - g)$. (10.15) gives the means

$$(10.16) \quad \bar{u} = \frac{115}{72}, \quad \bar{v} = \frac{43}{18}, \quad \bar{c} = \frac{13}{24}, \quad \bar{g} = \frac{1}{3}, \quad \overline{c - g} = \frac{5}{24}.$$

In particular the mean original degree is $2\bar{u} + \bar{v} = 67/12$, proving the asserted $S_X(E_p)$.

Put $H_x = (x = 0)$, $H_z = (z = 0)$ and $C_\tau = (w - \tau x^2 z = 0)$ for $\tau \neq 0$. The limiting primary moments are

$$S(H_x) = 37/72, \quad S(H_z) = 47/72, \quad S(D) = 5/24, \quad S(C_1) = 13/24, \quad S(C_\tau) = 5/24 \quad (\tau \neq 1).$$

The first two use the highest- y basis, the third the lowest- y basis. The orders on C_1 run from g_r to h_r ; on the other C_τ they run from zero to $h_r - g_r$.

Let $P_z = [0 : 1 : 0]$, $P_x = [1 : 0 : 0]$, $P_w = [0 : 0 : 1]$. After removing order d on C_τ , its restriction is the monomial $X_0^{u-2d}Z_0^{v-2d}$. The endpoint orders are $u - 2d$ at P_z and $(v - 2d)/2$

at P_x . Thus

	P_z	P_x
C_1	37/72	47/72
$C_\tau, \tau \neq 1$	85/72	71/72.

At every other point the moment is at most the smaller endpoint moment. To see this, fix the original degree and curve order. The residual space is monomial on a rational curve. Jets of orders $0, \dots, r_0 - 1$ for r_0 distinct exponents at a nonzero coordinate form a Vandermonde matrix times invertible diagonal matrices. The jumps are $0, \dots, r_0 - 1$. Their sum is at most either endpoint sum; nonnegative rational fixed offsets only increase the latter. Take a Cartier multiple to make this argument for the rational divisors.

Adjunction on the coordinate curves gives

flag	different	endpoint moments	other-point bound
H_x	$\frac{1}{2}P_z + \frac{2}{3}P_w$	(13/24, 47/216)	47/216
H_z	$\frac{1}{2}P_x + \frac{2}{3}P_w$	(13/24, 37/216)	37/216
D	0	(85/72, 71/72)	71/72.

For H_x, H_z , the first moment is half the residual y exponent; the second is one third the other coordinate exponent. The quotient chart at P_w is of type $\frac{1}{3}(1, 1)$, yielding different $2/3$. These pairs are plt. For D , replace its coefficient $1/2$ by one in the flag: its primary budget is $1/2$ and its curve different is zero. We use this flag away from P_z .

Step 5. We treat P_z . In the smooth chart $z = 1$, extract B_0 with weights $(x, w) = (1, 2)$. Its discrepancy for (E_p, Δ) is $1 + 2 - \frac{1}{2} \cdot 2 = 2$. On $x = s, w = s^2Q_0$, its different at $Q_0 = 0$ is $1/2$. The other chart is the quotient $w = t^2, x = tr$ by $(t, r) \mapsto (-t, -r)$, giving different $1/2$ at infinity. Thus B_0 is a coarse $\mathbb{P}^1$ with these two boundary points. Its primary moment is $\bar{u} = 115/72$. The full secondary moments are

Q_0	0	∞	1	other
S	5/24	37/144	13/24	5/24.

They use, respectively, the lowest y exponent divided by two, $\overline{u/2 - c}$, orders $g_r, \dots, h_r$, and orders $0, \dots, h_r - g_r$. The largest ratio to its primary or secondary budget is

$$\max\{115/144, 5/12, 37/72, 13/24\} = 115/144.$$

For each tested valuation, choose a basis compatible with it and B_0 . Write its pullback as $s_r B_0 + \Theta_r$ with effective Θ_r having no B_0 component. The point bounds control $\Theta_r|_{B_0}$ uniformly. For every scale below $144/115$, the primary coefficient and all residual coefficients fit their discrepancy and different budgets in sufficiently large degree. Effective-boundary inversion of adjunction proves the bound for all valuations centered at P_z , including those that are not monomial.

Step 6. We cover the surface and transfer back. When x, z, w are nonzero, use C_τ with $\tau = w/(x^2z)$; the primary bound is $13/24$ and the secondary bound is at most $71/72$, with budgets one. Use H_x on $x = 0$ except at P_z , where Step 5 applies. The remaining H_x ratios are at most $47/72$. Use H_z for $z = 0, w \neq 0$, with ratios at most $47/72$. Finally use D for $w = 0$ away from P_z , with primary ratio $5/12$ and secondary ratio at most $71/72$. This covers the quotient vertex and all coordinate points.

For each flag use two compatible filtrations, then the effective residual boundary. The uniform $O(1/r)$ comparisons show that, for every $\lambda < 72/71$ and all sufficiently large r , every basis divisor of the complete degenerate series $\mathcal{A}'_r$ gives a log canonical pair with Δ after scaling by λ . The quantifier on all bases follows by testing each valuation with a compatible basis, which attains its maximal basis weight.

The auxiliary action $(x, y, z) \mapsto (x, ty, t^2z)$ induces an automorphism of the coarse surface preserving Δ . In each original exceptional-degree piece, take a basis compatible with a tested

valuation and the auxiliary weight filtration. Rescale each transformed polynomial by its lowest power of t . Its initial polynomials form a basis of the full space $\mathcal{A}'_r$. After a common Cartier multiple the divisor family is effective with no vertical component; all nonzero fibers are the original divisor under the action. Lemma 2.10 now yields

$$S(W; v) \leq \frac{71}{72} A_{E_p, \Delta}(v)$$

for every divisor over E_p , with the corresponding uniform finite-degree bound.

Step 7. We retain the discrepancy on X . For a basis compatible with G, E_p , write $\pi^* D_r = s_r E_p + \Theta_r$, where $s_r \rightarrow 67/12$. The preceding full-series bound and plt adjunction give, for $\lambda < 72/71$,

$$A_X(G) \geq \lambda S_r(G) + (7 - \lambda s_r) \text{ord}_G(E_p).$$

Let $r \rightarrow \infty$ and then $\lambda \rightarrow 72/71$. The resulting coefficient is $67/12 - 7(71/72) = -95/72$, proving (10.10). For $G = E_p$ the conclusion of (10.10) follows directly from its two invariants. Since every point-centered G has $\text{ord}_G(E_p) > 0$, the asserted strict inequality follows. $\square$

11. CURVILINEAR TRIPLE INCIDENCE IN DEGREE EIGHTEEN

We now assume that the entire line-incidence scheme at p is a curvilinear scheme of length three supported at one line. Its osculating contact plane gives two forms of the local equations. We treat both forms, keeping the coefficients of the original smooth threefold.

Lemma 11.1. *In the adjoint coordinates of Setup 8.1, one may arrange that $p = [h]$, the supporting line is $[h : i]$, and the two equations of X are*

$$(11.1) \quad \begin{aligned} F_1 &= n + \lambda f + \beta(g + k) + \sum_{\xi \in J} A_\xi \xi, \\ F_2 &= g + \mu f + \kappa(g + k) + \sum_{\xi \in J} B_\xi \xi, \quad J = \{c, l, b, d, m, j\}. \end{aligned}$$

If $\mu = 0$, then $\lambda \neq 0$ and $\kappa \neq -1$. In this case the weighted blowup of the parameters (i, a, e) with weights $(1, 2, 3)$ is plt, with exceptional surface $E = \mathbb{P}(1, 2, 3)$, zero different, and

$$(11.2) \quad A_X(E) = 6, \quad S_X(E) = 21/4.$$

Its entire refined ring is the bigraded ring

$$(11.3) \quad R_Y / (n + \lambda f, g + \kappa(g + k)),$$

where the section degree of each coordinate is one and their exceptional degrees are

$$(11.4) \quad h : 0, i : 1, a : 2, e : 3, n : 4, f : 4, g : 5, k : 5, c : 6, l : 6, b : 7, d : 8, m : 9, j : 10.$$

On each fixed section degree, the map obtained by setting $h = 1$ is injective, with image in $\mathbb{C}[i, a, e]$.

Proof. We normalize the actual section equations and identify the weighted refinement when $\mu = 0$.

Step 1. In this step, we obtain the normal form for every smooth member under consideration.

The noncollinear length-three incidence forces the tangent plane into the contact $\mathbb{P}^3$. Otherwise its intersection with this contact space would be a line meeting the twisted cubic in length three. Every defining quadric of the cubic would then vanish on the whole line, whereas the twisted cubic contains no line. Its intersection with the twisted cubic is a triple point, so it is the osculating plane. The SL_2 contact action is transitive on those planes. Normalize this plane to $n = 0$, with coordinates (i, a, e) . Every defining linear form then involves only $n, f, g, k, c, l, b, d, m, j$. Its linear differential is a combination of dn, dg , since $dk = -dg$ and the remaining terms have no linear part. Smoothness and row operations give (11.1).

Use the chart $i = 1$ and its exact identities (8.10). Along $[h : i]$ the two differentials are

$$\begin{aligned} dF_1 &= \lambda df + (\beta - h) dg + (\beta h + A_l - h^2) dl, \\ dF_2 &= \mu df + (1 + \kappa) dg + (\kappa h + B_l) dl. \end{aligned}$$

If $\lambda = \mu = 0$, their determinant in (dg, dl) is $h^2 - (\beta + B_l)h + \beta B_l - (1 + \kappa)A_l$. It has a complex root, contradicting smoothness at that actual point of the line. Thus $\lambda \neq 0$ when $\mu = 0$. If also $\kappa = -1$, dF_2 vanishes at $h = B_l$, again contradicting smoothness.

Step 2. In this step, we describe the weighted extraction and prove that the global leading section is integral. With weights (11.4), the leading equations for $\mu = 0$ are $n + \lambda f$ and $g + \kappa(g + k)$. They eliminate n, g from the affine chart, leaving $\mathbb{C}[i, a, e]$. They form a regular sequence. The filtered complete-intersection assertion follows directly from Koszul syzygies: a cancellation between their initial forms is a multiple of the Koszul relation, and subtraction of the exact relation raises the weight. Induction in the completed local ring proves that these generate the full initial ideal. Equivalently, weighted formal implicit substitution gives the ordinary weights $(1, 2, 3)$ on (i, a, e) . The cyclic quotient charts of that weighted blowup are plt, have no codimension-one ramification on E , and give zero different. The volume form on $i = u, a = u^2v, e = u^3w$ has factor u^5 , so the discrepancy is six.

We check the global leading section $Z = Y \cap (n + \lambda f = g + \kappa(g + k) = 0)$ is integral. Its $h = 1$ chart is $\mathbb{A}^3$. On $h = 0, i = 1$ the identities give $f = a^3/\lambda$ and $(1 + \kappa)g + \kappa a^4/\lambda = 0$. The boundary has dimension two; this remains true even for $\kappa = -1$, when its reduced support has $a = f = 0$. On $h = i = 0$, the Pfaffians successively force $a = e = f = n = g = k = c = 0$ on the reduced support. The remaining coordinates satisfy

$$ld + b^2 = 0, \quad lm + bd = 0, \quad bm - d^2 = 0.$$

This is the two-dimensional cone over the twisted cubic $(l, b, d, m) = (s^3, s^2t, -st^2, t^3)$ with independent coordinate j . Lemma 8.2 gives pure dimension three and the Cohen–Macaulay property. Every component therefore meets the integral h -chart; generic reducedness and absence of embedded associated primes prove integrality.

Step 3. In this step, we identify the complete graded ring and compute its first moment. The coordinate weights arise from the adjoint cocharacter whose weights on the seven matrix indices are $(-3, 2, 1, 3, -2, -1, 0)$. It preserves the defining four-form: every term has weight zero. Multiply terms of F_1 of weight $r > 4$ by t^{r-4} , and terms of F_2 of weight $r > 5$ by t^{r-5} . The leading equations remain independent for every t . Lemma 8.2 gives a flat projective family with all nonzero fibers isomorphic to the fixed X , and section base change in every nonnegative degree.

The equivariant Koszul sequence identifies its central section ring with (11.3). Degree-one generation follows from the Cartan multiplication description of the adjoint section ring used above. Lift a homogeneous central eigenbasis in every section degree. The equivariant determinant of these lifts is nonzero at zero; its zero locus is invariant, and the punctured base is one orbit. It is consequently empty. In the h -chart each lift of weight j is a sum of terms t^k whose variable weights are $j + k$. Its initial local function is its nonzero central function. These are independent by integrality and density of the chart. Thus the lifts are adapted to the actual valuation filtration, and multiplication agrees as well. This proves the entire ring assertion, not only its degree-one part.

Let $h_Y(r) = h^0(Y, rH_Y)$, with value zero for $r < 0$. The equivariant character sum on $H^0(Y, rH_Y)$ for the natural G_2 linearization is zero. Relative to h , the total local weight is $5rh_Y(r)$. Applying the Koszul sequence with equation weights 4, 5 gives

$$\sum_j j \dim W_{r,j} = 5rN_r + h_Y(r-1) - h_Y(r-2).$$

Since $h_Y(r) = \frac{3}{20}r^5 + O(r^4)$ and $N_r = 3r^3 + O(r^2)$, division by rN_r yields $21/4$. $\square$

For the rest of the $\mu = 0$ case, write $q = [1 : 0 : 0]$ on E . In the chart $i = 1$, put $f = e - a^2$ and $K = ae - \lambda f$. Write M, J_0 for the original coordinates m, j . The full ring of Lemma 11.1 is generated by the following sections, with their exceptional degrees:

$$(11.5) \quad (1, 0), (1, 1), (a, 2), (e, 3), (f, 4), (K, 5), (c, 6), (l, 6), (b, 7), (d, 8), (M, 9), (J_0, 10),$$

where

$$(11.6) \quad \begin{aligned} c &= e^2 - \lambda af, & l &= -af - (\kappa + 1)K, \\ b &= -\kappa aK - c, & d &= \kappa eK + ac, \\ M &= -ec - \kappa \lambda fK, & J_0 &= fc - \kappa(\kappa + 1)K^2. \end{aligned}$$

The first two constants retain their different exceptional degrees. Every degree- r polynomial here has degree $O(r)$ after each fixed coordinate change used below.

Lemma 11.2. *For every prime divisor v over E whose center contains q ,*

$$(11.7) \quad S(W; v) \leq \frac{23}{24} A_E(v).$$

For the weighted valuation ν_3 at q with weights $(1, 3)$ in (a, f) , we obtain

$$(11.8) \quad S(W; \nu_3) = \begin{cases} 13/4, & \kappa \neq 0, \\ 10/3, & \kappa = 0. \end{cases}$$

Proof. We use a weighted extraction at q in each coefficient case and estimate its complete refinement.

Step 1. In this step, we construct the two weighted extractions and compute their semigroup polytopes.

Suppose first $\kappa \neq 0, -1$. Put $z = \lambda f - a^3$, and extract B with weights $(a, z) = (1, 3)$. Its discrepancy is four, and its coarse exceptional curve is $\mathbb{P}^1$ with coordinate z/a^3 and different $\frac{2}{3}p_\infty$. Denote its zero by p_* . Filter the generators first by this valuation and then by order at p_* . Their triples (j, u, v) , recording exceptional degree and exponents of a, z , are

$$\begin{aligned} &(0, 0, 0), (1, 0, 0), (2, 1, 0), (3, 2, 0), (4, 3, 0), (5, 0, 1), \\ &(6, 1, 1), (6, 0, 1), (7, 1, 1), (8, 2, 1), (9, 3, 1), (10, 0, 2). \end{aligned}$$

The required primary leading forms are

$$\begin{aligned} e &: a^2, & f &: (a^3 + z)/\lambda, & K &: -z, & c &: -az, & l &: (\kappa + 1)z, & b &: (\kappa + 1)az, \\ d &: -(\kappa + 1)a^2z, & M &: ((\kappa + 1)a^3 + \kappa z)z, & J_0 &: -\kappa(\kappa + 1)z^2. \end{aligned}$$

All specified secondary coefficients are nonzero. There exists also the actual section of bidegree $(3, 18)$

$$\mathcal{P}_3 = hM^2 + 2(\kappa + 1)\lambda MKf - \frac{\kappa + 1}{\kappa}\lambda^2 J_0 f^2.$$

Its primary leading form is z^4 , because substitution gives

$$[(a^3 + \kappa \lambda f)^2 - 2(\kappa + 1)\lambda f(a^3 + \kappa \lambda f) + (\kappa + 1)^2 \lambda^2 f^2]z^2 = (a^3 - \lambda f)^2 z^2.$$

The full lattice semigroup of these tuples has normalized polytope with vertices

$$\begin{aligned} v_0 &= (0, 0, 0), & v_1 &= (1, 0, 0), & v_2 &= (4, 3, 0), \\ v_3 &= (6, 0, 4/3), & v_4 &= (6, 0, 1), \\ v_5 &= (9, 3, 1), & v_6 &= (10, 0, 2). \end{aligned}$$

In coordinates (j, u, v) its facets are

$$\begin{aligned} u, v &\geq 0, & -9j + 14u + 45v &\leq 6, & -6j + 8u + 27v &\leq 0, & -3j + 5u + 18v &\leq 6, \\ j - u - 5v &\leq 1, & j - u - 4v &\leq 2. \end{aligned}$$

The following tetrahedra, coned from v_0 , give a complete integration certificate:

other vertices	vol	$\int u$	$\int v$
(5, 3, 6)	2/3	1/2	13/18
(5, 3, 2)	1/3	1/2	7/36
(4, 5, 6)	1	3/4	1
(4, 1, 2)	1/2	3/8	1/8
(4, 5, 2)	1/2	3/4	1/4

The facets show disjoint interiors and coverage. Each volume is the absolute determinant divided by six; each linear integral is the volume times the vertex average. Hence the volume is three and the moments are $23/8, 55/24$.

For $\kappa = 0$, instead extract B with weights $(a, f) = (1, 2)$. Its discrepancy is three and its different is $\frac{1}{2}p_\infty$. The triples are now

$$(0, 0, 0), (1, 0, 0), (2, 1, 0), (3, 2, 0), (4, 0, 1), (5, 0, 1), \\ (6, 1, 1), (6, 0, 1), (7, 1, 1), (8, 2, 1), (9, 3, 1), (10, 1, 2).$$

For example the leading forms of e, K, c, l, b, d, M, J_0 are $a^2 + f, -\lambda f, -\lambda a f, \lambda f, \lambda a f, -\lambda a^2 f, \lambda a f(a^2 + f)$, and $-\lambda a f^2$. Their secondary orders are the f orders. The vertices of the normalized polytope are

$$v_0 = (0, 0, 0), \quad v_1 = (1, 0, 0), \quad v_2 = (3, 2, 0), \\ v_3 = (4, 0, 1), \quad v_4 = (6, 0, 1), \\ v_5 = (9, 3, 1), \quad v_6 = (10, 1, 2).$$

Its facets are

$$u, v \geq 0, \quad -u + v \leq 1, \quad -3j + 5u + 13v \leq 1, \quad -2j + 3u + 8v \leq 0, \\ j - u - 5v \leq 1, \quad j - u - 3v \leq 3,$$

and its tetrahedron certificate is

(4, 5, 6)	3/2	3/2	3/2
(3, 4, 6)	1/3	1/12	1/3
(2, 4, 1)	1/3	1/6	1/12
(2, 4, 5)	1/2	5/8	1/4
(3, 2, 5)	1/6	5/24	1/12
(3, 5, 6)	1/6	1/6	1/6

with the columns just specified. Its volume is three and moments are $11/4, 29/12$.

Step 2. In this step, we recover the complete ranks and bound the secondary orders. In both cases the generators span the full lattice in (r, j, u, v) ; the tuples of h, i, a, K already supply an integral triangular basis. Distinct product tuples give independent sections adapted to the two successive filtrations. Lemma 10.4 gives $3r^3 + O(r^2)$ such products. The full space has dimension $N_r = 3r^3 + O(r^2)$. All primary and secondary orders are $O(r)$ by the polynomial degree bound. Lemma 2.9 then computes the complete moments, with error $O(1/r)$:

	$S(W; B)$	$S(T; p_*)$
$\kappa \neq 0, -1$	13/4	55/72
$\kappa = 0$	91/36	29/36.

Here T is the actual refinement on B . This also proves the first value of (11.8), since the change $z = \lambda f - a^3$ is invertible and homogeneous for weight $(1, 3)$. When $\kappa = 0$, use weight $(1, 3)$ in (a, f) and minimum a exponent for ties. The second displayed list of exponent triples remains leading: K has leading form $a^3 - \lambda f$, c has leading form $a^4 - \lambda a f$, and M has secondary leading term $a^3 f$. The polytope for this second list therefore gives $(11/4 + 3(29/12))/3 = 10/3$.

A residual section of B -order k has rational divisor degree k/s on B , where $s = 3$ or 2 is the second blowup weight. This includes its fractional order at infinity. Choosing a basis compatible with p_* and any other point gives the finite-level inequality

$$S_r(T; p) + S_r(T; p_*) \leq S_r(W; B)/s.$$

Consequently the moment at any $p \neq p_*$ is at most $23/72 + O(1/r)$ in the first case, and $11/24 + O(1/r)$ in the second. Their infinity budgets are $1/3$ and $1/2$, respectively; all other budgets are one. For any scale below $24/23$ in the first case or $12/11$ in the second, every residual basis divisor is log canonical with the different for all sufficiently large r , uniformly in its point.

Step 3. In this step, we apply surface adjunction while retaining the extraction coefficient. Finally choose bases compatible with a tested surface valuation and B , and remove the whole B coefficient from their pullbacks. The residual boundary is effective. Its restriction is the actual series T , so adjunction applies. The limiting primary products are $78/23 < 4$ and $91/33 < 3$; hence the actual pullback coefficient of B is at most one. This proves (11.7), with the stronger coefficient $11/12$ when $\kappa = 0$. The finite-degree argument is uniform over compatible bases. Testing maximal basis weights gives this local log canonicity for every basis divisor of W . $\square$

Lemma 11.3. *Let $C_i = (i = 0)$ and $p_2 = [0 : 1 : 0]$. Every divisor whose center meets $(E \setminus C_i) \cup \{p_2\}$ satisfies*

$$(11.9) \quad S(W; v) \leq \frac{47}{48} A_E(v).$$

This includes every prime curve on E .

Proof. We use ν_3 from Lemma 11.2. For effective divisors A, C without common component on the smooth chart of q ,

$$(A \cdot C)_q \geq \frac{1}{3} \nu_3(A) \nu_3(C).$$

To prove this, make the finite flat substitution $a = s, f = t^3$. Intersection length is multiplied by three, since the resulting power-series ring is free with basis $1, t, t^2$. The multiplicities of the pulled-back equations are their weighted orders. Blowing up the origin shows that the intersection of two plane curves is at least the product of their multiplicities. No common component is introduced by the finite map. Division by three proves the inequality, also for rational divisors.

Let $C \sim d\mathcal{O}_E(1)$ be prime and smooth at q , put $t = \nu_3(C)$, and write $s = S(W; C)$. Set

$$e_r = \frac{\sum_j j \dim W_{r,j}}{r N_r}, \quad r_r = S_r(W; \nu_3), \quad s_r = S_r(W; C).$$

Choose a basis compatible with C, ν_3 and write its divisor as $s_r C + \Theta_r$. The intersection inequality gives $S_r(T_C; q) \geq (t/3)(r_r - t s_r)$. Separately choose bases of T_C compatible with the two points q, p on its normalization. Since $\mathcal{O}_E(1)^2 = 1/6$,

$$S_r(T_C; p) + S_r(T_C; q) \leq d(e_r - d s_r)/6.$$

As $e_r \rightarrow 21/4$ and $r_r \geq 13/4 + o(1)$, it follows that

$$(11.10) \quad \limsup S_r(T_C; p) \leq \frac{7d}{8} - \frac{13t}{12} + \left(\frac{t^2}{3} - \frac{d^2}{6} \right) s \quad (p \neq q).$$

The finite-level bounds are uniform in p on each chosen curve. The two choices of compatible bases are separate.

For $\tau \neq 0$, let $C_\tau = (ae - \tau i f = 0)$. On $i = 1$ it has equation $(a - \tau)e + \tau a^2 = 0$. It is integral and smooth there: $a = \tau$ is impossible, and the derivative in e is $a - \tau$. It has no i component. Here $(d, t) = (5, 3)$. The marked-point bound gives $s \leq 23/24$. The actual degree-two section

$hK + (\lambda - \tau)if$ restricts to the defining equation of C_τ with multiplicity one. Multiplication by its k th power injects the full degree $r - 2k$ space into the C_τ -order-at-least- k space. Thus

$$s \geq \lim_{r \rightarrow \infty} \frac{\sum_{k \geq 1} N_{r-2k}}{rN_r} = \int_0^{1/2} (1-2u)^3 du = \frac{1}{8}.$$

(11.10) now gives $S(T_{C_\tau}; p) \leq 9/8 - 7s/6 \leq 47/48$ away from q .

Use also the integral curves $C_a = (a = 0)$, $C_e = (e = 0)$, and $C_f = (ie - a^2 = 0)$. They are smooth at q and have $(d, t) = (2, 1), (3, 2), (4, 3)$. The last curve is the coarse $\mathbb{P}(1, 3)$ parametrized by $[i : a : e] = [U^2 : UV : V^2]$, with polarization $\mathcal{O}(2)$. Their primary coefficients lie in $[0, 23/24]$. The respective secondary bounds from (11.10) are

$$2/3 - s/3 \leq 2/3, \quad 11/24 - s/6 \leq 11/24, \quad 1/4 + s/3 \leq 41/72.$$

The curve $C_e = \mathbb{P}(1, 2)$ has different $\frac{1}{2}p_2$ in its plt inclusion. This follows from the smooth index cover at p_2 , where the quotient weights on (i, e) are $(1, 1)$. Its bound $11/24$ fits the budget $1/2$.

Every point of $i = 1$ other than q is on one of C_a, C_e, C_f if $aef = 0$, and otherwise on $C_{ae/f}$. These flags are smooth at the tested point. For a scale $c < 48/47$, their primary coefficients fit budget one, since $(48/47)(23/24) = 46/47 < 1$. The secondary bounds fit all point budgets, including $(48/47)(11/24) = 22/47 < 1/2$ at p_2 . Use a basis compatible with the tested valuation and the chosen curve, remove its whole coefficient, and apply effective-boundary adjunction. This proves (11.9). The stronger marked-point bound covers q . The estimates also give local log canonicity for every basis divisor at each fixed point in sufficiently large degree, by maximality of compatible basis weights. Every prime curve meets $i \neq 0$, unless it is C_i itself; that curve contains p_2 . $\square$

Proposition 11.4. *In the case $\mu = 0$ of Lemma 11.1, every divisor v over E satisfies $S(W; v) \leq (47/48)A_E(v)$. Consequently $\delta_p(X) \geq 48/47$, and every G centered at p satisfies*

$$(11.11) \quad S_X(G) \leq \frac{47}{48}A_X(G) - \frac{5}{8}\text{ord}_G(E).$$

Proof. We treat the centers left by Lemma 11.3 and then apply threefold adjunction.

Step 1. In this step, we bound the remaining smooth centers and construct an extraction at the quotient point.

The only centers left by Lemma 11.3 are closed points of C_i other than p_2 . For a smooth point of C_i , use C_i as flag. Its coefficient s_r tends to a number at most $47/48$. The total residual degree is $(e_r - s_r)/6 \leq e_r/6 \rightarrow 7/8$. Thus, at scale below $48/47$, the primary and secondary coefficients fit budget one for large degree, uniformly in the smooth point. Effective-boundary adjunction handles all these centers.

It remains to treat $p_3 = [0 : 0 : 1]$. Normalize $\lambda = 1$ by substituting $a = \lambda a'$, $e = \lambda^2 e'$. The generators f, K, c, l, b, d, M, J_0 change by the nonzero factors $\lambda^2, \lambda^3, \lambda^4, \lambda^3, \lambda^4, \lambda^5, \lambda^6, \lambda^6$, respectively. This preserves their exceptional degrees and the valuation at p_3 . Drop the primes. Homogeneously,

$$\begin{aligned} f &= ie - a^2, & K &= ae - if, & c &= e^2 - af, & l &= -af - (\kappa + 1)iK, \\ b &= -\kappa aK - ic, & d &= \kappa eK + ac, & M &= -ec - \kappa fK, & J_0 &= fc - \kappa(\kappa + 1)K^2. \end{aligned}$$

These, together with $h = 1, i, a, e$, generate the full series.

The chart at p_3 is the quotient of (i, a) , $e = 1$, by μ_3 with weights $(1, 2)$. Subdivide its lattice $\mathbb{Z}^2 + \mathbb{Z}(1/3, 2/3)$ by the primitive ray $(1/3, 2/3)$. The resulting plt extraction has exceptional curve B and valuation ν with

$$A_E(B) = 1, \quad B^2 = -3/2, \quad \text{Diff}_B(0) = \frac{1}{2}p_\infty.$$

The two cone determinants are one and two. Equivalently, the cover has weighted blowup $(1, 2)$ with self-intersection $-1/2$, transverse ramification three, and degree one on its exceptional curve. This gives the displayed square. The coarse coordinate is a/i^2 ; denote its zero by p_0 .

For a homogeneous polynomial, ν is one third of the smallest $u + 2v$ of its monomials $i^u a^v e^w$. Refine by smallest u among ties. Since $j = u + 2v + 3w$ is fixed, the resulting order has one-dimensional leaves and is multiplicative. The primary leading forms used below are

$$\begin{aligned} f &: ie, & K &: e(a - i^2), & c &: e^2, & l &: ie[(\kappa + 1)i^2 - (\kappa + 2)a], \\ b &: -ie^2, & d &: e^2[(\kappa + 1)a - \kappa i^2], & M &: -e^3, & J_0 &: ie^3. \end{aligned}$$

The secondary leading monomial of l is iae if $\kappa \neq -2$, and i^3e if $\kappa = -2$. The other required monomials do not change.

Step 2. In this step, we construct actual sections and compute the limiting semigroup moment. We need two families of actual sections. Put $P_1 = -b$, $P_2 = lM - (\kappa + 2)KJ_0$, and, for $m \geq 2$,

$$P_{m+1} = (\kappa + 2)dP_m + (\kappa + 1)lJ_0P_{m-1}.$$

Then P_m has bidegree $(m, 8m - 1)$ and primary leading form $i^{2m-1}e^{2m}$. For P_2 , the iae^4 terms cancel and the remaining coefficient is $-(\kappa + 1) + (\kappa + 2) = 1$. In the recurrence the a terms cancel and the remaining coefficient is $(\kappa + 1)^2 - \kappa(\kappa + 2) = 1$. All omitted terms have larger primary order, proving the induction. The normalized exponents approach $(2, 0, 2)$.

For the second family set

$$\begin{aligned} U &= -il, & V &= aK, & L_0 &= \frac{\kappa U - (\kappa + 2)^2 V}{\kappa + 1}, \\ Q_0 &= i, & Q_1 &= \frac{i^2 l + (\kappa + 2)(a^2 f - iaK)}{\kappa + 1}, & Q_{r+1} &= L_0 Q_r + \frac{U^2 Q_{r-1}}{\kappa + 1}. \end{aligned}$$

All denominators are nonzero by Lemma 11.1. The bidegree of Q_r is $(2r + 1, 7r + 1)$ and its primary form is $i^{4r+1}e^r$. For Q_1 , use the exact identity $a^2 f - iaK = -a^4 + i^2 a f$. For the induction, write $t = i^4 e$ and $z = a/i^2$ only for checking the primary forms. Then

$$U_0 = t[(\kappa + 2)z - (\kappa + 1)], \quad V_0 = t(z^2 - z), \quad t^2 - (L_0)_0 t - U_0^2/(\kappa + 1) = 0.$$

This verifies the recurrence and its nonzero leading term. The sections themselves are the polynomial expressions above. Their normalized exponents approach $(2, 0, 1/2)$.

If $\kappa \neq -2$, the convex hull of the degree-one leading exponents and these two limits has vertices

$$\begin{aligned} v_0 &= (0, 0, 0), & v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (2, 0, 1/2), & v_4 &= (1, 1, 1), \\ v_5 &= (2, 0, 2), & v_6 &= (0, 1, 2), & v_7 &= (0, 0, 3), & v_8 &= (1, 0, 3). \end{aligned}$$

It is defined by

$$\begin{aligned} u, v, w &\geq 0, & v &\leq 1, & v + w &\leq 3, & u + v - 2w &\leq 1, & u + v &\leq 2, \\ u + 2v + w &\leq 4, & 2u + 3v - 2w &\leq 3. \end{aligned}$$

All the displayed section exponents lie in it. The full tetrahedron certificate, coned from v_0 , is

other vertices	vol	$\int u$	$\int v$	$\int w$
(4, 2, 3)	1/4	3/16	1/8	3/32
(1, 2, 3)	1/12	1/16	1/48	1/96
(5, 4, 3)	1/2	5/8	1/8	7/16
(6, 4, 2)	1/3	1/12	1/4	1/4
(8, 6, 7)	1/2	1/8	1/8	1
(8, 5, 4)	2/3	2/3	1/6	1
(8, 6, 4)	2/3	1/3	1/3	1

The facets give disjoint interiors and coverage, and determinants and vertex averages verify the entries. Thus its volume is three and $\int u = 25/12$, $\int v = 55/48$, $\int w = 91/24$.

For each fixed R , use only the degree-one generators and $P_1, \dots, P_R, Q_0, \dots, Q_R$. Their full lattice semigroup has polytope $\mathcal{P}_R$. These polytopes increase to the displayed polytope $\mathcal{P}$, including their volumes and linear moments. Lemma 10.4 counts independent products in

each fixed $\mathcal{P}_R$, preserving every exceptional degree. All full-series ν orders lie between zero and $10m/3$. The missing dimensions can therefore contribute at most that bound times their number. Consequently

$$\frac{1}{3} \int_{\mathcal{P}_R} \frac{u+2v}{3} \leq S(W; \nu) \leq \frac{1}{3} \int_{\mathcal{P}_R} \frac{u+2v}{3} + \frac{10}{9} (3 - \text{vol}(\mathcal{P}_R)).$$

Letting $R \rightarrow \infty$ proves $S(W; \nu) = 35/72$. No finite generation assertion about the full leading algebra is needed here.

If $\kappa = -2$, the degree-one exponents alone have polytope with vertices

$$\begin{aligned} w_0 &= (0, 0, 0), & w_1 &= (1, 0, 0), & w_2 &= (0, 1, 0), \\ w_3 &= (3, 0, 1), & w_4 &= (0, 1, 2), \\ w_5 &= (0, 0, 3), & w_6 &= (1, 0, 3). \end{aligned}$$

Its facets are $u, v, w \geq 0$, $v + w \leq 3$, $u + v - 2w \leq 1$, $u + 2v + w \leq 4$, and $u + 3v \leq 3$. The certificate is

$$\begin{array}{c|cccc} (4, 2, 3) & 1 & 3/4 & 1/2 & 3/4 \\ (1, 2, 3) & 1/6 & 1/6 & 1/24 & 1/24 \\ (6, 4, 5) & 1/2 & 1/8 & 1/8 & 1 \\ (6, 4, 3) & 4/3 & 4/3 & 1/3 & 2 \end{array}$$

with the columns just specified. Its volume is three, $\int u = 19/8$, $\int v = 1$, and $\int w = 91/24$. The full lattice count leaves only $O(m^2)$ missing dimensions, so it also gives the moment $(19/8 + 2)/9 = 35/72$.

Step 3. In this step, we apply adjunction on the quotient extraction and then on the threefold. Let T be the actual refinement on B . Its total residual rational degree is $\frac{3}{2}S_m(W; \nu) \rightarrow 35/48$. Multiplication by powers of the actual degree-one section a gives

$$S(W; C_a) \geq \lim_{m \rightarrow \infty} \frac{\sum_{\ell \geq 1} N_{m-\ell}}{mN_m} = \frac{1}{4}.$$

On the extraction, the strict transform of C_a meets B transversely at p_0 . A basis compatible with ν, C_a therefore gives $S_m(T; p_0) \geq S_m(W; C_a)$, also for semi-invariant sections, whose residual divisors retain their rational orders. Separately compare the point orders at p_0 and any $p \neq p_0$. Their sum is at most the residual degree. It follows uniformly that

$$\limsup S_m(T; p) \leq 23/48 \quad (p \neq p_0), \quad \limsup S_m(T; p_0) \leq 35/48.$$

The infinity budget is $1/2$, while all other budgets are one. At every scale below $24/23$, these inequalities make every residual basis divisor log canonical with the different in sufficiently large degree. The primary coefficient also fits: $(24/23)(35/72) = 35/69 < 1 = A_E(B)$. Effective-boundary adjunction on this plt extraction proves $S(W; v) \leq (23/24)A_E(v)$ for every center containing p_3 .

All points and prime curves of E are now covered with coefficient $47/48$. The preceding finite-degree arguments are uniform over all basis divisors at each fixed tested point. Choose a basis of $H^0(X, mH)$ compatible with G, E and write its pullback as $S_m(E)E + \Theta_m$. Its residual restriction is an actual basis divisor of W . The zero different and plt adjunction give

$$A_X(G) \geq cS_m(G) + (6 - cS_m(E)) \text{ord}_G(E), \quad c < 48/47.$$

Pass to the limits, using $S_X(E) = 21/4$. The coefficient $21/4 - 6(47/48)$ is $-5/8$. This proves (11.11); the divisor $G = E$ is included directly by its two invariants. For the local threshold defined using centers containing p , apply [AZ22, Theorem 3.2, (3.2)] with $Z = \{p\}$, the plt divisor E , its zero different, and the complete actual refinement. The all-surface bound controls every term of its infimum, giving $\delta_p(X) \geq \min\{6/(21/4), 48/47\} = 48/47$. $\square$

Lemma 11.5. *Suppose $\mu \neq 0$ in (11.1). Let E_0 be the exceptional divisor of the regular-parameter weighted blowup (i, a, e) with weights $(1, 2, 3)$. This extraction is plt, has zero different, and satisfies*

$$(11.12) \quad E_0 = \mathbb{P}(1, 2, 3), \quad A_X(E_0) = 6, \quad S_X(E_0) = 59/12.$$

Put

$$(11.13) \quad \begin{aligned} \eta &= 2\lambda - \mu\beta + \lambda\kappa, \\ \theta &= \lambda\mu\beta - \lambda^2\kappa - \lambda^2 + \mu^2A_l - \lambda\mu B_l, \\ T_1 &= e^3 - \eta a e f - \theta i f^2, \quad f = i e - a^2. \end{aligned}$$

The entire associated graded section ring is

$$(11.14) \quad \mathbb{C}[t, ti, ta, te, tf, tae, tif, te^2, taf, tef, tf^2, tT_1] \subseteq \mathbb{C}[t, i, a, e],$$

where t has section degree one and exceptional degree zero. The auxiliary highest-weight filtration $(-1, 0, 1)$ on (i, a, e) preserves both gradings and has full associated graded ring (11.14) with $\eta = \theta = 0$.

Proof. The weight- $(1, 2, 3)$ blowup of these regular parameters has plt quotient charts, discrepancy six, and zero different. We identify its complete series directly.

Step 1. In this step, we compute the actual initial sections.

The implicit equations for n, g in (11.1) give, with subscripts denoting regular-parameter weight,

$$\begin{aligned} n_4 &= -\lambda f, \quad g_4 = -\mu f, \quad Q_5 = ae - \lambda i f, \quad l_5 = \mu i f, \\ n_5 &= \beta Q_5 - \mu A_l i f, \quad g_5 = \kappa Q_5 - \mu B_l i f, \quad Q = ae + in = -(g + k). \end{aligned}$$

All occurrences of n, g beyond their identity linear coefficients carry positive-weight factors, so recursive substitution proves these initial terms for all coefficients of the original member. Only l among c, l, b, d, m, j has weight five.

The sections h, i, a, e, f give $1, i, a, e, f$. Since $\mu \neq 0$, $g + k, l$ span the two initials ae, if . Also

$$c_6 = e^2 - \lambda a f, \quad b_6 = \mu a f, \quad d_7 = -\mu e f, \quad (J_0)_8 = -\mu^2 f^2, \quad M_8 = \lambda \mu f^2.$$

They supply e^2, af, ef, f^2 . Using the exact identity $J_0 = fc - g^2 - gQ$, one obtains

$$M + (\lambda/\mu)J_0 = -ec + g(n - (\lambda/\mu)g) + (\lambda/\mu)fc - (\lambda/\mu)gQ.$$

Its weight-nine part is $-T_1$ with (11.13). Its e^3 coefficient is -1 . These twelve initials are independent in their weights $0, 1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 9$ and give an inclusion of the displayed algebra into the actual graded ring.

Step 2. In this step, we count the monomials and prove completeness in every degree. Use highest auxiliary weights $(0, 4, 7)$. The leading monomials of f, T_1 are $-a^2, e^3$, since the terms of T_1 have weights $21, 19, 16$. Thus product initials contain the degree-one monomial list

$$(11.15) \quad 1, i, a, e, a^2, ae, ia^2, e^2, a^3, ea^2, a^4, e^3.$$

We count all degree- m monomials exactly. Fix the i exponent x , put $q = m - x$, and write $z = 3h + r$, $0 \leq r \leq 2$, for the e exponent contributed by the q factors without i . For $q \geq 1$ and $0 \leq h \leq q - 1$, their a exponents are every integer

$$0 \leq y \leq 4q - 4h - 2r.$$

For $z = 3q$ only $y = 0$ occurs. The bound follows by maximizing the number of e^3 factors: the capacity is $4q - 2z + 2\lfloor z/3 \rfloor$. It is attained by h factors e^3 , one factor of degree r if $r > 0$, and the remaining degree-zero factors. Their a exponents range in complete intervals $[0, 4]$ for e -degree zero and $[0, 2]$ for e -degree one, which proves every intermediate exponent occurs.

The x factors i or ia^2 add every even integer in $[0, 2x]$. If the previous capacity is positive, the translated intervals cover all integers up to $2x + 4q - 4h - 2r$. If it is zero, only the even values occur. The zero-capacity slices are $z = 3q - 1, 3q$. For fixed $q \geq 1$ the count is therefore

$$\sum_{h=0}^{q-1} (6x + 12q - 12h - 3) + (2x + 1) - 2x = (6m + 3)q + 1.$$

For $q = 0$ the count is $m + 1$. The total is

$$(m + 1) + \sum_{q=1}^m ((6m + 3)q + 1) = 3m^3 + \frac{9}{2}m^2 + \frac{7}{2}m + 1 = N_m.$$

The independent product symbols already attain the whole actual dimension. Equality follows in every exceptional degree as well as every section degree, proving (11.14).

Step 3. In this step, we compute the first moment and identify the full auxiliary degeneration. The normalized exponent body of (11.15) is

$$x, y, z \geq 0, \quad 3x + z \leq 3, \quad 6x + 3y + 4z \leq 12.$$

The slice description differs from all its lattice points only in bounded-width boundary layers, including the two parity slices. Their count is $O(m^2)$ and their total weight is $O(m^3)$. Use the iterated ranges $0 \leq x \leq 1$, $0 \leq z \leq 3(1 - x)$, $0 \leq y \leq 4 - 2x - 4z/3$. The volume and coordinate integrals are

$$\text{vol} = 3, \quad \int x = 1, \quad \int y = 7/2, \quad \int z = 9/4.$$

For example the slice area is $6(1 - x)$; its z moment is $(6 + 3x)(1 - x)^2$, and its y moment is $6(1 - x) + 2(1 - x)^3$. The original exceptional weight is $x + 2y + 3z$, giving $S_X(E_0) = (1 + 7 + 27/4)/3 = 59/12$.

Finally, under highest auxiliary weights $(-1, 0, 1)$ the polynomial f is an eigenvector of weight zero and all generators except T_1 are eigenvectors. The three terms of T_1 have weights $3, 1, -1$. Thus the full auxiliary graded ring contains the algebra with $\eta = \theta = 0$. The count just proved gives dimension N_m for both rings, with the exceptional grading preserved. The inclusion is therefore equality in every bidegree. $\square$

Proposition 11.6. *In the case $\mu \neq 0$, every divisor over E_0 satisfies $S(W_0; v) \leq (8/9)A_{E_0}(v)$. For every G with $p \in c_X(G)$,*

$$(11.16) \quad S_X(G) \leq \frac{8}{9}A_X(G) - \frac{5}{12}\text{ord}_G(E_0).$$

In particular $\delta_p(X) \geq 9/8$.

Proof. We estimate eigenbasis divisors on the auxiliary limit and transfer the bounds to the actual refinement.

Step 1. In this step, we describe complete binary blocks containing the eigenweight spaces.

We first work with the auxiliary limit of Lemma 11.5, namely

$$\mathcal{B}_0 = \mathbb{C}[t, ti, ta, te, tf, tae, tif, te^2, taf, tef, tf^2, te^3].$$

The relation $a^2 = ie - f$ makes $\mathbb{C}[i, a, e]$ a free rank-two module over $\mathbb{C}[i, e, f]$ with basis $1, a$. Every degree- m product in $\mathcal{B}_0$ lies in the span of

$$i^p a^\epsilon e^q f^c, \quad p, q, c \geq 0, \quad \epsilon \in \{0, 1\},$$

subject to

$$(11.17) \quad 3p + q + 2\epsilon \leq 3m, \quad 3p + 2q + 3c + 3\epsilon \leq 6m.$$

Check these inequalities on the twelve generators. They add under products. Replacing two a factors by ie leaves the first left side unchanged and decreases the second by one; replacement by $-f$ decreases them by four and three. This proves the containing assertion by induction.

Put $P = p + c$, $r = q + c$. The exceptional and auxiliary weights are $j = P + 3r + 2\epsilon$, $\ell = r - P$. For fixed j, ℓ , at most one ϵ is possible, since $j + \ell = 4r + 2\epsilon$. The allowed c form the complete integer interval

$$L \leq c \leq U, \quad L = \max \left\{ 0, \left\lceil \frac{3P + r + 2\epsilon - 3m}{4} \right\rceil, \left\lceil \frac{3P + 2r + 3\epsilon - 6m}{2} \right\rceil \right\}, \quad U = \min\{P, r\}.$$

When nonempty, the row is exactly

$$(11.18) \quad a^\epsilon i^{P-U} e^{r-U} f^L H^0(\mathbb{P}^1, \mathcal{O}(U-L))|_{(s,t)=(ie,f)}.$$

The monomial $s^{U-c} t^{c-L}$ gives the required basis element. Independence follows from the free-module description. Each actual simultaneous eigenweight space is a subspace of this row.

Let $I = (i = 0)$, $A = (a = 0)$, $Q = (e = 0)$, $C_0 = (f = 0)$, and $C_\zeta = (f - \zeta ie = 0)$ for $\zeta \neq 0, 1$. Every auxiliary eigenvector factors into these curves. The missing value $\zeta = 1$ gives $a^2 = 0$, and the infinite parameter gives $I + Q$. For a row of dimension $n = U - L + 1$, its maximal component totals are

$$\begin{array}{l|l} I & (P - U)n + n(n - 1)/2 \\ Q & (r - U)n + n(n - 1)/2 \\ A & \epsilon n + n(n - 1) \\ C_0 & Ln + n(n - 1)/2 \\ C_\zeta & n(n - 1)/2. \end{array}$$

These follow from the complete binary vanishing sequence; a subspace has no larger total order, including the nonnegative fixed coefficient. The factor two for A retains $f - ie = -a^2$.

At $p_i = [1 : 0 : 0]$, the weight- $(1, 2)$ order on (a, e) is $\epsilon + 2r$ on every element of a simultaneous eigenweight row. At $p_e = [0 : 0 : 1]$, use its index-three cover with (i, a) ; the weight- $(2, 1)$ order there is $2P + \epsilon$. Both cover discrepancies are three. These nonnegative orders may also be bounded with the containing row dimension.

Step 2. In this step, we compute the component and weighted-order bounds from six integration cells. Scale P, r by m and set

$$L_0 = \max\{0, (3P + r - 3)/4, (3P + 2r - 6)/2\}, \quad U_0 = \min\{P, r\}, \quad H_0 = U_0 - L_0.$$

The domain $H_0 \geq 0$ lies in $[0, 2] \times [0, 3]$. The two parity choices give conversion factor $2/3$ after division by the original mN_m . The integrals in $dP dr$ are

$$(11.19) \quad \begin{aligned} \int H_0 &= 3/2, & \int H_0^2/2 &= 45/128, & \int (U_0^2 - L_0^2)/2 &= 7/8, \\ \int ((P - U_0)H_0 + H_0^2/2) &= 1/2, & \int ((r - U_0)H_0 + H_0^2/2) &= 9/8, \\ \int 2rH_0 &= 4, & \int 2PH_0 &= 11/4. \end{aligned}$$

Here is the full rational certificate. Each last entry lists the six integrals $H_0, H_0^2/2, (U_0^2 - L_0^2)/2, (P - U_0)H_0 + H_0^2/2, (r - U_0)H_0 + H_0^2/2, 2rH_0$. The selected affine bounds give disjoint interiors and cover the whole domain.

(L_0, U_0)	Vertices (P, r)	Integral tuple
$(0, P)$	$(0, 0), (3/4, 3/4), (0, 3)$	$(9/32, 27/512, 27/512, 27/512, 135/512, 81/128)$
$(0, r)$	$(0, 0), (1, 0), (3/4, 3/4)$	$(3/32, 9/512, 9/512, 21/512, 9/512, 9/128)$

Cell certificate (continued)

(L_0, U_0)	Vertices (P, r)	Integral tuple
$((3P + r - 3)/4, P)$	$(3/2, 3/2), (0, 3), (3/4, 3/4)$	$(9/16, 81/512, 81/256, 81/512, 297/512, 459/256)$
$((3P + r - 3)/4, r)$	$(1, 0), (2, 1), (3/2, 3/2), (3/4, 3/4)$	$(5/16, 39/512, 53/256, 95/512, 39/512, 145/256)$
$((3P + 2r - 6)/2, P)$	$(2, 2), (0, 3), (3/2, 3/2)$	$(3/16, 9/256, 51/256, 9/256, 45/256, 3/4)$
$((3P + 2r - 6)/2, r)$	$(2, 1), (2, 2), (3/2, 3/2)$	$(1/16, 3/256, 21/256, 7/256, 3/256, 3/16)$

The last integral in (11.19) also follows by integrating $c \in [L_0, U_0]$: $\int p = 1/2$ and $\int c = 7/8$, so $\int 2P = 2(1/2 + 7/8) = 11/4$.

Consequently every normalized auxiliary eigenbasis divisor D_m satisfies, with uniform errors $O(1/m)$,

$$(11.20) \quad \begin{aligned} \text{coeff}_I D_m &\leq 1/3, & \text{coeff}_A D_m &\leq 15/32, & \text{coeff}_Q D_m &\leq 3/4, \\ \text{coeff}_{C_0} D_m &\leq 7/12, & \text{coeff}_{C_\zeta} D_m &\leq 15/64, & \nu_i(D_m) &\leq 8/3, & \nu_e(D_m) &\leq 11/6. \end{aligned}$$

The last order is on the index-three cover. Bounded parity terms and the finitely many rational cell boundaries give total numerator errors $O(m^3)$. No expression depends on the nonzero root parameter, so the errors are uniform even when the basis roots vary with m .

Step 3. In this step, we control all surface centers and deduce the threefold bound. Fix $b < 9/8$. For large degree all prime-component coefficients of bD_m are less than one, and $b\nu_i(D_m) < 3$, $b\nu_e(D_m) < 3$. We check the complete support. Away from the three coordinate points, its coordinate curves and distinct pencil members are disjoint and smooth. At $p_a = [0 : 1 : 0]$ only I, Q occur; on the index-two cover they are transverse coordinate axes. At p_i the branches are $a = 0$, $e = 0$, and $e = ca^2$ for distinct nonzero constants c . The weight- $(1, 2)$ blowup separates them: on $e = a^2t$, they meet the exceptional curve at distinct finite parameters, transversely. The other chart has a smooth index-two cover with the exceptional divisor and strict $a = 0$ as axes. Its crepant exceptional coefficient is $b\nu_i(D_m) - 2 < 1$. At p_e , first use the index-three cover and then the weight- $(2, 1)$ blowup. The branches $i = 0, a = 0, i = ca^2$ have the identical separation, with exceptional coefficient $b\nu_e(D_m) - 2 < 1$. All supports on these smooth covers are simple normal crossing with coefficients less than one. The covers are étale in codimension one, so klt descends. This proves that every auxiliary eigenbasis divisor is b -klt, uniformly in sufficiently large degree. It includes all further valuations over the quotient charts.

In each original exceptional-degree piece of the actual ring, choose a basis compatible with a tested valuation and the highest auxiliary weight. Its initial polynomials form an eigenbasis of the entire $\mathcal{B}_0$ piece by Lemma 11.5. The diagonal action produces a family of effective divisors on the fixed weighted plane with no zero initial section. Multiplication by $6mN_m$ makes the family Cartier, and its support is proper over the base. All nonzero fibers are automorphic images of the original divisor. Lemma 2.10, equivalently [BL21, Definition 7 and Proposition 10(2)] on this $\mathbb{Q}$ -Gorenstein flat family of klt surfaces, transfers the uniform eigenbasis bound to the actual series. Thus $bS_m(W_0; v) \leq A_{E_0}(v)$ for every tested valuation. Maximality gives this finite-degree bound for every basis divisor. Letting $m \rightarrow \infty$ and $b \rightarrow 9/8$ proves the asserted surface estimate.

For the threefold choose a basis compatible with G, E_0 and write its pullback as $S_m(E_0)E_0 + \Theta_m$. The actual residual restriction is controlled by the surface bound. Plt adjunction yields

$$A_X(G) \geq bS_m(G) + (6 - bS_m(E_0)) \text{ord}_G(E_0).$$

Its limits and $S_X(E_0) = 59/12$ give $59/12 - 6(8/9) = -5/12$, proving (11.16). If the center has positive dimension and contains p , its strict center meets E_0 and is not contained in it. Log canonicity in a neighborhood of that intersection holds at the generic point of the strict center; its exceptional order is zero. Thus the stated center convention is retained. $\square$

12. ORDINARY-LINE FLAGS IN DEGREE TWENTY-TWO

We return to an ordinary line L on a degree-twenty-two threefold. Write $Y = \text{Bl}_L X$, $E = \mathbb{F}_1$, with negative section σ , and set $A = H - E$, $B = H - 2E$, $R = H - 3E$. The inverse link is the blowup of a smooth rational normal quintic $\Gamma \subseteq V_5$. The actual fixed divisor $R|_E$ will control the terminal terms in every flag used below.

Lemma 12.1. *Let Z be the length-three base scheme of $U = \text{im}[H^0(Y, B) \rightarrow H^0(E, 2\sigma + 3f)]$, and resolve its three simple free base points by $p_T : T \rightarrow E$. Writing e_i for total exceptional classes, put $M = p_T^*(2\sigma + 3f) - \sum e_i$. The unique effective divisor R has restriction R_0 such that*

$$(12.1) \quad D_R = p_T^* R_0 - 2 \sum e_i = C + N, \quad M \cdot C = 5,$$

where C occurs with coefficient one, maps isomorphically to Γ , and N is effective and M -trivial. Moreover $\dim U = 6$.

At most one successive center lies on σ and at most two lie on any fiber. If σ is a component of R_0 , it contains one such center. A component fiber contains two such centers; at most one fiber can be a component. Every fiber coefficient is at most two, and is at most one if all centers avoid σ .

The following configurations cannot occur: a length-two subscheme of a fiber, disjoint from σ , and a third proper point on σ outside that fiber; two proper points on a fiber, one on σ , with the third center infinitely near that point; or a length-three free chain rooted on σ , whose second center lies on the fiber and whose third leaves it.

Proof. We use the component mapping to the inverse quintic to control the fixed divisor and exclude the stated clusters.

Step 1. In this step, we identify the fixed quintic component and bound the multiplicities.

Across the flop B^+ is the hyperplane pullback and R^+ the exceptional divisor over Γ , while $E^+ = B^+ - R^+$. Thus the image of E is the degree-five hyperplane surface $S \subseteq \mathbb{P}^5$ containing Γ . The latter spans this $\mathbb{P}^5$. The blowup pushforward gives $h^0(R) = 1$ and $h^0(B) = 7$. Since $B - E = R$, the restriction image has dimension six.

At the generic point of Γ , S has multiplicity one, as expressed by $E^+ = B^+ - R^+$. Its leading equation cuts one point on the exceptional generic $\mathbb{P}^1$. Therefore $E^+ \cap R^+$ has a component dominating Γ . It is not a flopped curve, since those map to degree-one bisecants whereas Γ has degree five. It gives a component of R_0 dominating Γ .

On a common resolution, write $G_b = \alpha^* B - \beta^* B^+$. This is the effective fixed divisor of the base system, with successive multiplicity one at each of the three surface centers. The equality of pullbacks of A and A^+ , together with $R = 2B - A$, gives $\alpha^* R = \beta^* R^+ + 2G_b$. Restrict to the strict transform of E and push to T . No term contains that surface, so this proves actual effectivity in (12.1). Now

$$M \cdot D_R = (2\sigma + 3f)(3\sigma + 4f) - 6 = 5.$$

Since M is nef, the component dominating the quintic has coefficient and map degree one, and every other component has degree zero. Its finite birational map to smooth Γ is an isomorphism. Neither σ nor a fiber can be this component: their degrees under U are at most one and two. The intersections $M \cdot \sigma_T = 1 - k_\sigma$ and $M \cdot f_T = 2 - k_f$ prove the claims about successive centers. Distinct fibers have disjoint sets of centers, so only one can contain two of the three.

For the coefficient bound, if a fiber had coefficient $\mu \geq 3$, the remainder would have class $3\sigma + (4 - \mu)f$. Every integral component other than σ and a fiber has class $a\sigma + bf$ with $0 \leq a \leq b \leq 1$, hence map degree at most $a + 2b \leq 3$. No component could dominate Γ , a contradiction. If $k_\sigma = 0$, the section is absent from R_0 . A fiber coefficient $\mu \geq 2$ would then give $(R_0 - \mu f) \cdot \sigma = 1 - \mu < 0$ without a shared component, which is impossible.

Step 2. In this step, we exclude the three stated cluster configurations. We prove the three exclusions using the spanning quintic. In each case we will produce a nonzero $t \in U$ with $R_0 = \sigma + f_x + D_t$. The quintic component is neither σ nor f_x , so its image would lie in the hyperplane $t = 0$ in $\mathbb{P}^5$, a contradiction.

For the first configuration, the condition $R_0 \in I_Z^2$ forces f_x to be a component, since its restriction degree is three and the imposed length is four. Similarly σ is a component: its restriction degree is one and the proper point on it imposes multiplicity two. After division by $\sigma + f_x$, the remaining section satisfies I_Z . At a double point on the fiber this follows from $((z, u^2)^2 : z) = (z, u^2)$. The three conditions on $H^0(E, 2\sigma + 3f)$ are independent. Restriction to the fiber is complete of degree two, separating the length-two subscheme; its kernel contains the sections divisible by f_x from the globally generated class $2\sigma + 2f$, one of which is nonzero at the third point. The containing space therefore has dimension $9 - 3 = 6$ and equals U . This gives the prohibited decomposition.

For the second configuration, nefness excludes both the σ and fiber directions for the third center. Again R_0 contains $\sigma + f_x$. At their meeting point write the length-two ideal as $J = (v, u^2)$, with tangent $v = 0$. Both curves are transverse to this tangent. Modulo J^2 , the product of their equations is $g = cu^2 + du^3 + euv$, $c \neq 0$. If the residual section is $t \equiv a + bu \pmod{J}$, the coefficients of u^2, u^3 in gt are $ca, da + cb$. As $gt \in J^2$, they vanish, so $t \in J$. The other proper point imposes one condition after fiber division. The two tangent-vector conditions are independent, since the ruled-surface monomials contain the constant and both local coordinates. A section with divisor 2σ plus three fibers different from f_x vanishes on that tangent vector and is nonzero at the other point. Thus all three conditions are independent, and again the residual section belongs to U .

For the last configuration use global ruling coordinates z, u , with $f_x = (z = 0)$, $\sigma = (u = 0)$ and $I_Z = (z - u^2, u^3)$. The nine monomials of $H^0(E, 2\sigma + 3f)$ have $0 \leq \ell \leq 2$, $0 \leq i \leq 1 + \ell$ in $z^i u^\ell$. The subspace satisfying I_Z has basis

$$z - u^2, zu, z^2u, zu^2, z^2u^2, z^3u^2.$$

For $H^0(E, 3\sigma + 4f)$ add the five monomials $u^3, zu^3, z^2u^3, z^3u^3, z^4u^3$. Put $a = z - u^2$. The quotient by $I_Z^2 = (a^2, au^3, u^6)$ has basis $1, u, u^2, u^3, u^4, u^5, a, au, au^2$. The fourteen monomials have eight independent nonzero images, spanned by

$$1, a + u^2, u, au + u^3, u^2, au^2 + u^4, u^3, u^5.$$

Their kernel has basis $z^2u - zu^3, z^2u^2, z^3u^2, z^2u^3, z^3u^3, z^4u^3$. Consequently

$$H^0(E, (3\sigma + 4f) \otimes I_Z^2) = zu H^0(E, (2\sigma + 3f) \otimes I_Z).$$

The unmultiplied space $H^0(E, (2\sigma + 3f) \otimes I_Z)$ has dimension six and is U ; the displayed right-hand side is its image under multiplication by zu . Applying the identity to the actual section of R_0 gives the excluded decomposition $R_0 = \sigma + f_x + D_t$. $\square$

Lemma 12.2. *The full refinement along E satisfies*

$$(12.2) \quad \begin{aligned} W_{m,j} &\subseteq H^0(E, j\sigma + (m+j)f), & 0 \leq j \leq m, \\ W_{m,j} &\subseteq H^0(E, (j\sigma + (m+j)f) \otimes I_Z^{j-m}), & m \leq j \leq 2m, \\ W_{m,j} &= R_0^{j-2m} U_{3m-j}, & 2m \leq j \leq 3m, \end{aligned}$$

where U_q is the actual restriction image of $H^0(Y, qB)$, contained in $H^0(E, q(2\sigma + 3f) \otimes I_Z^q)$. There exist no pieces beyond $3m$, and

$$(12.3) \quad S_X(E) = 17/11, \quad J = \lim_m \frac{\sum_{j>2m} (j-2m) \dim W_{m,j}}{mN_m} = 5/88.$$

Proof. The full spaces across the flop agree by their rational-function divisorial inequalities. For the middle range the common-resolution identity is

$$\alpha^*(mH - jE) = \beta^*((2m - j)A^+ + (j - m)B^+) + (j - m)G_b.$$

The moving class is globally generated. Restriction to the surface resolution gives every successive multiplicity $j - m$, exactly the ideal power of the free cluster. Restriction of the globally generated moving class likewise imposes the stated multiplicities on U_q . The divisorial contraction on the inverse side and $mH - jE = (j - 2m)R + (3m - j)B$ give the exact tail. Multiplication by R_0 is injective, as E is not a component of R . The invariants and support were computed in Proposition 4.8. For the tail weight, telescope the dimensions to the complete section counts and use $(B^+)^3 = 5$:

$$J = \frac{1}{22} \int_2^3 5(3 - t)^3 dt = \frac{5}{88}.$$

□

Proposition 12.3. *Suppose Z consists of a length-two subscheme $Z_f \subseteq f_p$ and a third proper point outside f_p . Put $c = 88/83$ if Z_f avoids σ , and $c = 1$ otherwise. Then every G with $p \in c_X(G)$ satisfies*

$$(12.4) \quad A_X(G) \geq cS_X(G) + (2 - 17c/11) \operatorname{ord}_G(E).$$

In particular this inequality is strict against $S_X(G)$ for a center equal to p .

Proof. We estimate the marked-fiber refinement in the off-section and negative-section cases, retaining the actual fixed tail.

Step 1. In this step, we bound the fixed tail along the marked fiber.

The third point r is off σ . If Z_f avoids σ , this follows from Lemma 12.1; otherwise two centers on σ would contradict nefness. Write $R_0 = \mu f_p + R'$. Lemma 12.1 gives $\mu \leq 2$, improved to $\mu \leq 1$ if all base points avoid σ . For every $q \in f_p$ we obtain $\operatorname{ord}_q(R'|_{f_p}) \leq 3$. The full tail consequently adds μJ to the fiber moment and at most $3J$ to a secondary point moment, using actual ranks.

Step 2. In this step, we estimate the moments when the cluster avoids the negative section. First assume Z_f avoids σ . Take base coordinate z with the marked fiber at zero and r over infinity. A translation by a section of $\mathcal{O}_{\mathbb{P}^1}(1)$ sets r to the positive zero corner and prescribes independently the positive fiber coordinate u at zero. Then Z_f has ideal $(z, Q(u))$ for a monic quadratic Q , possibly with a double root. A section of $a\sigma + bf$ has unique expansion $\sum_i z^i g_i(u)$ with $\deg g_i \leq d_i = \min\{a, b - i\}$. The ideal power imposes divisibility by Q^{h_i} , $h_i = \max\{0, w - i\}$. At infinity its monomial order is $b - i$, so the condition at infinity is $i \leq b - w$. Thus the complete containing rows are

$$z^i Q^{h_i} \mathbb{C}[u]_{\leq d_i - 2h_i}, \quad 0 \leq i \leq b - w.$$

They are empty when their last degree is negative.

The fiber moment depends only on the row dimensions and is independent of Q . At a finite point, its fixed multiplicity is at most $2h_i$, attained at the double root in the model $Q = u^2$. At the negative endpoint it is $a - d_i$. The complete row contribution is its dimension times this fixed multiplicity plus the triangular number of its moving dimension. Actual row inclusions and Lemma 2.8 therefore reduce all moments to the two endpoints of this double-root model.

Its monomials satisfy

$$0 \leq \ell \leq a, \quad 0 \leq i \leq b - \ell, \quad 2i + \ell \geq 2w, \quad i \leq b - w.$$

The three ranges use $(a, b, w) = (j, m + j, 0), (j, m + j, j - m), (2q, 3q, q)$, $q = 3m - j$. Scale $t = j/m$, $z = i/m$, $v = \ell/m$. For $t \in [0, 1]$ the fiber, positive-endpoint, and negative-endpoint moment functions are

$$t/2 + t^2/2 + t^3/6, \quad t^2/2 + t^3/6, \quad t^2/2 + t^3/3,$$

with integrals $11/24, 5/24, 1/4$. For $t = 1 + w \in [1, 2]$, the polygon is

$$0 \leq v \leq 1 + w, \quad \max\{0, w - v/2\} \leq z \leq \min\{2, 2 + w - v\}.$$

Splitting at $v = w, 2w$ gives moment functions

$$2t - 5/6 - w^3/3, \quad w^2 + 3w/2 + 2/3 - 2w^3/3, \quad 5/6 + 2w - w^3/3.$$

Their integrals are $25/12, 19/12, 7/4$. In the tail the polygon is scaled by $\rho = 3 - t$ from its endpoint, so the moments are $(17/6)\rho^3, (5/2)\rho^3, (5/2)\rho^3$, with integrals $17/24, 5/8, 5/8$. After multiplying the sums by $3/11$, the free bounds are $39/44, 29/44, 63/88$. The actual fixed shifts prove

$$S(W; f_p) \leq 39/44 + 5/88 = 83/88, \quad S(V; q) \leq 63/88 + 15/88 = 39/44.$$

Step 3. In this step, we estimate the doubled and reduced clusters meeting the negative section. Now suppose Z_f is doubled at $\sigma \cap f_p$. Use the negative coordinate v , so its ideal is (z, v^2) . The opposite proper point is again a toric corner. The exact monomial inequalities become

$$0 \leq \ell \leq a, \quad 0 \leq i \leq b - a + \ell, \quad 2i + \ell \geq 2w, \quad i \leq b - w.$$

For each i the residual exponents are $\max\{0, i - b + a, 2(w - i)\} \leq \ell \leq a$. Their maximum point moment occurs at $v = 0$ and bounds all others. The early polygon has $0 \leq v \leq t$, $0 \leq z \leq 1 + v$, with fiber and maximal point integrals $11/24, 1/4$. The middle polygon has $\max\{0, w - v/2\} \leq z \leq \min\{1 + v, 2\}$. Splitting at $v = 1, 2w$ gives moment functions $2t - 5/6 - w^3/3$ and $t^2 - 1/6 - 2w^3/3$, with integrals $25/12, 2$. The tail contributes $17/24, 19/24$. Thus the free bounds are $39/44$ and $73/88$. Adding $2J, 3J$ gives one for both full moments.

It remains to compare a reduced pair on f_p with one point on σ . Translate its finite point to $u = 0$ and the residual point to the positive corner at infinity. At fiber order i , put $h = \max\{0, w - i\}$. The positive exponents satisfy $h \leq \ell \leq \min\{a - h, b - i\}$ and $i \leq b - w$. In the negative coordinate they run from $\max\{h, c_i\}$ to $a - h$, where $c_i = \max\{0, i - b + a\}$. The doubled-negative-point row runs from $\max\{2h, c_i\}$ to a . The first row has dimension n at most the second's n' , since $h + \max\{h, c_i\} \geq \max\{2h, c_i\}$. The largest total point order of an n -dimensional subspace of $H^0(\mathbb{P}^1, \mathcal{O}(a))$ is at most $n(2a - n + 1)/2$, the sum of the largest n distinct integers in $[0, a]$. This increases with $n \leq a + 1$. The doubled-negative-point row attains it for n' at its negative endpoint. It therefore bounds the reduced row at every point, as well as its fiber contribution in . The preceding bounds by one apply.

Step 4. In this step, we pass from the uniform moment estimates to the bound for arbitrary valuations. All comparisons used actual associated graded injections. The fixed shifts were weighted by the original tail ranks. Boundary lattice errors total $O(m^3)$ in the numerator $mN_m = (11/3)m^4 + O(m^3)$, uniformly in the fiber point. Two successive smooth flags, with zero differentials, give the constant c above. Since $A_X(E)/S_X(E) = 22/17 > c$, Lemma 2.6 yields (12.4). The remaining coefficient is $30/83$ or $5/11$. For a point center, $\text{ord}_G(E) > 0$, proving strictness. $\square$

Lemma 12.4. *For two ordinary lines L_i, L_j meeting at p , let p_{ij} be the point representing L_j on the exceptional surface over L_i , and let n_{ij} be its width in that link. Then*

$$(12.5) \quad n_{ij} > 1 \iff p_{ji} \text{ lies on the negative section over } L_j.$$

Proof. For an ordinary line M , $H^0(N_{M/X})$ is one-dimensional and its evaluation is the nowhere-zero trivial subline of $\mathcal{O} \oplus \mathcal{O}(-1)$. The negative section parametrizes this subline: its normal bundle in $\mathbb{P}(N_{M/X})$ is $\text{Hom}(\mathcal{O}, \mathcal{O}(-1))$. For tangent vectors e_L, e_M at the intersection, the first-order condition for a normal deformation σ of M to meet L is

$$\text{ev}_p(\sigma) + ae_M = be_L.$$

For $\sigma = 0$, both marked parameters vanish. For nonzero σ , it is solvable exactly when the normal direction of L is that evaluation subline. This is the tangent space of the embedded incidence functor: writing a deformed line as a normal graph over the dual numbers gives the

equation, with a its moving marked parameter. By Lemma 7.1, its completed local algebra is $\mathbb{C}[t]/(t^{n_{ij}})$. Its tangent space is nonzero precisely when $n_{ij} > 1$, which proves (12.5). $\square$

Proposition 12.5. *Suppose the entire incidence scheme at p consists of two distinct reduced lines. In a chosen ordinary-line link let q be the root representing the other line, and let n be its width. If q is off the negative section, put $c = 44/43$. If q is on the negative section and $n = 2$, put $c = 88/83$. In either case, every G with $p \in c_X(G)$ satisfies*

$$(12.6) \quad A_X(G) \geq cS_X(G) + (2 - 17c/11) \operatorname{ord}_G(E).$$

Proof. We treat the off-section configurations and the doubled root on the negative section, keeping the fixed tail in both cases.

Step 1. In this step, we bound the fiber and point moments for the off-section configurations.

The incidence dictionary gives $Z \cap f_p = \{q\}$ reduced. A cluster of length at least two at q is therefore transverse to the fiber. If q lies on σ , nefness makes it transverse to σ as well. Exactly one successive base center lies on f_p . Lemma 12.1 shows that f_p is absent from R_0 , while $R_0|_{f_p}$ is an effective degree-three divisor containing $2q$. We use all pieces of Lemma 12.2.

First let q be off σ and set it to $(z, u) = (0, 0)$ in the positive chart. Its local ideal is $(u - P(z), z^n)$. Choose infinity away from the supports and act by $z \mapsto \lambda^{-1}z$, $u \mapsto \lambda^B u$, with a fixed positive integer B . The marked ideal tends to (u, z^n) . At infinity (ζ, u_∞) have weights $(1, B + 1)$. For $n = 3$ there exists no residual cluster; for $n = 2$ it is a simple off-section point r or a simple on-section point t ; for $n = 1$ it is a horizontal double at r , a vertical double at r , a vertical double at t , or $r + t$. Two off-section points with different base coordinates give the horizontal limit, while equal ruling coordinates give the vertical one. A double on the section is transverse to it, so its graph slope tends to zero in the indicated negative chart. The one-center bound on σ excludes the other possibilities.

All powers are flat in these degenerations. A local graph ideal $(a, Q(b))$ with Q monic of degree h has free quotient basis $a^i b^j$, $0 \leq i < w$, $0 \leq j < h(w - i)$, modulo its w th power. For colliding off-section points the interpolating graph has a regular limiting slope, with a factor λ^B . Use the fiber coordinate instead when the ruling positions coincide. The limiting supports otherwise stay disjoint. Thus kernel semicontinuity with an imposed fiber order bounds every first filtered dimension.

For the middle range put $w = j/m - 1$, $y = \ell/m$. The lower fiber-exponent bound is $\max\{0, n(w - y)\}$, and the upper bound is the minimum of $2 + w - y$ and the extra entries below, with $0 \leq y \leq 1 + w$ and nonempty intervals only. Integrating half the difference of their squares gives

n , residual	extra upper bounds	middle	tail	full free moment
3, $\emptyset$	$\emptyset$	13/6	17/24	10/11
2, r	2	2	5/8	37/44
2, t	$3 + w - 2y$	53/24	19/24	83/88
1, r horizontal	$2 - w + y$	11/6	13/24	17/22
1, r vertical	$2 - y/2$	11/6	13/24	17/22
1, t vertical	$5/2 + w/2 - 3y/2$	25/12	17/24	39/44
1, $r + t$	2, $3 + w - 2y$	47/24	5/8	73/88

The initial integral is 11/24, the tail is one fourth of its unit endpoint integral, and normalization is 3/11. All subdivisions are the equalities of the displayed affine bounds; for the vertical double at t the extra equality is $y = 1 - w$, meeting $y = w$ at $(1/2, 1/2)$. The actual fixed divisor has no fiber component. Hence $S(W; f_p) \leq 83/88$.

For secondary orders use the actual rows. At $q_\sigma = \sigma \cap f_p$, keep only simple multiplicity w at q . The order- k row injects into $\langle u^h, \dots, u^d \rangle$, $h = \max\{0, w - k\}$, $d = \min\{a, b - k\}$. Its endpoint orders are $a - \ell$. Their initial, middle, and tail integrals are 1/4, 13/6, 23/24, giving 81/88. Since $R_0|_{f_p}$ contains $2q$ and $q \neq q_\sigma$, its coefficient at q_σ is at most one. The full moment

is at most $86/88 = 43/44$. At every finite point discard the base conditions and use the complete polynomial row of degree d . Its total order is at most $d(d+1)/2$. The unnormalized continuous moment for (a, b) is $a^2b/2 - a^3/3$. Its early and middle integral is two and its tail integral is $5/6$, giving $17/22$ after normalization. The full point bound is at most $17/22 + 3J = 83/88$. These two flags prove the off-section constant $44/43$.

Step 2. In this step, we bound the eigenbasis divisors for the doubled root on the negative section. Now let $q = q_\sigma$ and $n = 2$. The residual simple point is off both σ and f_p . Use the negative coordinate v and global automorphisms $u \mapsto au + b + c_1z$ to make the marked ideal exactly $(v - z, z^2)$ and the residual point the positive corner at infinity. This is possible because the tangent is transverse to both axes: if $v = cz \bmod z^2$, choose $a = c$; then c_1 sets the residual coordinate to zero. These changes preserve the global monomial bounds.

Group the negative-chart monomials by $d = i + \ell$. Before the marked condition, their v exponents range from

$$L = \max\{0, \lceil (d - b + a)/2 \rceil, w + d - b\} \quad \text{to} \quad U = \min\{a, d\}.$$

The last lower bound is exactly multiplicity w at the residual point, whose monomial order is $b - d + \ell$. In total degree d , membership in $(v - z, z^2)^w$ is equivalent to divisibility by $(v - z)^c$, $c = \max\{0, 2w - d\}$. A monomial $(v - z)^h z^{d-h}$ has ideal order condition $d + h \geq 2w$. Thus the entire containing block is

$$(12.7) \quad v^L z^{d-U} (v - z)^c \mathbb{C}[v, z]_Q, \quad Q = U - L - c \geq 0.$$

The fiber orders are $d - U, \dots, d - U + Q$ and the corresponding endpoint orders are $U, \dots, U - Q$. For an eigenbasis the average component orders on σ , $C_1 = (v = z)$, and $C_t = (v = tz)$, $t \neq 0, 1$, are bounded above, respectively, by $L + Q/2, c + Q/2, Q/2$. Every section has total multiplicity d .

The six required integrands are

$$Q, \quad (d - U)Q + Q^2/2, \quad UQ - Q^2/2, \quad LQ + Q^2/2, \quad cQ + Q^2/2, \quad Q^2/2.$$

Their complete integral table is

	rank	f_p	q	σ	C_1	C_t
initial	2/3	11/24	1/4	1/4	1/6	1/6
middle	13/6	193/96	199/96	2	17/16	95/96
tail	5/6	31/48	41/48	19/24	19/48	1/3.

For a full cell certificate in the middle use $0 \leq w \leq 1$ and

$$U = \min\{1 + w, d\}, \quad L = \max\{0, (d - 1)/2, d - 2\}, \quad c = \max\{0, 2w - d\}.$$

The six polygons and their selected affine expressions are

Vertices (w, d)	L	U	c
$(0, 0), (1/2, 1), (0, 1)$	0	d	0
$(0, 0), (1, 1), (1/2, 1)$	0	d	$2w - d$
$(1, 2), (1, 3), (0, 3), (0, 1)$	$(d - 1)/2$	$1 + w$	0
$(1, 2), (0, 1), (1/2, 1)$	$(d - 1)/2$	d	0
$(1, 1), (1, 2), (1/2, 1)$	$(d - 1)/2$	d	$2w - d$
$(1, 3), (1, 4), (0, 3)$	$d - 2$	$1 + w$	0

Triangulation and (8.26) give the table. The tail rank is one third of the endpoint rank integral and its moments one fourth of the endpoint moments. Normalization gives the free moments

$$299/352, \quad 305/352, \quad 73/88, \quad 39/88, \quad 13/32,$$

in the five non-rank columns. The free total multiplicity at q is $151/88$.

Retain the actual fixed cubic term. For R_0 , the block with $(a, b, w) = (3, 4, 2)$ has no degree below three, and its degree-three part is $v(v - z)\mathbb{C}[v, z]_1$. Every higher-degree term has a z factor.

Since f_p is absent from R_0 , its initial form is $v(v - z)(\alpha v + \beta z)$ with $\alpha \neq 0$. Its multiplicity is three, its σ, C_1 coefficients are at most two, its other line coefficients at most one, and its fiber coefficient zero. The limiting basis divisors, including the actual tail weight $J = 5/88$, therefore have bounds

component	f_p	σ	C_1	C_t	total multiplicity
bound	299/352	83/88	49/88	13/32 + 5/88	83/44.

A sum of distinct lines through a smooth point is klt if each coefficient is less than one and their total is less than two: one ordinary blowup separates the lines and gives exceptional coefficient equal to the total minus one. Thus, for any $b < 88/83$ and large m , all these limiting eigenbasis pairs are klt at q , uniformly in the basis.

Step 3. In this step, we transfer the finite-degree bounds and apply adjunction. To transfer this local assertion without assumptions away from q , choose, in each actual free space, a basis compatible with a tested point-centered valuation and ordinary order at q . The initial homogeneous forms are independent subspaces of (12.7). Their bases extend to bases of the complete binary blocks, and all added component orders and multiplicities are nonnegative. Thus the complete-block bounds apply with the original denominator. Multiplying the tail by the actual initial R_0 only shifts both filtrations; its fixed shifts remain weighted by the actual ranks. The global simultaneous scaling of z, v realizes the degeneration to those lowest homogeneous forms and fixes q . Let D_t be its normalized divisor family and put $M_m = mN_m$. For any positive integer N , the ideal

$$\mathfrak{a}_N = \mathcal{O}(-M_m D_t) + I_{q \times \mathbb{A}^1}^N$$

has zero set contained in the proper section $q \times \mathbb{A}^1$. The central local bound gives $\text{lct}(E; (\mathfrak{a}_N)_0) > b/M_m$. Apply [BL21, Definition 7 and Proposition 10(2)] to the constant smooth projective surface over the normal base. It yields this lower bound at a nonzero parameter; every such pair is isomorphic to the original one. For the tested valuation choose N so large that $Nv(I_q) > v(M_m D_1)$. Its ideal order is then $v(M_m D_1)$, giving $bS_m(W; v) \leq A_E(v)$. Curve coefficients through q follow by taking weights $(k, 1)$ along their strict transforms on a resolution and letting $k \rightarrow \infty$. This proves the local bound for every center containing q .

At every other point of f_p the complete polynomial-row comparison above gives secondary bound 83/88, including its fixed term. The first fiber bound is 299/352. Thus the entire fiber has threshold at least 88/83.

All rational regions have bounded support and uniform boundary errors $O(m^3)$ in the total order numerator. The local eigenbasis estimate was uniform in the basis. Testing maximal basis weights gives the required finite-degree assertions for all basis divisors. Two successive adjunctions, with zero differentials and retained $S_X(E) = 17/11$, now prove (12.6). $\square$

13. REDUCED INCIDENCE PAIRS AND TRIPLES IN DEGREE TWENTY-TWO

The ordinary-line estimates leave one pair configuration and one cyclic triple configuration. We identify their complete weighted section rings using the canonical alternating-net presentation.

Setup 13.1. Every smooth prime degree-twenty-two threefold has its canonical Mukai presentation

$$(13.1) \quad X = X_{\mathcal{N}} = \{U \in \text{Gr}(3, V) : \omega|_U = 0 \text{ for every } \omega \in \mathcal{N}\}, \quad \dim V = 7, \quad \dim \mathcal{N} = 3,$$

scheme-theoretically, with its anticanonical evaluation embedding. Every nonzero form of $\mathcal{N}$ has rank six. These assertions concern every smooth member.

More precisely, the rank-three Mukai bundle $\mathcal{E}_X$ has $\det \mathcal{E}_X = K_X$, vanishing cohomology, and globally generated dual with seven sections; see [BKM26a, Theorem 1.1 and Lemma 5.2]. On a smooth anticanonical K3 section S with $\text{Pic}(S) = \mathbb{Z}[H|_S]$ it restricts to the Mukai bundle; such a section exists by [BKM26a, Lemma 5.2]. Serre duality [Sta26, Tags 0FVV and 0FW0] applied

to $\mathcal{E}_X^*(-H) \rightarrow \mathcal{E}_X^* \rightarrow \mathcal{E}_S^*$ identifies the two seven-dimensional section spaces. The kernel on the K3 of $\bigwedge^2 V^* \rightarrow H^0(S, \bigwedge^2 \mathcal{E}_S^*)$ has dimension three and every nonzero element has rank six. Its three relations extend to X and give the closed evaluation embedding and scheme equality in (13.1); these are [BKM26b, Definition 4.1, Proposition 4.5, Lemma 6.9 and Corollary 6.10]. Thus this kernel is the canonical net on X . We use no assertion here about arbitrary singular limits.

At a point $[U]$ choose $V = U \oplus Q$, $\dim Q = 4$. In the graph chart $[I_3 \mid M]$, write a net form and its equation as

$$(13.2) \quad \Omega = \begin{pmatrix} 0 & C \\ -C^t & B \end{pmatrix}, \quad CM^t - MC^t + MBM^t = 0.$$

The entries of C and of the actual alternating block B are linear forms on the net. A Grassmannian line through this point has a rank-one tangent matrix; for such a matrix the quadratic term in (13.2) vanishes.

Proposition 13.2. *If the entire incidence scheme at p consists of two distinct reduced lines, then $\delta_p(X) \geq 88/87$. In the configuration where both marked roots are on their negative sections and both widths are three, there exists a plt extraction with exceptional divisor $E_p = \mathbb{P}(1, 1, 2)$, zero different, and*

$$(13.3) \quad A_X(E_p) = 4, \quad S_X(E_p) = 151/44.$$

Every G with $p \in c_X(G)$ then satisfies

$$(13.4) \quad S_X(G) \leq \frac{87}{88}A_X(G) - \frac{23}{44}\text{ord}_G(E_p).$$

Proof. If either marked root is off the negative section, Proposition 12.5 gives 44/43. If both are on the section, Lemma 12.4 makes both widths at least two. A width two gives 88/83 by Proposition 12.5. Both constants exceed 88/87. It remains to treat two negative-section roots with both widths three.

Step 1. We determine the actual net. The two rank-one directions have independent row factors and independent column factors, since otherwise their projective pencil would consist of rank-one directions, contradicting finite incidence. Normalize them to E_{11}, E_{22} . Their whole lines lie in X , so

$$C = \begin{pmatrix} a & 0 & u & r \\ 0 & b & v & s \\ 0 & 0 & w & t \end{pmatrix}.$$

Here all entries are linear forms on the three-dimensional net. The forms $a, b, k = B_{12}$ are independent. Otherwise a nonzero form annihilated by all three would have rank at most four. If its last two C columns have rank less than two, there exist at least two kernel vectors in U , hence at least three in the odd-dimensional alternating matrix. If those columns have rank two, pair them with two U vectors and eliminate the other pairings. The remaining Q_1, Q_2 block is $B_{12} = 0$ and there exists an unpaired U vector, again giving rank at most four. Both alternatives contradict Setup 13.1.

The forms w, t are independent. Otherwise a pencil of net forms annihilates the third U vector; the Pfaffian on its six-dimensional quotient is a homogeneous cubic on that pencil and vanishes somewhere over $\mathbb{C}$, giving rank at most four.

In fact $\langle w, t \rangle = \langle a, b \rangle$. A nonzero first-order deformation of the first whole line has normal evaluation at p in the second-line direction, by the negative-section hypothesis. It can be written

$$M = \epsilon E_{22} + \tau(E_{11} + \epsilon N), \quad \epsilon^2 = 0.$$

Rank one of its direction gives $N_{ij} = 0$ for $i \neq 1, j \neq 1$. The $\epsilon\tau$ coefficient in entry 13 is $aN_{31} = wN_{13} + tN_{14}$. If a, w, t were independent, these three constants would vanish, and entry 12 would put k in $\langle a, b \rangle$. Thus $a \in \langle w, t \rangle$. The second line gives $b \in \langle w, t \rangle$.

Choose a third coordinate c on the net. At $a = b = 0, c = 1$, the last two C columns in the first two rows form an invertible matrix, since their determinant times B_{12} is the six-dimensional quotient Pfaffian. Change Q_3, Q_4 to make this matrix the identity. Write $w = A_0a + B_0b$, $t = C_0a + D_0b$. The two negative-section conditions force $B_0C_0 \neq 0$. For example $B_0N_{13} + D_0N_{14} = 0$, while the c coefficient in entry 12 forces $N_{14} \neq 0$; if $B_0 = 0$, independence forces $D_0 \neq 0$, a contradiction. The other line is identical. Rescale to get $w = Aa + b$, $t = a + Db$. Adding Q_1, Q_2 multiples to Q_3, Q_4 removes u_a, r_a, v_b, s_b ; adding U_3 multiples to U_1, U_2 removes u_b, s_a ; repeat the first additions if needed. These operations preserve the two directions and the c pivots. The result is

$$(13.5) \quad C = \begin{pmatrix} a & 0 & c & \gamma b \\ 0 & b & \beta a & c \\ 0 & 0 & Aa + b & a + Db \end{pmatrix}, \quad B = aB_a + bB_b + cB_c, \quad \kappa = (B_c)_{12} \neq 0.$$

All other entries of the three actual alternating blocks are retained.

Step 2. We use the full incidence widths. The kernel of the nine linear equations in (13.2) is, with free x, y, z ,

$$\begin{pmatrix} x & -\gamma Az & -Dz & z \\ -\beta Dz & y & z & -Az \\ (1-AD)z & (1-AD)z & 0 & 0 \end{pmatrix}.$$

If $AD = 1$, set $z = 1, x = \beta D^2, y = \gamma A^2$. The matrix has rank one and gives a third line, a contradiction. Hence $1 - AD \neq 0$.

A nearby line meeting the first has graph

$$M = \epsilon E_{11} + \tau \mathbf{b} \mathbf{v}^t, \quad \mathbf{b} = (b_1, 1, b_3)^t, \quad \mathbf{v} = (v_1, 1, v_3, v_4)^t.$$

This parametrization is valid over every local Artin algebra, since the two normalized components are units. The coefficient linear in τ says that $C\mathbf{v} + \epsilon(B_{12} + B_{13}v_3 + B_{14}v_4)e_1$ is proportional to $\mathbf{b}$. Its third-component equations are

$$Av_3 + v_4 = \beta b_3 v_3, \quad v_3 + Dv_4 = b_3, \quad b_3 v_4 = 0.$$

The first-component equations are

$$\begin{aligned} v_1 + \epsilon(k_a + l_a v_3 + n_a v_4) &= \beta b_1 v_3, \\ \gamma v_4 + \epsilon(k_b + l_b v_3 + n_b v_4) &= b_1, \\ v_3 + \epsilon(\kappa + l_c v_3 + n_c v_4) &= b_1 v_4, \end{aligned}$$

where $k_j = (B_j)_{12}$, $l_j = (B_j)_{13}$, $n_j = (B_j)_{14}$. The five equations except $b_3 v_4 = 0$ have invertible linear system in $(v_1, b_1, v_3, v_4, b_3)$ with parameter ϵ . Their formal solution starts as

$$v_3 = -\kappa\epsilon + O(\epsilon^2), \quad v_4 = A\kappa\epsilon + O(\epsilon^2), \quad b_3 = -(1-AD)\kappa\epsilon + O(\epsilon^2).$$

If $A \neq 0$, the remaining equation has exact order two. If $A = 0$, then $v_4 = \beta b_3 v_3$ and its product with b_3 starts as $-\beta\kappa^3\epsilon^3$. Finiteness forces $\beta \neq 0$, as otherwise this equation vanishes identically. Thus width three is equivalent to $A = 0$. Interchange the lines to get $D = 0$ and $\gamma \neq 0$. Our configuration has

$$A = D = 0, \quad \beta\gamma\kappa \neq 0.$$

Step 3. We identify initial sections. The entries $x = M_{11}$, $y = M_{22}$, $z_0 = M_{14}$ are regular algebraic parameters. Give them weights $(1, 1, 2)$, and put $h_0 = z_0 - \kappa xy$. With $Q_j = MB_j M^t$, the nine equations are

$$\begin{aligned} M_{21} &= \beta M_{13} - (Q_a)_{12}, & M_{12} &= \gamma M_{24} + (Q_b)_{12}, & M_{23} &= z_0 - (Q_c)_{12}, \\ M_{31} &= z_0 - (Q_a)_{13}, & M_{13} &= \gamma M_{34} + (Q_b)_{13}, & M_{33} &= -(Q_c)_{13}, \\ M_{24} &= \beta M_{33} + (Q_a)_{23}, & M_{32} &= M_{23} - (Q_b)_{23}, & M_{34} &= -(Q_c)_{23}. \end{aligned}$$

Implicit iteration preserves the weighted bounds

$$\begin{aligned} M_{12}, M_{21} &= O_w(2), & M_{23} &= h_0 + O_w(3), & M_{31} &= z_0 + O_w(3), & M_{32} &= h_0 + O_w(3), \\ M_{13}, M_{24} &= O_w(3), & M_{33} &= -\kappa x h_0 + O_w(4), & M_{34} &= \kappa y z_0 + O_w(4). \end{aligned}$$

The only relevant lowest products in $(Q_c)_{12}, (Q_c)_{13}, (Q_c)_{23}$ are respectively $\kappa xy, \kappa x h_0, -\kappa y z_0$; the other terms have larger weights. This verifies the bounds for every remaining block entry.

Let P_0 be the nonvanishing Plücker section. Entries, 2-minors, and 3-minors of M are Plücker sections divided by P_0 , up to signs. The following actual sections have the indicated symbols:

P_0	1	M_{11}	x
M_{22}	y	M_{14}	z_0
$M_{14} - M_{23}$	xy	M_{33}	$x h_0$
M_{34}	yz_0	$(13; 13)$	$x^2 h_0$
$(23; 24)$	$y^2 z_0$	$(13; 24)$	$z_0 h_0$
$(13; 34)$	$x z_0 h_0$	$(23; 34)$	$y z_0 h_0$
$(123; 134)$	$z_0 h_0^2$	$(123; 234)$	$z_0^2 h_0$

Here $(I; J)$ denotes the minor with the indicated rows and columns; nonzero scalar factors have been suppressed. For the last two, the leading determinants are $\kappa x y z_0 h_0 - z_0^2 h_0 = -z_0 h_0^2$ and $-\kappa x y z_0 h_0 - z_0 h_0^2 = -z_0^2 h_0$. Rescale $z = z_0/\kappa$, $h = h_0/\kappa = z - xy$. The generated section algebra is

$$(13.6) \quad \mathcal{B} = \mathbb{C}[T, Tx, Ty, Tz, Txy, T x h, T y z, T z h, T x z h, T y z h, T x^2 h, T y^2 z, T z h^2, T z^2 h].$$

Step 4. We prove completeness in all degrees. Filter by h order. Since z, xy occur, their difference h also occurs. The fourteen independent initial exponent vectors of $x^a y^b h^c$, numbered $0, \dots, 13$, are

$$\begin{aligned} &(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (1, 2, 0), \\ &(1, 1, 1), (2, 1, 1), (1, 2, 1), (2, 0, 1), (1, 3, 0), (2, 2, 1), (1, 1, 2). \end{aligned}$$

Their convex hull $\mathcal{P}$ has facets

$$(13.7) \quad a, b, c \geq 0, \quad a - c \leq 1, \quad c - b \leq 1, \quad a + c \leq 3, \quad b + c \leq 3, \quad -2a + b + c \leq 1.$$

Here is a full unimodular triangulation certificate. Lift the listed vectors to the respective integer heights

$$982, 599, 1592, 1049, 1894, 2462, 5611, 2071, 5004, 6379, 4877, 10286, 8323, 5448.$$

The lower tetrahedra are the following vertex quadruples:

$$(13.8) \quad \begin{aligned} &(0, 1, 3, 4), (0, 2, 3, 4), (1, 3, 4, 7), (1, 3, 7, 8), (1, 4, 5, 7), \\ &(1, 5, 7, 10), (1, 7, 8, 10), (2, 3, 4, 7), (2, 3, 6, 9), (2, 3, 7, 9), \\ &(2, 4, 7, 13), (2, 6, 9, 11), (2, 7, 9, 13), (3, 6, 9, 12), \\ &(3, 7, 8, 12), (3, 7, 9, 12), (4, 5, 7, 13), (5, 7, 10, 13), \\ &(6, 9, 11, 12), (7, 8, 10, 13), (7, 8, 12, 13), (7, 9, 12, 13). \end{aligned}$$

For each quadruple the edge determinant has absolute value one. The affine function interpolating its heights is at most every listed height. Each triangular face belongs to two tetrahedra or to one and lies on a facet of (13.7). These are finite checks with the displayed integer coordinates: solve the matrix with rows $(1, v_i)$ for each height function and test it on all fourteen vectors. The determinants are ± 1 . The face counts in dimensions $0, 1, 2, 3$ are $14, 46, 55, 22$. The lower cells have disjoint interiors; the face test leaves no exposed interior boundary. Their intersection with the connected interior of the convex hull is both open and closed, so they cover it. This also proves the stated facet description.

Unimodularity implies integer decomposition: a lattice point of $m\mathcal{P}$ in one tetrahedron has nonnegative integral barycentric coefficients, summing to m , after multiplying its normalized

coordinates by m . It is therefore a sum of m listed vectors. Counting the relatively open faces gives

$$\#(m\mathcal{P} \cap \mathbb{Z}^3) = 14 + 46(m-1) + 55\binom{m-1}{2} + 22\binom{m-1}{3} = \frac{11}{3}m^3 + \frac{11}{2}m^2 + \frac{23}{6}m + 1 = N_m.$$

At $m = 0$ the count is one. Distinct monomials give independent products in (13.6), which already embeds in the actual associated graded section ring of dimension N_m . Thus both inclusions are equalities in every degree and retained weight. In particular, the further h refinement is exactly all lattice monomials of $m\mathcal{P}$.

Averaging the centroids of the twenty-two equal-volume tetrahedra gives

$$\bar{a} = \bar{b} = 87/88, \quad \bar{c} = 8/11.$$

Explicitly each mean is the sum of the corresponding coordinates of the four vertices of every quadruple in (13.8), divided by $4 \cdot 22$. The original weight is $a + b + 2c$, so $S_X(E_p) = 151/44$. The weighted blowup is plt: its x, y charts are smooth and its z chart has smooth cover $x = ua, y = ub, z = u^2$ with simultaneous sign change on (u, a, b) . There exists no codimension-one stabilizer on $u = 0$, giving zero different. The Jacobian gives discrepancy four.

Step 5. We identify complete binary blocks. The algebra has two characters $\deg x = (1, 0)$, $\deg y = (0, 1)$, $\deg z = \deg h = (1, 1)$. Write them as (A, B') ; then $A = a + c$, $B' = b + c$. For fixed m, A, B' , (13.7) gives exactly

$$c_{\min} = \max\{0, \lceil (A - m)/2 \rceil\} \leq c \leq \min\{A, B', \lfloor (B' + m)/2 \rfloor, \lfloor (m + 2A - B')/2 \rfloor\}.$$

Also $A, B' \leq 3m$. Put $D = \min\{A, B'\}$ and $e_{\min} = \max\{0, \lceil (B' - m)/2 \rceil\}$. The upper endpoint is $D - e_{\min}$, by checking the cases $A \leq B'$ and $A \geq B'$.

The algebra has the involution $x \mapsto y, y \mapsto x, z \mapsto xy - z = -h, h \mapsto -z$. It preserves the generator spans, so the minimal z order in the block is $e_{\min}$. Every bidegree polynomial is divisible by $x^{A-D}y^{B'-D}$, leaving a degree- D binary form in (z, xy) . The two minimal orders therefore put the actual block inside

$$(13.9) \quad x^{A-D}y^{B'-D}z^{e_{\min}}h^{c_{\min}}\mathbb{C}[z, xy]_Q, \quad Q = D - e_{\min} - c_{\min}.$$

The exact h refinement gives dimension $Q + 1$. Thus this is equality, with negative Q interpreted as zero. The ample-series condition is explicit: for $0 \leq j \leq m$, T, Tx, Ty, Tz supply every weight- j polynomial on E_p .

Step 6. We bound every eigenbasis divisor. Its support consists of D_x, D_y, D_z, D_h and finitely many $C_t = (z - txy = 0)$, $t \neq 0, 1$. For a degree- Q complete binary space, the maximal sum of root orders is $Q(Q + 1)/2$. Applied to (13.9), the average D_x order is at most $A - D + Q/2$. The mean h -initial x exponent is $A - c_{\min} - Q/2$; their difference is $e_{\min} \geq 0$. Hence the full D_x, D_y coefficients are at most $87/88 + O(1/m)$. The D_h coefficient is at most $8/11 + O(1/m)$, and the involution gives the identical coefficient bound for D_z . Every other pencil coefficient is bounded blockwise by $Q/2$, hence also by $8/11 + O(1/m)$, uniformly in t .

At $P_x = [1 : 0 : 0]$ every eigenvector has ordinary local degree B' in (y, z) ; at $P_y = [0 : 1 : 0]$ it has degree A . Both multiplicities of the normalized eigenbasis divisor are $151/88 + O(1/m)$. All pencil branches at P_x are distinct lines $z = ty$, together with the applicable coordinate lines, and one ordinary blowup separates them. An ordinary blowup at P_y also separates its distinct branches. At the quotient point $P_z = [0 : 0 : 1]$ only D_x, D_y occur, as transverse axes on its index-two cover. Everywhere else the support is smooth and disjoint.

For any $b_0 < 88/87$ all component coefficients become less than one for large m , and $b_0(151/88) < 2$ fits the two ordinary-blowup discrepancies. The quotient chart is klt on its smooth cover. Thus every auxiliary eigenbasis divisor is globally b_0 -klt. The errors are uniform: the fixed rational polytope and its finitely many change planes meet $O(m^2)$ boundary cubes, and each order is $O(m)$. Total errors are $O(m^3)$.

Choose a basis compatible with any tested valuation and the auxiliary A weight. The global action $(x, y, z) \mapsto (tx, y, tz)$ gives its full independent eigenbasis as initial polynomials. The effective divisor family becomes Cartier after multiplication by $2mN_m$, has proper support, and has automorphic nonzero fibers. Lemma 2.10 transfers the uniform bound to every basis divisor of the actual full series. Hence $S(W; v) \leq (87/88)A_{E_p}(v)$ for every divisor over E_p .

Step 7. We conclude on X . Choose a basis compatible with G, E_p , retain the full exceptional coefficient, and apply plt adjunction to its effective remainder. It gives

$$A_X(G) \geq \frac{88}{87}S_X(G) + \left(4 - \frac{88}{87} \frac{151}{44}\right) \text{ord}_G(E_p).$$

Rearrangement is (13.4). For a positive-dimensional center containing p , its strict transform meets E_p , and log canonicity near that intersection also holds at its generic point; its exceptional order is zero. The two previously covered configurations have larger constants, so every reduced pair has threshold at least $88/87$. $\square$

Lemma 13.3. *Suppose the incidence scheme at p consists of three distinct reduced lines L_1, L_2, L_3 . Either one ordinary-line link has two width-one roots on its marked fiber and a third proper point off that fiber and the negative section, or, after relabeling,*

$$(13.10) \quad n_{12} = n_{23} = n_{31} = 2, \quad n_{21} = n_{32} = n_{13} = 1.$$

In the cyclic case each link has a double root off its negative section, transverse to the marked fiber, and its other simple root at the intersection of that section and fiber.

Proof. The directions are noncollinear and the lines are ordinary. In each row the two roots on the marked fiber are distinct with reduced fiber intersection. Their widths, whose total is at most three, are either $(1, 1)$ or $(2, 1)$. For $(1, 1)$ the remaining point is proper and off the fiber; Lemma 12.1 places it off the section.

Otherwise draw $i \rightarrow j$ when $n_{ij} = 2$. Every vertex has one outgoing arrow. Opposite arrows would put the doubled root on the negative section by Lemma 12.4. Together with the other proper root on the fiber, this is the second forbidden configuration in Lemma 12.1. There exist therefore no opposite arrows, and the three arrows form the cycle (13.10). The doubled root is off the section by the excluded clusters in Lemma 12.1. The incoming arrow puts the other simple root on the section. The doubled root has length-one intersection with the fiber. The Artin incidence dictionary therefore makes that double cluster transverse to the fiber. $\square$

Proposition 13.4. *At every point with three distinct reduced incident lines, $A_X(G) \geq S_X(G)$ for $p \in c_X(G)$, strictly when $c_X(G) = p$. In the directed-cycle case,*

$$(13.11) \quad \delta_p(X) = 88/73.$$

It is computed by the ordinary point blowup $E_p = \mathbb{P}^2$, whose invariants are $A_X(E_p) = 3$, $S_X(E_p) = 219/88$.

Proof. The first alternative in Lemma 13.3 is covered by Proposition 12.3, including its strict retained surplus. We assume the directed cycle.

Step 1. We normalize the tangent tensor. In Setup 13.1, write the three rank-one directions as $\alpha_i \otimes k_i$. No two factors in either family are proportional, since their joining projective line would then consist of rank-one directions. The two factor spans each have dimension two or three. If both have dimension two, the determinant on their tangent plane in a 2-by-2 matrix space gives a conic, or vanishes identically, contradicting finite incidence.

Suppose the α_i span dimension two and the k_i dimension three. A nonzero vector u_0 in their common kernel pairs to zero, under every net form, with $U + \langle k_1, k_2, k_3 \rangle$. A pencil of forms also kills its one remaining pairing. The Pfaffian on the resulting six-dimensional quotient vanishes somewhere on this pencil, contradicting rank six. If the first span has dimension three and the second two, rescale $k_3 = k_1 + k_2$. Then $Ck_3 = c_3\alpha_3 = c_1\alpha_1 + c_2\alpha_2$ forces all scalars to be zero.

Choose a nonzero net form whose alternating block vanishes on $\langle k_1, k_2 \rangle$. If $\text{rank } C < 2$, the kernel in U already has dimension at least two; if $\text{rank } C = 2$, eliminate its paired directions to get

$$\text{rank } \Omega = 2 \text{rank } C + \text{rank}(B|_{\ker C}) = 4.$$

Both are impossible.

Thus both spans have dimension three, and the directions may be set to E_{11}, E_{22}, E_{33} . Tangency diagonalizes the first three columns of C . Their diagonal coefficients are independent, since a nonzero form killing all three would have $\text{rank } C \leq 1$ and total rank at most four. Call them a, b, c . Adding multiples of the first three Q vectors to the fourth gives

$$C = [\text{diag}(a, b, c) \mid (u_1, u_2, u_3)^t], \quad u_1 = rb + sc, \quad u_2 = ta + uc, \quad u_3 = va + wb.$$

On each coordinate axis of the net, C has rank two, since rank one would again give total rank at most four. The rank-six condition then forces

$$d_{23} = [a]B_{23} \neq 0, \quad d_{13} = [b]B_{13} \neq 0, \quad d_{12} = [c]B_{12} \neq 0.$$

For clarity, compute the first whole-line deformation. A normal evaluation $v_2 E_{22} + v_3 E_{33}$ along L_1 has graph

$$M = \epsilon \text{diag}(0, v_2, v_3) + \tau(E_{11} + \epsilon S), \quad \epsilon^2 = 0.$$

The rank-one slope makes S supported in row one or column one. Set $h_1 = S_{14}$. The $\epsilon\tau$ coefficients in entries 12, 13 are

$$aS_{21} - bS_{12} - u_2 h_1 + v_2 B_{12} = 0, \quad aS_{31} - cS_{13} - u_3 h_1 + v_3 B_{13} = 0.$$

Their c, b coefficients give $d_{12}v_2 = uh_1$, $d_{13}v_3 = wh_1$. The other equations solve the remaining entries; S_{11} is reparametrization. These are actual families of whole lines: the slopes are rank one over the dual numbers and their Plücker coordinates homogenize across infinity. The nonzero ordinary normal section has nonzero evaluation, so $h_1 \neq 0$. The cycle says this evaluation is toward L_3 , giving $u = 0, w \neq 0$. Cyclically $v = 0, s \neq 0$ and $r = 0, t \neq 0$. Rescaling the first three Q vectors makes

$$(13.12) \quad C = \begin{pmatrix} a & 0 & 0 & c \\ 0 & b & 0 & a \\ 0 & 0 & c & b \end{pmatrix}.$$

The actual alternating block B remains unrestricted apart from the three nonvanishings just proved.

Step 2. We find fourteen actual initials. Write

$$M = \begin{pmatrix} x & b_0 & c_0 & d \\ e & y & f_0 & g \\ h & i & z & j \end{pmatrix}.$$

The linear isotropy entries are

$$a(e - d) - bb_0 + cg, \quad ah - bd + c(j - c_0), \quad aj + b(i - g) - cf_0.$$

Their nine coefficients have kernel the diagonal matrices, so x, y, z are regular parameters and every other entry has ordinary order at least two. The quadratic equations give exactly

$$g_2 = -d_{12}xy, \quad d_2 = d_{13}xz, \quad j_2 = -d_{23}yz.$$

Only the diagonal entries contribute at that order. The nonvanishing Plücker section, the three diagonal entries, the three entries d, g, j , the minors $(12; 14), (23; 24), (13; 34)$, and the four minors $(123; 123), (123; 124), (123; 134), (123; 234)$ have independent initials

$$(13.13) \quad 1, x, y, z, xy, xz, yz, x^2y, y^2z, z^2x, xyz, x^2yz, xy^2z, xyz^2.$$

For the three 2-minors the leading products are $xg, yj, -dz$. For the last three 3-minors they are $xyj, -xgz, dyz$. Every other product uses enough off-diagonal entries to have larger order. Thus these are actual symbols for every allowed coefficient of the net.

Step 3. We prove the complete monomial ring. The exponent polytope $\mathcal{P}$ is

$$(13.14) \quad \begin{aligned} a, b, c &\geq 0, & a + b + c &\leq 4, \\ a + b, b + c, c + a &\leq 3, & a - b, b - c, c - a &\leq 1. \end{aligned}$$

Its ten vertices are 000, 100, 010, 001, 210, 021, 102, 211, 121, 112, where concatenation denotes an exponent triple. For example its $c = 0$ facet is 000, 100, 210, 010. A vertex with all coordinates positive and total less than four would require three pair-sum or difference planes to be active. Three pair-sum planes give total $9/2$; two with a difference give total at least four or violate an inequality; one with two differences gives a zero coordinate or total at least $9/2$; and three differences are inconsistent. On total four all coordinates are at least one, giving the triangle 211, 121, 112. This exhausts the vertices and proves the facet description.

Each coordinate is at most two, by its corresponding pair-sum and difference inequalities. Their integer points are exactly the fourteen in (13.13). The only interior integer point is $c_* = (1, 1, 1)$, at lattice distance one from every facet. Triangulate each facet using all lattice points, compatibly on its edges, and star to c_* . Every empty lattice triangle is unimodular; the height-one condition therefore makes every resulting tetrahedron unimodular. There exist thirteen boundary vertices. Euler's relation for the triangulated sphere and $3f_2 = 2f_1$ give thirty-three boundary edges and twenty-two triangles. The star has face numbers 14, 46, 55, 22. Its unimodular barycentric coordinates imply integer decomposition in every degree, exactly as in Step 4 of Proposition 13.2. Counting relatively open faces gives

$$\#(m\mathcal{P} \cap \mathbb{Z}^3) = 14 + 46(m-1) + 55 \binom{m-1}{2} + 22 \binom{m-1}{3} = N_m.$$

Products of the actual sections already provide this many independent initials. They are therefore the entire associated graded section ring, in every section degree and ordinary vanishing degree. The full refinement on $E_p = \mathbb{P}^2$ is precisely its homogeneous monomial pieces.

For an explicit moment certificate, cone the following facet triangles from c_* . Here d is the absolute edge determinant and s the total coordinate sum at the tetrahedron centroid.

facet type	multiplicity	triangles	(d, s)
$a + b + c = 4$	1	(211, 121, 112)	(1, 15/4)
$a + b = 3$	3	(121, 211, 210)	(1, 7/2)
$a - b = 1$	3	(102, 100, 210), (102, 210, 211)	(2, 5/2), (1, 13/4)
$c = 0$	3	(210, 010, 000), (210, 000, 100)	(2, 7/4), (1, 7/4)

The cyclic copies exhaust all facets. Summing $d/6$ gives $11/3$, and summing $ds/6$ gives $5/8 + 7/4 + 33/8 + 21/8 = 73/8$. Cyclic symmetry yields mean $73/88$ for each coordinate. Thus $S_X(E_p) = 219/88$. Its discrepancy is three, as for an ordinary point blowup. The support is bounded by ordinary degrees $0, \dots, 4m$; for degrees $0, \dots, m$, the generators $1, x, y, z$ contain every homogeneous polynomial, giving an ample series.

Step 4. We control every surface valuation. The monomial basis divisor is exactly

$$D_m^0 = b_m(L_x + L_y + L_z), \quad b_m = \frac{\sum_{\mathbf{u} \in m\mathcal{P} \cap \mathbb{Z}^3} u_1}{mN_m} \rightarrow 73/88.$$

Its coordinate lines are simple normal crossing, so its threshold is $1/b_m$. For a tested divisor v , choose in each ordinary-degree piece a basis compatible with v and the torus filtration from

$$(x, y, z) \mapsto (tx, t^{2m+1}y, z).$$

The weights on its monomials are distinct, since $0 \leq u_i \leq 2m$ and the total degree fixes u_3 . After individual rescaling by the lowest power of t , the limits give the complete monomial basis. All

nonzero fibers are automorphic images of the original divisor. Multiplying the normalized divisor family by mN_m gives an effective Cartier divisor family on the fixed projective $\mathbb{P}^2$, with proper support and nonzero restrictions to all fibers. At this fixed degree m , threshold semicontinuity applies by [BL21, Proposition 10(2)]. The action is allowed to depend on m in this finite-degree argument. Apply it first at any coefficient below $1/b_m$ and then pass to that endpoint. This gives

$$S_m(W; v) \leq b_m A_{\mathbb{P}^2}(v).$$

Maximality of compatible basis weights makes this inequality valid for every other basis divisor, uniformly in v . The limit proves the full surface coefficient $73/88$.

For a threefold divisor G with $p \in c_X(G)$, choose a basis compatible with G, E_p and remove its full E_p coefficient. Smooth adjunction yields

$$A_X(G) \geq b_m^{-1} S_m(G) + (3 - b_m^{-1} S_m(E_p)) \text{ord}_G(E_p).$$

The identity $S_m(E_p) = 3b_m$ follows already from cyclic symmetry. For a positive-dimensional center, its strict transform meets E_p and this local log canonical statement holds at its generic point; its exceptional order is zero. Taking limits gives $A_X(G) \geq (88/73)S_X(G)$. The divisor E_p attains that ratio, proving (13.11). In particular $\beta_X(G) \geq (15/88)A_X(G) > 0$ in the cycle case. Together with the first alternative this proves the proposition. $\square$

14. DOUBLE AND MIXED INCIDENCE IN DEGREE TWENTY-TWO

We first establish two complete surface estimates. They include the base fibers that arise when a second line passes through the marked point. We then identify both grade parities of the threefold refinement with these surface systems.

Lemma 14.1. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, with ruling classes σ, f , and let $B \in |\sigma + 2f|$ be smooth. Let Δ be a degree-two effective divisor on B whose maps to both ruling bases are closed immersions. Denote the involution of the degree-two map with fibers of class σ by ι , and let R be the sum, with multiplicities, of the corresponding two σ fibers. Thus $R|_B = \Delta + \iota(\Delta)$. For $0 \leq k \leq \lfloor 5m/2 \rfloor$ put*

$$(14.1) \quad \mathcal{R}_{m,k} = \begin{cases} H^0(k\sigma + mf), & 0 \leq k \leq m, \\ H^0((k\sigma + mf) \otimes I_{\Delta}^{k-m}), & m \leq k \leq 2m, \\ s_B^{k-2m} s_R^{2k-4m} H^0(q(2\sigma + f) \otimes I_{\Delta}^q), & 2m \leq k \leq \lfloor 5m/2 \rfloor, \quad q = 5m - 2k. \end{cases}$$

All spaces retain their displayed fixed factors. If $N_m^{\mathcal{R}} = \sum_k \dim \mathcal{R}_{m,k}$, then

$$(14.2) \quad N_m^{\mathcal{R}} = \frac{11}{6} m^3 + O(m^2), \quad S_{\mathcal{R}}(v) \leq A_{S,B/2}(v)$$

for every prime divisor v over S , with normalization $mN_m^{\mathcal{R}}$ in the definition of $S_{\mathcal{R}}$.

Proof. We compute the flags away from the base fibers, then use ordinary and weighted point extractions at their boundary points.

Step 1. The two-point resolution gives the ranks. Blow up the two simple successive centers of Δ , writing e_1, e_2 for total exceptional classes. For distinct points the negative curves are $\sigma - e_i, f - e_i, e_i$. For a double point they are $\sigma - e_1, f - e_1, e_1 - e_2, e_2$; the total classes $\sigma - e_2, f - e_2$ remain effective. The double-point tangent differs from both ruling directions by the immersion hypothesis. For any other irreducible nonvertical curve $\alpha\sigma + \beta f - m_1 e_1 - m_2 e_2$, intersection with the rulings gives $m_i \leq \min(\alpha, \beta)$. The expression

$$m_1(\sigma - e_1) + m_2(f - e_2) + (\alpha - m_1)\sigma + (\beta - m_2)f$$

proves that the listed curves generate the effective cone. Its dual tests nefness. The anticanonical divisor is nef and big, of square six, so vanishing and Riemann–Roch compute integral nef spaces.

The powers of (u, t^r) , $r = 1, 2$, have multiplicity equal to the power at each of their r successive simple centers. Thus these resolution counts compute the actual ideal powers. For $a = k/m$ the rank density is

$$(14.3) \quad \rho(a) = \begin{cases} a, & 0 \leq a \leq 1, \\ a - (a - 1)^2, & 1 \leq a \leq 2, \\ (5 - 2a)^2, & 2 \leq a \leq 5/2. \end{cases} \quad \int_0^{5/2} \rho(a) da = 11/6.$$

The free divisors in all three ranges are nef by the cone test. Their dimensions are $m^2\rho(a) + O(m)$ uniformly. All subsequent double integrals have normalization $6/11$.

Step 2. We retain the full B coefficient. In the early and middle ranges remove a scaled free order b along B . The residual divisor is

$$(a - b)\sigma + (1 - 2b)f - (a - 1 - b)_+(e_1 + e_2).$$

Its negative part is $(a + b - 2)_+[(\sigma - e_1) + (\sigma - e_2)]$ in the middle range, and zero in the early range. In the tail, writing $q = 5 - 2a$, it is

$$(2q - b)\sigma + (q - 2b)f - (q - b)(e_1 + e_2),$$

with negative part $b[(\sigma - e_1) + (\sigma - e_2)]$. For a double point this last sum is $2(\sigma - e_1) + (e_1 - e_2)$, supported on a negative-definite chain. Zero intersection on the indicated support and nonnegative intersection with every cone generator verify each decomposition.

The complete table gives the free degree n on B , the actual order β , and the atoms H on Δ and K on $\iota(\Delta)$, counted with their scheme multiplicities:

a	b	n	β	H	K
$[0, \frac{1}{2}]$	$[0, a]$	$2a + 1 - 4b$	b	0	0
$[\frac{1}{2}, 1]$	$[0, \frac{1}{2}]$	$2a + 1 - 4b$	b	0	0
$[1, \frac{3}{2}]$	$[0, a - 1]$	$3 - 2b$	b	$a - 1 - b$	0
$[\frac{3}{2}, 2]$	$[a - 1, \frac{1}{2}]$	$2a + 1 - 4b$	b	0	0
$[\frac{3}{2}, 2]$	$[0, 2 - a]$	$3 - 2b$	b	$a - 1 - b$	0
$[\frac{3}{2}, 2]$	$[2 - a, \frac{1}{2}]$	$7 - 2a - 4b$	b	$a - 1 - b$	$a + b - 2$
$[2, \frac{5}{2}]$	$[0, q/2]$	$3q - 4b$	$a - 2 + b$	$q - b + 2(a - 2)$	$b + 2(a - 2)$

The base ideal contributes its residual order on Δ ; every removed σ fiber also contributes at the conjugate point. At a branch point these atoms add. The fixed tail uses exactly $R|_B = \Delta + \iota(\Delta)$. Beyond the upper edges the cone inequalities leave no positive-volume residual divisor. Integration over these domains gives

$$\int n = \frac{11}{6}, \quad \int \beta n = \frac{19}{48}, \quad \int \frac{n^2}{2} = \frac{47}{24}, \quad \int nH = \frac{1}{2}, \quad \int nK = \frac{5}{48}.$$

For example, the tail contributes $1/6, 3/64, 13/96, 9/64, 13/192$, respectively. It follows that

$$S_{\mathcal{R}}(B) = 19/88, \quad S_{\mathcal{R}|B}(z) \leq (94 + 5r)/88 \quad (z \notin \Delta, r = \text{mult}_z \iota(\Delta)).$$

For a fiber C of class f disjoint from Δ , remove order u from the free part. The negative coefficients on these two σ curves are $(a + u - 2)_+$ in the middle and u in the tail. The full table, including the fixed tail, is

a	u	n	K on $R _C$	H on $B _C$
$[0, 1]$	$[0, 1]$	a	0	0
$[1, 2]$	$[0, 2 - a]$	a	0	0
$[1, 2]$	$[2 - a, 1]$	$4 - a - 2u$	$a + u - 2$	0
$[2, \frac{5}{2}]$	$[0, q]$	$2(q - u)$	$u + 2(a - 2)$	$a - 2$

Here

$$\int n = \frac{11}{6}, \quad \int un = \frac{19}{24}, \quad \int n^2/2 = 1, \quad \int nK = \frac{5}{24}, \quad \int nH = \frac{1}{48}.$$

Consequently

$$S_{\mathcal{R}}(C) = 19/44, \quad S_{\mathcal{R}|C}(z) \leq \frac{6}{11} + \frac{5}{44} \text{mult}_z(R|_C) + \frac{1}{88} \text{mult}_z(B|_C).$$

These secondary bounds require only containment in the complete curve series of degree $mn + O(1)$. Lemma 2.8 bounds its order sum by $m^2n^2/2 + O(m)$, with each fixed atom added to every row. No equality of terminal image ranks is required. The bounded-width chamber-wall rows number $O(m)$, each with weight $O(m^2)$; their total error is $O(m^3)$. The interior errors total $O(m^3)$ and become $O(1/m)$ after division by $mN_m^{\mathcal{R}}$.

At $z = C \cap B$ outside the base fibers, put $r = \text{mult}_z \iota(\Delta) \leq 2$. The secondary bound is $(49 + 10r)/88$. Choose bases compatible with the ordinary point-order valuation E_z and the fiber order C . For each section, its point order is at most its fiber order plus the order of the first nonzero coefficient. Summing on these compatible bases gives

$$S_{\mathcal{R}}(E_z) \leq (87 + 10r)/88.$$

Choose a basis compatible with a tested valuation and B , and write its normalized divisor as $b_m B + \Omega_m$. The preceding maximal point-order bound applies to this basis too. Thus

$$b_m \rightarrow 19/88, \quad \text{mult}_z \Omega_m \leq (68 + 10r)/88 + o(1) \leq 1 + o(1), \quad (B \cdot \Omega_m)_z \leq (94 + 5r)/88 + o(1).$$

For fixed $\lambda < 1$ sufficiently close to one, the coefficient $1/2 + \lambda b_m$ is below one, and

$$\lambda((94 + 5r)/88 + b_m) < 3/2$$

for large m , since the limiting sum is at most $123/88$. Lemma 4.5 makes $B/2 + \lambda(b_m B + \Omega_m)$ log canonical. Off B the fiber flag gives the secondary bound $17/22 < 1$. This proves the assertion away from the base fibers.

Step 3. A base fiber is controlled away from B . Let C meet Δ , and let $z \in C \setminus B$. Its early and middle orders are at most m ; its free tail orders are at most $q = 5m - 2k$. Neither prescribed tail divisor contains C . Using the actual ranks for the first moment gives

$$S_{\mathcal{R}}(C) \leq \frac{6}{11} \left(\frac{1}{2} + \frac{7}{6} + \frac{1}{8} \right) = \frac{43}{44}.$$

After dividing by a C power, the early and middle images lie in degree k on C , with at most $m + 1$ orders; the tail lies in degree $2q$, with at most $q + 1$ orders. The free secondary moment is therefore at most

$$\frac{6}{11} \left(\int_0^1 \frac{a^2}{2} da + \int_1^2 \frac{a^2}{2} da + \int_2^{5/2} 2(5 - 2a)^3 da \right) = \frac{19}{22}.$$

The remaining fixed contribution at z is

$$\frac{6}{11} \text{mult}_z(R|_C) \int_2^{5/2} (2a - 4)(5 - 2a)^2 da = \frac{\text{mult}_z(R|_C)}{44} \leq \frac{1}{22}.$$

The total is at most $10/11$. The fiber flag proves the required inequality there.

Step 4. We treat unramified base points. At $p \in \Delta$ where the σ ruling is unramified on B , choose ruling coordinates (x, y) with B tangent to $y = cx$, $c \neq 0$, and let $r = \text{mult}_p \Delta \in \{1, 2\}$. For a free rectangular system of degrees (M, K) and base exponent t , the ordinary order- d row has

$$L = \max(0, d - K) \leq i \leq U = \min(M, d), \quad d \geq t$$

in the monomials $x^i y^{d-i}$. For $r = 2$, the ideal is $(y - \phi(x), x^2)$ with $\phi'(0) = c$. Its power has multiplicity t at each successive center. After first order d is removed, the row is divisible by $(y - cx)^h$, $h = (2t - d)_+$. This factor is coprime to x, y , so it decreases the row dimension by h while retaining both coordinate orders. For $r = 1$ put $h = 0$.

Scale all variables by m . Use

$$(M, K, t) = (1, a, 0), (1, a, a - 1), (q, 2q, q)$$

in the three ranges. Put $b = 0$ before the tail and $b = a - 2$ in the tail, and $n = (U - L - h)_+$, on $d \geq t$ and $U - L \geq h$. The full first order includes $(1 + 2r)b$, from B and R . The five integrands for rank, first order, and secondary orders at the B tangent, the x axis and the y axis are

$$n, \quad [d + (1 + 2r)b]n, \quad (h + b)n + n^2/2, \quad Ln + n^2/2, \quad (d - U + 2rb)n + n^2/2.$$

Their raw integrals are

r	range	rank	E	B	x	y
1	$[0, 1]$	1/2	5/12	1/8	1/4	1/6
1	$[1, 2]$	4/3	11/6	13/24	17/24	9/8
1	$[2, 5/2]$	1/4	41/96	11/96	5/48	7/24
2	$[0, 1]$	1/2	5/12	1/8	1/4	1/6
2	$[1, 2]$	7/6	5/3	1/2	29/48	1
2	$[2, 5/2]$	1/6	17/48	1/12	1/16	1/4

The endpoints are $d = K, M, t, 2t$ and $U - L = h$. For instance the simple tail, with $d = qD$, has row length q on $1 \leq D \leq 2$ and $q(3 - D)$ on $2 \leq D \leq 3$; $b = (1 - q)/2$ and $da = -dq/2$ give its displayed row. These formulas determine the entire rational integration.

The normalized first and three secondary bounds are, respectively,

r	E	B	x	y
1	257/176	75/176	51/88	19/22
2	117/88	17/44	1/2	17/22

The ordinary exceptional curve has log discrepancy $3/2$ for $(S, B/2)$ and different $1/2$ at the B tangent. Both first bounds are below $3/2$, both B bounds below $1/2$, and the other bounds below one. Other directions have at most the free row moment, hence no larger bound than the B column. Effective adjunction proves the claim at these points.

Step 5. Weighted rows control ramified base points. At a ramification point p , the immersion condition forces $\text{mult}_p \Delta = 1$, so $\Delta = p + p'$ with $p' \neq p$. Normalize the degree-two map by putting both ramification points and values at zero and infinity. Then $B = (y = x^2)$. The weight- $(1, 2)$ extraction has discrepancy 2 for $(S, B/2)$, and different $(p_\infty + p_B)/2$ on its coarse exceptional line, where $T = y/x^2$ and $p_B = (T = 1)$. The x chart is smooth; the other chart has index two and gives the different at infinity.

For its first moment only, use the automorphism $(x, y) \mapsto (\lambda^{-1}x, \lambda^{-2}y)$ to move p' to infinity. It preserves B, p and the weighted valuation. The disjoint two moving point sections have flat ideal-power quotients, with the monomial bases of all terms of total degree below the stated ideal exponent in translated parameters. Intersection with a fixed weighted-order space is a kernel condition, so upper semicontinuity bounds the actual first moment by this limit. In that limit the free monomials satisfy

$$0 \leq i \leq M, \quad 0 \leq \ell \leq K, \quad i + \ell \geq t, \quad i + \ell \leq M + K - t$$

in the middle and tail; the last bound is absent in the early range. The weighted degree is $i + 2\ell$, with fixed tail order $6b$. Its three raw integrals are $7/12, 7/3, 7/16$. For the tail the polygon is $0 \leq i \leq q, 0 \leq \ell \leq 2q, q \leq i + \ell \leq 2q$; integrating $q(I + 2J) + 3(1 - q)$ after $(i, \ell) = q(I, J)$ gives $7/16$. Thus

$$S_{\mathcal{R}}(E) \leq 161/88 < 2.$$

For secondary orders return to the original system and discard only the other point p' . The actual weighted degree- d row is contained in

$$L = \max(0, (d - M)/2) \leq \ell \leq U = \min(K, d/2, d - t), \quad n = (U - L)_+.$$

At $T = 1, \infty, 0$ the complete secondary integrands are

$$bn + n^2/2, \quad (d/2 - U)n + n^2/2, \quad (L + 2b)n + n^2/2.$$

The fractional term $d/2 - U$ retains the weighted polarization at the index-two point. The raw integrals are

$$\begin{array}{c|ccc} [0, 1] & 17/192 & 1/8 & 1/6 \\ [1, 2] & 19/64 & 17/48 & 9/8 \\ [2, 5/2] & 7/96 & 5/96 & 7/24 \end{array}$$

over $d \geq t$, $L \leq U$. Split at $d = 2K, M, 2t$ and the equations $L = U$. In the tail the scaled intervals are $1 \leq D \leq 2$, $2 \leq D \leq 4$, $4 \leq D \leq 5$. The normalized bounds are $1/4, 51/176, 19/22$. The first two lie below the respective half-different budgets; the last lies below one. Every other point has bound at most $1/4$.

Step 6. The estimates apply to every valuation. For a tested v , choose in each summand a basis compatible with v and the indicated fiber or extraction. Actual subsequent order subspaces inject into the containing binary rows. Thus their maximal sums satisfy the displayed bounds. For fixed $\lambda < 1$, all sufficiently large degrees satisfy the effective adjunction inequalities on the extracted curve. Raise its coefficient to one, apply inversion of adjunction, and then decrease it to the actual first coefficient. This gives $\lambda S_m(\mathcal{R}; v) \leq A_{S, B/2}(v)$.

All integrations have finitely many bounded rational polygons. Rounding changes each two-parameter row numerator by $O(m)$; there exist $O(m^2)$ rows, and chamber walls again contribute $O(m^3)$. The original denominator gives uniform $O(1/m)$ errors. The first-moment specialization in Step 5 makes no assertion about degeneration of secondary quotients. Let $m \rightarrow \infty$ and $\lambda \rightarrow 1$. Steps 2–5 cover every closed point, and a curve center is covered at a closed point on it. This proves the lemma. $\square$

Lemma 14.2. *Let $S = \mathbb{F}_2$, let σ be its negative section and f a ruling fiber, and let $B \in |\sigma + 3f|$ be smooth. Choose $q_\sigma \in \sigma \setminus B$ and $q_B \in B \setminus \sigma$ on different fibers C_σ, C_B , and put $\Delta = q_\sigma + q_B$. Let C_0 be the fiber through $\sigma \cap B$. Define*

$$(14.4) \quad \mathcal{R}_{m,k} = \begin{cases} H^0(k\sigma + (m+k)f), & 0 \leq k \leq m, \\ s_\sigma^{k-m} H^0((m\sigma + (m+k)f) \otimes I_\Delta^{k-m}), & m \leq k \leq 2m, \\ s_\sigma^{2k-3m} s_B^{k-2m} s_{C_\sigma+C_B}^{2k-4m} H^0(q(\sigma + 3f) \otimes I_\Delta^q), & 2m \leq k \leq \lfloor 5m/2 \rfloor, \quad q = 5m - 2k. \end{cases}$$

Then, with the original normalization $m \sum_k \dim \mathcal{R}_{m,k}$,

$$(14.5) \quad \sum_k \dim \mathcal{R}_{m,k} = \frac{11}{6} m^3 + O(m^2), \quad S_{\mathcal{R}}(v) \leq A_{S, B/2}(v) \quad \text{if } c_S(v) \notin C_0.$$

Both base fibers are included.

Proof. We compute each fiber flag, retaining the colon ideal and all fixed atoms at its two distinguished points.

Step 1. In this step, we determine the effective cone and the complete free classes. Blow up q_σ, q_B , with exceptional curves e_1, e_2 . The effective cone is generated by

$$s = \sigma - e_1, \quad c_1 = f - e_1, \quad e_1, \quad c_2 = f - e_2, \quad e_2.$$

Their squares are $-3, -1, -1, -1, -1$; the nonzero edges are $s - e_1, s - e_2, c_1 - e_1, c_2 - e_2$. For another nonvertical irreducible curve $\alpha\sigma + \beta f - m_1 e_1 - m_2 e_2$, we obtain $\beta \geq 2\alpha$ and $m_i \leq \alpha$. The expression

$$\alpha s + m_2 c_2 + (\beta - m_2) f + (\alpha - m_1) e_1$$

proves completeness of the cone. The pair with boundary $s/2$ is klt and its negative log canonical class is nef and big. Vanishing therefore gives Riemann–Roch for integral nef divisors.

Write $a = k/m$, $v = a - 1$ and $q = 5 - 2a$. The free classes are $a\sigma + (a+1)f$, $\sigma + (a+1)f - v(e_1 + e_2)$, and $q(\sigma + 3f - e_1 - e_2)$. They are nef by the cone test and give exactly (14.3). For a nonbase fiber, removal of free order u subtracts uf . For C_σ replace the e_1 exponent by $(v-u)_+$ in the middle and $(q-u)_+$ in the tail; for C_B make that replacement at e_2 . These are

the exact colon identities by the fiber equation. The tail's actual primary order on a base fiber is $\tau = u + 2a - 4$; elsewhere it is $\tau = u$.

Step 2. In this step, we compute the three full chamber tables. The following three tables specify every chamber. Their entries are the support I of the negative part, free degree n on the fiber, full atom ζ at its intersection with σ and full atom b at its intersection with B . The letters E, M, T denote the early, middle and tail ranges. Every listed polygon is the triangle with the indicated vertices. For $C \cap \Delta = \emptyset$:

	(a, u) vertices	I	n	ζ	b
E	$(0, 0), (1, 0), (0, 1)$	$\emptyset$	a	0	0
E	$(1, 0), (1, 2), (0, 1)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
M	$(1, 0), (2, 0), (3/2, 3/2)$	s	$1 - u/3$	$a + u/3 - 1$	0
M	$(1, 0), (3/2, 3/2), (1, 2)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
M	$(2, 0), (2, 2), (3/2, 3/2)$	s, c_2	$2 - (a + u)/2$	$(3a + u)/2 - 2$	0
M	$(2, 2), (1, 2), (3/2, 3/2)$	s, e_1, c_2	$2 - u$	$a + u - 2$	0
T	$(2, 0), (5/2, 0), (2, 2)$	s, c_2	$5 - 2a - u/2$	$2a + u/2 - 3$	$a - 2$

For $C = C_\sigma$:

	(a, u) vertices	I	n	ζ	b
E	$(0, 0), (1, 0), (0, 1)$	$\emptyset$	a	0	0
E	$(1, 0), (1, 2), (0, 1)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
M	$(1, 0), (2, 0), (2, 1)$	$\emptyset$	$2 - a + u$	$2a - u - 2$	0
M	$(1, 0), (2, 1), (1, 2)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
M	$(2, 1), (2, 2), (1, 2)$	s, e_1, c_2	$2 - u$	$a + u - 2$	0
T	$(2, 0), (5/2, 0), (2, 1)$	$\emptyset$	u	$2 - u$	$a - 2$
T	$(5/2, 0), (2, 2), (2, 1)$	s, e_1, c_2	$10 - 4a - u$	$4a + u - 8$	$a - 2$

For $C = C_B$:

	(a, u) vertices	I	n	ζ	b
E	$(0, 0), (1, 0), (0, 1)$	$\emptyset$	a	0	0
E	$(1, 0), (1, 2), (0, 1)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
M	$(1, 0), (2, 0), (2, 1)$	s	$2 - a + 2u/3$	$a + u/3 - 1$	$a - u - 1$
M	$(1, 0), (2, 1), (2, 3)$	s	$1 - u/3$	$a + u/3 - 1$	0
M	$(1, 0), (2, 3), (1, 2)$	s, e_1	$(a - u + 1)/2$	$(a + u - 1)/2$	0
T	$(2, 0), (5/2, 0), (2, 1)$	s	$2u/3$	$2a + u/3 - 3$	$3 - a - u$
T	$(5/2, 0), (2, 3), (2, 1)$	s	$5 - 2a - u/3$	$2a + u/3 - 3$	$a - 2$

These decompositions can be checked directly from the five-curve intersection matrix: on a support I , the negative coefficients are $M_I^{-1}(D \cdot C_i)_{i \in I}$. They are nonnegative and the resulting positive part meets all five cone generators nonnegatively. At the outer edges its square vanishes; beyond them no positive volume satisfies the cone inequalities. For the base fibers, add the residual colon multiplicity and any exceptional negative coefficient at that base point. Adding the prescribed σ and tail factors gives exactly the displayed ζ, b . In the early rows the decompositions are pulled back from S , including the exceptional contributions.

Step 3. In this step, we compute the moments, control the errors uniformly, and apply adjunction. Integration in $da du$ gives

C	$\int n$	$\int \tau n$	$\int n^2/2$	$\int n\zeta$	$\int nb$
$C \cap \Delta = \emptyset$	11/6	59/48	7/12	5/4	1/48
C_σ	11/6	17/12	7/12	5/4	1/48
C_B	11/6	13/8	1/2	4/3	5/48

Each entry is the integral of the displayed polynomial over the listed rational triangles, so the tables specify the calculation and all its endpoints. After multiplication by 6/11 the primary moments are 59/88, 17/22, 39/44. The secondary bounds at σ and B are, in all three cases, at

most 1 and 29/88. At any other point they are at most 7/22 for the first two types and 3/11 for C_B .

Only complete-curve upper bounds are used for secondary images: rank at most $mn + O(1)$ and order sum at most $m^2n^2/2 + O(m)$, plus the full fixed atoms. Terminal rank equality is unnecessary. The $O(m)$ wall rows contribute $O(m^3)$; an $O(m^3)$ bound also holds for interior errors, giving $O(1/m)$ after the original normalization.

If $C \neq C_0$, its intersections with B and σ are distinct. For a tested valuation, choose bases compatible with it and C , and write the normalized divisor as $a_m C + \Omega_m$. For fixed $\lambda < 1$ and large m , raise λa_m to one. The boundary $B|_C/2 + \lambda \Omega_m|_C$ has all coefficients at most one: its B coefficient is at most $1/2 + \lambda(29/88 + o(1))$, and its σ coefficient at most $\lambda(1 + o(1))$. Inversion of adjunction, followed by decreasing the C coefficient, proves log canonicity of $(S, B/2 + \lambda(a_m C + \Omega_m))$ near C . Pass to $m \rightarrow \infty$ and $\lambda \rightarrow 1$. Every point outside C_0 lies on such a fiber, and a generic curve center not contained in C_0 meets that locus. $\square$

Proposition 14.3. *Let $p \in X$ have incident-line scheme either a curvilinear double point supported at one line L , or the union of that double point and a reduced point supported at another line. Then every prime divisor G with $c_X(G) = p$ satisfies*

$$(14.6) \quad A_X(G) \geq S_X(G).$$

More precisely, let $Y = \text{Bl}_L X$, let s be the negative section of its exceptional surface, and let F be the exceptional divisor of $\text{Bl}_s Y$. Let M be the corresponding inverse bisecant in the degree-five line link. If M is special and its contact with the inverse quintic is nonreduced, then X is K -semistable. In all other cases

$$(14.7) \quad A_X(F) = 3, \quad S_X(F) = 117/44, \quad A_X(G) - S_X(G) \geq \frac{15}{44} \text{ord}_G(F) > 0,$$

where F in the last term is the rational Cartier divisor on the blowup of the doubled-line ideal.

Proof. We identify the extraction over the whole marked fiber, transport the complete section spaces through the inverse link, and then apply the two surface lemmas.

Step 1. The doubled-line extraction retains every center above p . An ordinary line has $H^0(N_{L/X}(-p)) = 0$, so its pointed incidence is reduced. Thus L is special and its first exceptional surface is $\mathbb{F}_3$. In transverse parameters (u, v) the doubled-line ideal is (v, u^2) , with u the positive normal direction. Its blowup has charts $v = u^2 t$ and $u^2 = vt$. The second is a transverse A_1 chart, so the blowup U is normal, (U, F) is plt, and $2F$ is Cartier. The index-two chart gives different $B/2$, where B is the image of the intersection of the strict first exceptional divisor E_1 with F . The log discrepancy is 3.

The possible normal types of s give $S = F \simeq \mathbb{F}_0$ or $\mathbb{F}_2$. With f the ruling over L , their restrictions on $\text{Bl}_s Y$ are

$$(14.8) \quad \begin{array}{c|ccc} S & H|_S & F|_S & B \\ \hline \mathbb{F}_0 & f & -\sigma - f & \sigma + 2f \\ \mathbb{F}_2 & f & -\sigma - 2f & \sigma + 3f. \end{array}$$

The normal sequence $0 \rightarrow \mathcal{O}(-3) \rightarrow N_{s/Y} \rightarrow \mathcal{O}(1) \rightarrow 0$ makes B the section of self-intersection four. It maps isomorphically to L . The pullback of the rational Cartier divisor on U is $F + E_1/2$. Thus even grade $2k$, after removal of E_1^k , has the surface classes in the two lemmas. Odd grade $2k + 1$, after removal of E_1^{k+1} , has one less f and the intrinsic fractional term $B/2$. The fiber of $U \rightarrow X$ over p is the entire fiber S_p . The contraction of E_1 to B puts also the old centers outside s onto this same fiber.

Step 2. The inverse line determines the surface. The all-line link and its disjoint flops are given by [KPS18, Theorem 5.2.2, Remark 5.2.4 and Lemma 5.2.5]. Projection from either inverse line type is given by [Fuj26a, Theorem 3.6(3)–(6), Theorem 4.4(1) and Proposition 4.5]. Write

$Z = \text{Bl}_\Gamma V_5$, and blow up its inverse bisecant $M \subset V_5$ with exceptional divisor D_0 . With B_V the hyperplane pullback, projection to the smooth quadric has hyperplane and exceptional classes

$$C = B_V - D_0, \quad J = B_V - 2D_0.$$

The surface J normalizes the hyperplane containing Γ . To identify that hyperplane, use $E \cdot s = 1$ and the pagoda pullback formula of Lemma 4.11. Its first exceptional coefficient is one on the E side, hence two on the inverse side. Since M is generically disjoint from Γ , the image hyperplane has multiplicity two along M . The system $|B_V - 2D_0|$ consists of J alone, as intersection with its contracted fibers forces J as a component. This proves the identification.

The lift $\Gamma_1 \subset J$ maps isomorphically to Γ , because the ideal of M on the smooth curve is invertible. Its intersection Z_0 with D_0 has length two and projects as a closed immersion to M , including a repeated contact. For an ordinary M , write $D_0 = \mathbb{F}_0$ with classes u, η , where η is the fiber over M and $-D_0|_{D_0} = u$; then $J = \mathbb{F}_1$. For a special M , write $D_0 = \mathbb{F}_2$ with negative section g and fiber η , with $-D_0|_{D_0} = g + \eta$; then $J = \mathbb{F}_3$. The restrictions are

M	$B_V _J$	$D_0 _J$	Γ_1
ordinary	$s_0 + 3f_J$	s_0	$s_0 + 3f_J$
special	$s_0 + 4f_J$	$s_0 + f_0$	$s_0 + 4f_J$

For the first row, degree and intersection give $2a + b = 5$, $-a + b = 2$ for $\Gamma_1 = as_0 + bf_J$. For the second, $b \geq 3a$ and $a + b = 5$ force $(a, b) = (1, 4)$. The conductors on D_0 are respectively $2u + \eta$ and $g + C_1$, with $C_1 \in |g + 3\eta|$ smooth. In the $\mathbb{F}_3$ case $g = f_0$ on J , since the quadric pullback has degree zero there; Γ_1 meets each conductor component in length one.

For completeness, the normal modification also retains a double contact. Write locally $M = (x = y = 0)$ and $\Gamma = (x = f(t), y = h(t))$, with f of minimal order. A normal displacement (v_x, v_y) lifts precisely when $v_y - (h/f)v_x$ is divisible by f . Thus the modified normal bundle is obtained by the two contact conditions on its projectivized direction curve, counting zeros of the section. For ordinary M , a nonzero constant section has direction class u , whose intersection with $2u + \eta$ is one. It cannot meet both conditions, so the modified degree -2 bundle has no section and is $\mathcal{O}(-1)^2$. For special M , a section of the positive subline $\mathcal{O}(1)$ vanishing at the residual contact annihilates the length-one discrepancy along g . The permitted modified type is therefore $\mathcal{O} \oplus \mathcal{O}(-2)$. The pagoda is symmetric, giving these two normal types for s . Hence $S = \mathbb{F}_0$ exactly in the ordinary case.

If the special contact is nonreduced, Lemma 5.3 and Proposition 5.2 give K-semistability of the whole member. In the remaining $\mathbb{F}_2$ case there exist two distinct contacts, one on each conductor component and away from their crossing. The ordinary case still allows either contact scheme.

Step 3. Reversing the blowups identifies the full section spaces. Let $W = \text{Bl}_{\Gamma_1}(\text{Bl}_M V_5)$, let N be its quintic exceptional divisor, and write D for the transform of D_0 . The strict transform T of J corresponds to E , and the strict anticanonical class from X is M_X , with

$$T = B_V - 2D - N, \quad A = -K_W = 2B_V - D - N, \quad M_X = 3B_V - D - 2N.$$

The last identity follows from $H = 3B^+ - 2R^+$ or from $K_W = -M_X + T + 2D$ with a common canonical rational form. The full spaces of sections with F order at least j are $H^0(W, mM_X - jD)$. This is equality by rational-function divisorial inequalities: T, D are the only prime divisors on W exceptional over X , whereas N corresponds to a divisor on X . It requires no morphism $W \rightarrow X$. In particular

$$N = M_X - 3T - 5D, \quad R = H - 3E_1 - 5F$$

on the two sides.

Resolve Z_0 on D_0 by its two successive simple blowups, with total classes e_1, e_2 and $J_e = e_1 + e_2$. The elementary transformation $b : P \rightarrow S$ has total exceptional divisor $J_b = 2\eta - J_e$

and inverse cluster Δ of length two. In the ordinary case

$$b^*\sigma = \eta, \quad b^*f = u + \eta - J_e, \quad A|_P = b^*(\sigma + f), \quad C|_P = b^*(2\sigma + f) - J_b.$$

The smooth conductor maps to B . Its inverse cluster is conjugate to Z_0 under the conductor double cover. The first ruling immersion follows from that of Z_0 ; the other is the isomorphism $B \rightarrow L$. The image of $N|_P = J_e$ is exactly R_S , the two σ fibers, and $R_S|_B = \Delta + \iota(\Delta)$.

In the special case the inverse-side calculation is

$$b^*\sigma = g + \eta - J_e, \quad b^*f = \eta, \quad A|_P = b^*(\sigma + 2f), \quad C|_P = b^*(\sigma + 3f) - J_b.$$

The inverse points are $q_\sigma \in \sigma \setminus B$ and $q_B \in B \setminus \sigma$ on different fibers. The contact on g transforms to the other conductor component, and that on C_1 transforms to g . The remaining pagoda reflects restrictions by

$$D_S \longmapsto D_S + (D_S \cdot \sigma)\sigma$$

as in [Rei83, Theorem 6.3 and Claim 6.5]. It sends the different curve to $\sigma + B$ on the inverse side. The inverse image of J_e is $C_\sigma + C_B$, so its actual X -side image is

$$R_S = 2\sigma + C_\sigma + C_B.$$

The quadric restriction has the additional fixed σ , with free class $\sigma + 3f$ and the two base conditions. These are identities of divided rational sections: they agree away from σ , and the reflected class determines the remaining multiplicity along σ .

The transverse inequality $v_F \leq 2v_E$ forces $E_1^{[j/2]}$. For $j = 2k$ and $m \leq k \leq 2m$ the divided class is

$$mM_X - 2kD - kT = (2m - k)A + (k - m)C.$$

The early and middle images therefore lie in precisely the first two systems of (14.1) or (14.4). Repeated contacts retain the true powers of (u, t^2) , with equal multiplicities at both resolution centers.

For $j \geq 4m$ the order condition on $\text{Bl}_M V_5$ is

$$3mB_V - (m + j)D_0 = (5m - j)C + (j - 2m)J.$$

Intersection with the contracted fibers forces J^{j-2m} . It already supplies order $2m$ along Γ_1 . The residual space is exactly $H^0(Q, \mathcal{O}_Q(5m - j))$, and vanishes for $j > 5m$. On the X model the fixed factors are $E_1^{j-2m}R^{j-4m}$, with residual class $(5m - j)(H - 2E_1 - 4F)$. For even $j = 2k$ their restrictions are exactly the final systems in the two surface lemmas, including every fixed factor.

Step 4. The odd grades have the even pieces' leading rank with their actual different. The odd divided class differs by $-D - T$. On P this restricts to $-b^*f$ in the balanced case and $-b^*(\sigma + f)$ in the unbalanced case. In the unbalanced case, the odd inverse free class is $(m-1)\sigma + (m+k-1)f$ with base exponent $k - m$. Its σ intersection is $k - m + 1$, so reflection produces σ^{k-m+1} . Multiplication by a fixed f fiber embeds it into σ^{k-m} times the even free system, using the remaining σ factor. The balanced embedding is multiplication by that same fiber. The base-ideal exponent is unchanged.

In the tail q decreases by one, the B power stays $k - 2m$, and the R power increases by one. On $\mathbb{F}_0$ the additional $s_{R_S}sf$ is a section of $2\sigma + f$ vanishing on Δ . On $\mathbb{F}_2$ the additional $s_\sigma s_{C_\sigma + C_B}sf$ is a section of $\sigma + 3f$ vanishing on both base points. These embed the odd tail into the even tail with exponent q . Choose the multiplying fiber different from S_p . Its local order at all tested centers is zero. The rational $B/2$ correction remains, bounded independently of m .

Each even containing family has rank $(11/6)m^3 + O(m^2)$. The actual anticanonical rank is $(11/3)m^3 + O(m^2)$. Thus the two containing copies and the full refinement have total deficit

$O(m^2)$, with the original denominator unchanged. Every first grade is at most $5m$, so bounded shifts and deficits contribute only $O(m^3)$ to moment numerators. Consequently

$$S_X(F) = \frac{6}{11} \int_0^{5/2} 2a\rho(a) da = \frac{6}{11} \left(\frac{2}{3} + \frac{7}{2} + \frac{17}{24} \right) = \frac{117}{44}.$$

For each local valuation, complete actual bases inside the containing spaces; nonnegative order sums only increase. The odd $B/2$ term is uniformly $O(1/m)$ relative to discrepancy, since $v(B) \leq 2A_{S,B/2}(v)$. This transfers the surface estimates for both parities.

Step 5. The marked fiber is covered and the surplus is positive. No step above assumes that L is the sole support of incidence. The full balanced estimate covers every fiber, including the base fibers that can occur when another line passes through p . The unbalanced estimate excludes only the fiber through $\sigma \cap B$. By Lemma 9.1, its image on L has entire incidence scheme of length three supported at L . Neither the length-two nor the $2+1$ point in the statement is that point. Thus S_p is covered in both cases.

For fixed $\lambda < 1$ and all sufficiently large m , the surface estimates and parity embeddings make $(S, B/2 + \lambda D_m^{\text{ref}})$ log canonical near S_p for every actual refined basis divisor. Choose an anticanonical basis compatible with G, F and write

$$\mu^* D_m = S_m(F)F + \Theta_m, \quad \Theta_m \geq 0, \quad F \not\subset \text{Supp } \Theta_m.$$

Its restriction is the actual refinement just bounded. Plt inversion of adjunction gives

$$A_X(G) \geq \lambda S_m(G) + (3 - \lambda S_m(F)) \text{ord}_G(F).$$

Take $m \rightarrow \infty$ and then $\lambda \rightarrow 1$. Since $2F$ is the Cartier exceptional divisor of an ideal supported on L and $c_X(G) = p \in L$, its order is positive. This proves (14.7). The whole-member alternative from Step 2 proves the remaining case. $\square$

15. CURVILINEAR TRIPLE INCIDENCE IN DEGREE TWENTY-TWO

Lemma 15.1. *Suppose that $\mathcal{L}_p$ is curvilinear of length three, supported at a line L . In the canonical net of Setup 13.1, choose $V = U \oplus Q$, $p = [U]$, and graph rows $[I_3 \mid M]$. There exist bases for which*

$$(15.1) \quad T_p X = \left\{ \begin{pmatrix} x & y & z & 0 \\ y & z & 0 & 0 \\ z & 0 & 0 & 0 \end{pmatrix} \right\}$$

and the three net forms have blocks $\Omega_j = \begin{pmatrix} 0 & C_j \\ -C_j^t & B_j \end{pmatrix}$, where each B_j is alternating and

$$(15.2) \quad C_0 = \begin{pmatrix} 1 & 0 & 0 & \alpha_0 \\ 0 & 1 & 0 & \beta_0 \\ 0 & 0 & 1 & \gamma_0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 & 0 & \alpha_1 \\ 0 & 0 & 1 & \beta_1 \\ 0 & 0 & 0 & \gamma_1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 & \alpha_2 \\ 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}.$$

The exact local equations are

$$(15.3) \quad C_j M^t - M C_j^t + M B_j M^t = 0, \quad j = 0, 1, 2,$$

and the matrix

$$(15.4) \quad \mathcal{J} = \begin{pmatrix} \gamma_1 - \beta_2 & \alpha_2 - \gamma_0 & \beta_0 - \alpha_1 \\ -\gamma_2 & \gamma_1 & \alpha_2 - \beta_1 \\ 0 & -\gamma_2 & \beta_2 \end{pmatrix}$$

is invertible. The entries $x = M_{11}, y = M_{12}, z = M_{13}$ are regular parameters, and every correction to (15.1) belongs to $(xz - y^2, yz, z^2) \subset \mathbb{C}[[x, y, z]]$.

Proof. We normalize the incidence jet, retain all net coefficients, and compute the correction ideal.

Step 1. In this step, we normalize the full length-three jet.

The canonical net gives the exact graph equations. The line scheme is the intersection of the tangent plane with the rank-one Segre variety: on a rank-one direction every quadratic term MB_jM^t vanishes modulo the minors, and the remaining equations are precisely the linear tangent equations. This is also the scheme-theoretic description in [KP23, Lemma 5.6 and Lemma A.5].

The length-three jet spans its plane. Otherwise every Segre quadric would vanish on its spanning line, because a quadratic on that line vanishing on a length-three subscheme is zero. The entire line would lie in the finite intersection, a contradiction. Both factor jets have nonzero first derivative, since otherwise the tangent line lies in a Segre fiber and again forces a positive-dimensional incident-line scheme. Normalize the two jets independently as

$$[1 : t : \epsilon_U t^2], \quad [1 : t : \epsilon_Q t^2 : 0], \quad \epsilon_U, \epsilon_Q \in \{0, 1\}.$$

To justify the normalization, expand a vector as $u_0 + tu_1 + t^2u_2$ with u_0, u_1 independent. If u_2 adds a third direction, use the three as basis. Otherwise a coordinate $t + bt^2$ on the factor line is made t by the fractional linear map $s \mapsto s/(1 + bs)$. The resulting tangent matrices are

$$\begin{pmatrix} x & y & \epsilon_Q z & 0 \\ y & z & 0 & 0 \\ \epsilon_U z & 0 & 0 & 0 \end{pmatrix}.$$

Both epsilon values cannot vanish, since the rank-one intersection would then contain the conic $xz - y^2 = 0$.

Step 2. In this step, we exclude the one-sided jets. If $(\epsilon_U, \epsilon_Q) = (1, 0)$, symmetry of CM^t forces the first two columns of every cross block C to be zero. Successive coefficients give $C_{21} = C_{31} = 0$, $C_{22} = C_{11}$, $C_{32} = 0$, and $C_{12} = C_{11} = 0$. The derivative kernel then contains all matrices supported in the first two columns, a six-dimensional space, contradicting smoothness of the threefold. If $(\epsilon_U, \epsilon_Q) = (0, 1)$, symmetry gives

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & d_1 \\ 0 & c_0 & c_1 & d_2 \\ 0 & 0 & 0 & d_3 \end{pmatrix}.$$

A pencil in the kernel of the net functional d_3 annihilates the third U vector. It descends to alternating forms on a six-dimensional quotient. Its cubic Pfaffian has a projective zero over $\mathbb{C}$, including the identically zero case, so some nonzero form has rank at most four. This contradicts the rank-six condition on the whole net. Thus both epsilon values are one, and its minors give $(xz - y^2, yz, z^2)$.

Step 3. In this step, we determine the cross blocks with all coefficients retained and compute their Jacobian. The general cross block is now

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & d_1 \\ 0 & c_0 & c_1 & d_2 \\ 0 & 0 & c_0 & d_3 \end{pmatrix}.$$

The map from the net to (c_0, c_1, c_2) is injective: a nonzero kernel member has only its fourth column and contributes at most two derivative equations, while each other form contributes at most three. The total derivative rank would be at most eight. The map is therefore an isomorphism, yielding (15.2).

Put $u = M_{14}, v = M_{24}, w = M_{34}$. Six equations of the linearized system give

$$\begin{aligned} M_{21} &= y + \beta_0 u - \alpha_0 v, & M_{22} &= z + \beta_1 u - \alpha_1 v, & M_{23} &= \beta_2 u - \alpha_2 v, \\ M_{31} &= z + \gamma_0 u - \alpha_0 w, & M_{32} &= \gamma_1 u - \alpha_1 w, & M_{33} &= \gamma_1 v - \beta_1 w. \end{aligned}$$

The remaining three are $\mathcal{J}(u, v, w)^t = 0$. Smoothness makes $\mathcal{J}$ invertible. The formal implicit-function theorem solves the nine eliminated entries uniquely as their tangent terms plus higher terms.

Step 4. In this step, we prove that every correction belongs to the stated ideal. Finally put $M = T + R$. Every entry of TB_jT^t belongs to $I = (xz - y^2, yz, z^2)$, as it is a linear combination of minors. The equations have the form

$$\mathcal{A}(R) + Q(T) + Q_1(T, R) + Q_2(R) = 0,$$

where $\mathcal{A}$ is the invertible constant linear map. Iteration from $R = 0$ preserves I and increases ordinary degree, so the unique convergent formal solution has every entry in I . This asserts no independence of the complete filtered section ring from the retained coefficients. $\square$

Lemma 15.2. *At the point of Lemma 15.1 there exist exactly one or two distinct conics through p , where conics include all subschemes of Hilbert polynomial $2t + 1$. Every such conic contains L . There exists one precisely when $\gamma_2 = 0$; otherwise there exist two, with net parameters*

$$(15.5) \quad [0 : 0 : 1], \quad [0 : \gamma_2 : -\gamma_1].$$

Proof. For the rank-six net, the radical map into $\mathbb{P}(V)$ is a closed embedding, X contains no planes, and its conics are exactly the loci of isotropic three-planes containing a net radical [BKM26b, Lemmas B.1, B.3 and B.4]. This parametrization has no identifications even for nonreduced conics. If two distinct radicals gave one conic, their span K would be a two-plane, and that conic would lie scheme-theoretically in the linearly embedded $\{U' \supset K\} \simeq \mathbb{P}(V/K) = \mathbb{P}^4$. The isotropy equations there are linear: their constant restrictions to K vanish because the conic is nonempty, and their remaining conditions pair the third vector with K . Thus X would contain the linear span of the conic. That span has dimension at least two: a one-dimensional closed subscheme of a reduced projective line must be the line itself, with polynomial $t + 1$. This would give a plane in X .

For parameters $[c_0 : c_1 : c_2]$, write the last column of C as d_i , where

$$d_1 = \sum \alpha_i c_i, \quad d_2 = \sum \beta_i c_i, \quad d_3 = \sum \gamma_i c_i.$$

A conic contains $p = [U]$ exactly when its radical lies in U , equivalently when C has a nonzero left kernel. Its rank must then equal two, since lower rank would give a radical of dimension at least two. The maximal minors are

$$c_0^3, \quad c_0^2 d_3, \quad c_0(c_1 d_3 - c_0 d_2), \quad c_1^2 d_3 - c_0 c_2 d_3 - c_0 c_1 d_2 + c_0^2 d_1.$$

Their support is $c_0 = 0, c_1^2(\gamma_1 c_1 + \gamma_2 c_2) = 0$. If $\gamma_2 = 0$, invertibility of $\mathcal{J}$ gives $\beta_2 \gamma_1(\gamma_1 - \beta_2) \neq 0$, so only $[0 : 0 : 1]$ occurs. Otherwise the two points in (15.5) are distinct.

The line L consists of three-planes spanned by $u_1 + xq_1, u_2, u_3$, with fixed two-plane $K = \langle u_2, u_3 \rangle$. At $[0 : 0 : 1]$ the left kernel of C has first coordinate zero. At the other parameter, $c_1 \neq 0$ and $d_3 = 0$ force the first two kernel coordinates to vanish. Both radicals therefore lie in K , so their conics contain the entire Grassmannian line L scheme-theoretically. $\square$

Proposition 15.3. *If exactly one conic passes through a curvilinear triple-incidence point, then the whole threefold X is K -semistable.*

Proof. We construct a smooth family with all nonzero fibers isomorphic to X and central fiber carrying a nontrivial multiplicative action. Use Lemma 15.2, so $\gamma_2 = 0$. Invertibility of $\mathcal{J}$ gives $\beta_2 \gamma_1(\gamma_1 - \beta_2) \neq 0$. Put $b = (B_2)_{12}$. The third U vector lies in the radical of Ω_2 . On its six-dimensional quotient the Pfaffian is, up to sign, $\beta_2 b$: the second U vector must pair with the fourth Q vector, and the first with the third. Rank six forces $b \neq 0$.

Step 1. We give the actual family of nets. In dual coordinates A, B, C, D, E, F, G , let $\nu(s)$ act on vectors with weights $(-1, -2, -3, 0, 1, 2, 3)$. The highest parts of the three alternating forms are

$$AD + BE + CF, \quad AE + BF + \gamma_1 CG, \quad AF + \beta_2 BG + bDE,$$

with weights $1, 0, -1$, respectively. Every undisplayed coefficient has smaller weight. For $d = (1, 0, -1)$, the forms $\Omega_j(s) = s^{d_j} \nu(s^{-1}) \Omega_j$ are polynomial: each coefficient has nonnegative exponent d_j minus its weight. They have rank three at every parameter, using the pivots AD, AE, AF . At a nonzero parameter the net is conjugate to the original one.

The diagonal change of dual coordinates with multipliers

$$(1, b^{-1}, b^{-2}, 1, b, b^2, b^3/\gamma_1)$$

and rescaling of the last two forms makes the central net

$$(15.6) \quad AD + BE + CF, \quad AE + BF + CG, \quad AF + \tau BG + DE, \quad \tau = \beta_2/\gamma_1 \notin \{0, 1\}.$$

Step 2. Every central member is smooth globally. The seven vector weights are distinct, so torus-fixed three-planes are coordinate planes. The forbidden pairs for isotropy are

$$AD, BE, CF, AE, BF, CG, AF, BG, DE.$$

The allowed triples are exactly ABC, BCD, DFG, EFG . With two U coordinates, only BC permits a Q coordinate, namely D ; with one U coordinate, its permitted Q choices give no allowed pair; with three Q coordinates, excluding DE leaves the last two listed triples.

In the graph chart of each ordered triple, take the complement in alphabetical order and write its twelve entries as $(x_1, \dots, x_4; y_1, \dots, y_4; z_1, \dots, z_4)$. The nine linear equations, in net order and pair order 12, 13, 23, are

$$\begin{array}{l|l} ABC & -x_2 + y_1, -x_3 + z_1, -y_3 + z_2; -x_3 + y_2, -x_4 + z_2, -y_4 + z_3; -\tau x_4 + y_3, z_3, \tau z_4 \\ BCD & -x_3 + y_2, x_1 + z_2, y_1 + z_3; -x_4 + y_3, z_3, z_4; \tau y_4, -x_2 + \tau z_4, -y_2 \\ DFG & x_3 - y_1, -z_1, -z_3; x_2, x_3, y_3 - z_2; x_1 + y_4, \tau x_2 + z_4, \tau y_2 - z_1 \\ EFG & x_3 - y_2, -z_2, -z_3; x_2 - y_1, x_3 - z_1, y_3 - z_2; x_1 - y_4, \tau x_2 - z_4, \tau y_2 - z_1. \end{array}$$

At ABC , elimination leaves free x_1, x_2, x_3 : $z_3 = z_4 = y_4 = 0$, $z_2 = y_3 = x_4$, and $(\tau - 1)x_4 = 0$. At BCD , it leaves x_4, z_1, z_2 : $z_3 = z_4 = y_4 = x_2 = y_2 = x_3 = y_1 = 0$, $x_1 = -z_2$, $y_3 = x_4$. At DFG , it leaves x_4, y_4, z_2 : $x_2 = x_3 = y_1 = z_1 = z_3 = y_2 = z_4 = 0$, $x_1 = -y_4$, $y_3 = z_2$. At EFG , it leaves x_2, x_4, y_4 : $z_2 = z_3 = y_3 = 0$, $x_3 = y_2 = z_1$, $(\tau - 1)y_2 = 0$, $y_1 = x_2$, $z_4 = \tau x_2$, $x_1 = y_4$. All four Jacobians have rank nine.

The rank-deficient locus is closed, projective and torus invariant. If nonempty, a limit of any point under the one-parameter subgroup would give a fixed point in that locus. All fixed points were just checked. Hence it is empty, and the entire central zero locus is smooth of dimension three.

Step 3. The family has the required Fano properties. Let $\mathcal{X}$ be the relative zero locus in $\text{Gr}(3, 7) \times \mathbb{A}^1$. Every nonzero fiber is scheme-theoretically isomorphic to X ; Step 2 checks the central fiber. The fiber-direction Jacobian has rank nine everywhere, so the relative Jacobian criterion gives a smooth projective morphism of relative dimension three. Its fibers are connected because their component count in a smooth proper family is locally constant and the nonzero fibers are connected.

For the Plücker bundle $H_{\text{Pl}} = \det \mathcal{U}^*$,

$$\omega_{\text{Gr}(3,7)} = H_{\text{Pl}}^{-7}, \quad \det(\Lambda^2 \mathcal{U}^*) = H_{\text{Pl}}^2.$$

Relative adjunction for the rank-nine regular zero locus gives $\omega_{\mathcal{X}/\mathbb{A}^1} = H_{\text{Pl}}^{-1}|_{\mathcal{X}}$. Thus the relative anticanonical bundle is Cartier and relatively ample. Smooth proper topological local triviality preserves the integral H^2 lattice and this bundle class. Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem] and the exponential sequence identify H^2 with the Picard group on every fiber. Hence the central fiber has Picard group generated by its primitive anticanonical class, with cube 22. Its torus action is nontrivial, since otherwise a three-dimensional variety would lie in the four-point fixed locus.

The central member is K-semistable by Theorem 5.1. This smooth family over the normal base $\mathbb{A}^1$ has connected normal Fano fibers and Cartier relative canonical class, so [BLX19,

Definition 2.1 and Corollary 1.2] applies. The semistable locus is open and contains zero, hence contains a nonzero parameter. All nonzero fibers are X . The valuative criterion then gives $A_X(G) \geq S_X(G)$ for every prime divisor over X . $\square$

Lemma 15.4. *Suppose instead that exactly two conics pass through the curvilinear triple-incidence point p . There exist algebraic regular parameters x, y, z and $\lambda \in \mathbb{C}^*$ such that, for*

$$(15.7) \quad D = y^2 - xz, \quad G = yz + \lambda x^2 D, \quad J = z^2 + \lambda xy D,$$

the weight-(1, 3, 5) extraction has exceptional $E = \mathbb{P}(1, 3, 5)$, zero different and $A_X(E) = 9$. If T records anticanonical section degree, its entire associated graded section ring is

$$(15.8) \quad \mathbb{C}[T, Tx, Ty, Tz, TD, TxD, TG, TyD, TJ, TzD, TD^2, TxD^2, TDG, TyD^2].$$

Its further associated graded along D has basis $T^m U^a V^{2r-a} h^c$, where $x = U^2, y = UV, z = V^2$ on $D = 0$ and

$$(15.9) \quad \begin{aligned} 0 \leq r \leq 2m, \quad 0 \leq a \leq 2r, \quad 0 \leq c \leq 2m, \\ r + a \leq 3m, \quad r + c \leq 3m, \quad r - a + c \leq 2m. \end{aligned}$$

Here h is the symbol of D , and the exceptional degree is $j = 5r - 2a + 6c$. The valuation Rees algebra is finitely generated, and

$$(15.10) \quad S_X(E) = 733/88.$$

Proof. We keep all coefficients while normalizing two forms, exhibit fourteen actual sections, and prove exhaustion by integer counts.

Step 1. The implicit equations give the first nonzero quadratic section. Change the fourth complement vector to $q_4 - \alpha_0 q_1 - \alpha_1 q_2 - \alpha_2 q_3$. This makes all α_j zero. Write

$$b = \gamma_2 \neq 0, \quad r_0 = \gamma_1, \quad s = \beta_2, \quad b_0 = \beta_0, \quad b_1 = \beta_1, \quad g_0 = \gamma_0.$$

The determinant condition is

$$J_0 = (r_0 - s)(r_0 s - b b_1) - g_0 b s + b_0 b^2 \neq 0.$$

Write the six entries of each B_j , in DE, DF, DG, EF, EG, FG order, as $k_j, l_j, m_j, n_j, o_j, p_j$. Put $k = k_2$. The cross block of Ω_2 has rank two. On the quotient by its radical, its two cross pairings use F, G and the remaining symplectic pairing is kDE . Thus the rank-six condition forces $k \neq 0$.

Write the actual matrix as

$$M = \begin{pmatrix} x & y & z & u \\ a & c_0 & v & w_0 \\ d & e & f & h \end{pmatrix}, \quad Q_{\alpha\beta}^j = (M B_j M^t)_{\alpha\beta}.$$

The nine exact equations are

$$(15.11) \quad \begin{aligned} a - y - b_0 u + Q_{12}^0 &= 0, & d - z - g_0 u + Q_{13}^0 &= 0, & e - v + b_0 h - g_0 w_0 + Q_{23}^0 &= 0, \\ c_0 - z - b_1 u + Q_{12}^1 &= 0, & e - r_0 u + Q_{13}^1 &= 0, & f + b_1 h - r_0 w_0 + Q_{23}^1 &= 0, \\ v - s u + Q_{12}^2 &= 0, & f - b u + Q_{13}^2 &= 0, & s h - b w_0 + Q_{23}^2 &= 0. \end{aligned}$$

Give the regular parameters x, y, z weights 1, 3, 5. All tangent minors have weight at least six. Iteration of the invertible ordinary implicit system therefore keeps every correction in weight at least six and increases the weight of subsequent corrections.

Set

$$L = \frac{r_0 s - b b_1}{b^2}, \quad \eta = \frac{k b^2}{J_0} \neq 0.$$

Solving weight six in (15.11) gives

$$(15.12) \quad h_6 = \eta D, \quad u_6 = L \eta D, \quad (w_0)_6 = (s/b) \eta D, \quad f_6 = b L \eta D, \quad e_6 = r_0 L \eta D, \quad v_6 = (s L \eta + k) D.$$

To check this, the 13, 23 quadratic entries vanish in weight six, whereas $Q_{12}^j = -k_j D$. The last, sixth and eighth equations give $w_0 = sh/b$, $f = bLh$, $u = Lh$. Then $e = r_0 u$ and $v = su + kD$. The third equation becomes $(r_0 - s)u + (b_0 - g_0 s/b)h = kD$, whose coefficient of h is J_0/b^2 .

Step 2. Two rank-six forms can be normalized without changing the regular filtration. Use dual coordinates $A, B, C, D_0, E_0, F_0, G_0$, reserving D for the polynomial. Replace Ω_1 by $\Omega_1 - (r_0/b)\Omega_2$ and put

$$E_1 = E_0 - (r_0/b)F_0, \quad F_1 = F_0 - bLG_0, \quad C_1 = C + LA + (s/b)B.$$

The two cross parts become $AE_1 + BF_1$ and $AF_1 + bC_1G_0$. Write the first quadratic block as

$$k'D_0E_1 + l'D_0F_1 + \kappa D_0G_0 + n'E_1F_1 + o'E_1G_0 + p'F_1G_0.$$

Replacing A by $A_2 = A + k'D_0 - o'G_0$ and B by $B_2 = B + l'D_0 + n'E_1 - p'G_0$ makes this form exactly

$$\Omega' = A_2E_1 + B_2F_1 + \kappa D_0G_0.$$

Its rank six forces $\kappa \neq 0$. The other form retains cross part $A_2F_1 + bC_1G_0$ and DE coefficient k : the preceding replacements add only the other quadratic products.

We verify the filtration through these changes. The change to C_1 replaces the first two graph rows by $\text{row}_1 - L \text{row}_3$ and $\text{row}_2 - (s/b) \text{row}_3$. The new first-row free entries before the shear are

$$x - Ld, \quad y - (r_0/b)z - L(e - (r_0/b)f), \quad z - bLu - L(f - bLh).$$

Their symbols remain x, y, z in weights 1, 3, 5 by (15.12). The third-row last entry retains symbol ηD . The complement shear has form $M \mapsto (I + MS)^{-1}M$ with the third column of S zero. Its first-row corrections in the three free positions have weights at least 2, 4, 6, respectively. The third-row last entry changes only above weight six, since the first two entries of that row of MS have weight at least five. These bounds persist through the inverse geometric series. All transformations are algebraic near p , as the inverted matrix equals the identity there. A substitution preserving the initial x, y, z has an inverse with unchanged initial symbols, so the entire weight filtration is preserved.

Relabel the normalized coordinates. The exact equations of Ω' are

$$c_0 = z - \kappa(xw_0 - ua), \quad e = -\kappa(xh - ud), \quad f = -\kappa(ah - w_0d).$$

The other form gives $v + Q_{12}^2 = 0$, $f - bu + Q_{13}^2 = 0$, $-bw_0 + Q_{23}^2 = 0$. Initially u, v, w_0, h have weight at least six. The first three equations raise e, f to weights at least seven and nine, with symbols $-\kappa\eta xD, -\kappa\eta yD$. They retain symbol z for c_0 . The second form then raises u, w_0 to at least seven and nine, while $v_6 = kD$. The symbols of a, d remain y, z . Put $\lambda = \kappa\eta \neq 0$.

Step 3. Fourteen Plücker sections have the required symbols. With $P_0 = P_{ABC}$, determinant expansion gives

P_I	ABC	BCD	BCE	BCF	ABG	BDG	BDE
$\text{in}(P_I/P_0)$	1	x	y	z	ηD	ηxD	$-G$
P_I	BEG	BDF	BFG	DEG	DFG	DEF	EFG
$\text{in}(P_I/P_0)$	ηyD	$-J$	ηzD	$-\eta D^2$	$k\eta xD^2$	kDG	$k\eta yD^2$

For the two mixed symbols use $xe - yd = -\lambda x^2D - yz$ in weight eight and $xf - zd = -\lambda xyD - z^2$ in weight ten. The DEF determinant groups as

$$v(yd - xe) + f(xc_0 - ya) + z(ae - c_0d).$$

Its first term has symbol kDG in weight fourteen; the other two have weights at least fifteen. For DEG the leading term is $h(xc_0 - ya)$, giving $-\eta D^2$. For DFG, EFG it is xvh, yvh . The remaining one-by-one or two-by-two minors follow from the weight bounds in Step 2. The weights of the fourteen nonzero symbols are 0, 1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. Thus the algebra $\mathcal{B}$ in (15.8) is contained in the full associated graded.

Step 4. Integer filling proves exhaustion in every degree. Filter $\mathbb{C}[x, y, z]$ by the prime polynomial D . Its associated graded is $\mathbb{C}[x, y, z, h]/(y^2 - xz)$. The initial exponents (r, a, c) of the fourteen generators are

$$\begin{array}{l|l} c = 0 & (0, 0), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1) \\ c = 1 & (0, 0), (1, 0), (1, 1), (1, 2), (2, 1) \\ c = 2 & (0, 0), (1, 1), (1, 2). \end{array}$$

Every generator satisfies (15.9). Conversely, when $r \leq m$, use r bins of r -value one and $m - r$ bins of value zero. A zero bin permits $c = 0, 1, 2$ and $a = 0$. A one bin permits $a = 0, 1, 2$, with c capacity one for $a = 0$ and two otherwise. Distribute a to make $\min(a, r)$ one bins positive, then use the remaining second units. The total capacity is $\min(2m, 2m - r + a)$. Lowering individual c values realizes every integer up to it, exactly the required conditions.

For $m \leq r \leq 2m$, use $r - m$ two bins and $2m - r$ one bins. A two bin permits $a = 0, 1$ with c capacity a . Total a capacity is $3m - r$. First put one unit in $\min(a, m)$ distinct bins, then put remaining units into one bins. This is possible because $a \leq m + (2m - r)$. The c capacity is $(2m - r) + \min(a, m) = \min(3m - r, 2m - r + a)$. Again all smaller integers occur. This proves filling at every boundary.

For fixed $r \leq m$ the number of pairs (a, c) is

$$(2r + 1)(2m + 1) - r(r + 1)/2.$$

For $m \leq r \leq 2m$ it is $(3m - r + 1)^2 - m(m + 1)/2$. Sum the first over $0, \dots, m$ and the second over $m + 1, \dots, 2m$ to obtain

$$(15.13) \quad N_m = \frac{11}{3}m^3 + \frac{11}{2}m^2 + \frac{23}{6}m + 1.$$

This is $h^0(X, mH)$ by Kodaira vanishing [DEL00, p. 142, Nadel Vanishing Theorem] and Riemann-Roch: $H^3 = 22$, $Hc_2(X) = 24$, and $\chi(\mathcal{O}_X) = 1$. The semigroup algebra injects into $\text{gr}_D \mathcal{B}$, so

$$N_m \leq \dim \mathcal{B}_m \leq \dim(\text{gr}_w R)_m = N_m.$$

All are equal, in every retained degree. This proves both the full ring and the full further refinement.

Lift a section's lowest symbol by a polynomial in the fourteen actual sections and subtract. Repetition strictly raises its integer order and stops before $15m + 1$. It proves finite generation of the extended Rees algebra. For nonnegative filtration indices, use the finitely many lifts $s_i t^q$, $0 \leq q \leq w(s_i)$; distribute a required total exponent among the factors within these capacities. Successive subtraction of the initial symbols, terminating in each degree, proves generation of the Rees algebra.

Step 5. The weighted extraction and the moment use the original normalization. The three weight- $(1, 3, 5)$ blowup charts have smooth index covers with exceptional coordinate hyperplane, modulo groups of orders 1, 3, 5. Thus the pair is plt, with one exceptional $\mathbb{P}(1, 3, 5)$ and relatively ample $-E$. The weights are pairwise coprime, so no curve has generic stabilizer and the different is zero. The weighted Jacobian gives discrepancy $1 + 3 + 5 = 9$, normalized by $w(x) = 1$. This local algebraic extraction glues to the identity away from p .

Let $\mathcal{P}$ be (15.9) at $m = 1$. Its volume and moments are

$$\int_{\mathcal{P}} 1 = \frac{11}{3}, \quad \int_{\mathcal{P}} r = \frac{89}{24}, \quad \int_{\mathcal{P}} a = 3, \quad \int_{\mathcal{P}} c = 3.$$

A complete integration prescription is: for $0 \leq r \leq 1$, use $0 \leq a \leq r$ with c height $2 - r + a$, then $r \leq a \leq 2r$ with height 2; for $1 \leq r \leq 2$, use $0 \leq a \leq 1$ with height $2 - r + a$, then $1 \leq a \leq 3 - r$ with height $3 - r$. Integrating 1, r , a and half the squared height gives the four entries. The

grading $j = 5r - 2a + 6c$ yields

$$S_X(E) = \frac{5(89/24) - 2 \cdot 3 + 6 \cdot 3}{11/3} = \frac{733}{88}.$$

The exponent differences generate the full lattice $\mathbb{Z}^3$ in (r, a, c) . The finitely many facets have only $O(m^2)$ lattice points in bounded-width boundary strips, each of weight $O(m)$. The normalized count and moment errors are therefore $O(1/m)$. This identifies the complete jumping-number moment with $S_X(E)$. $\square$

Proposition 15.5. *In the situation of Lemma 15.4, the full refined series W on E satisfies*

$$(15.14) \quad S(W; v) \leq \frac{125}{132} A_E(v) \quad \text{for every prime divisor } v \text{ over } E.$$

For every prime divisor G over X centered at p ,

$$(15.15) \quad A_X(G) - S_X(G) \geq \frac{7}{132} A_X(G) + \frac{17}{88} \text{ord}_G(E) > 0.$$

Proof. We degenerate the full finite-dimensional image spaces to a series whose eigenvectors factor into explicit curves, resolve their intersections, and transfer the resulting bound to every valuation.

Step 1. An auxiliary degeneration preserves every image space. Denote (15.8) by $\mathcal{B}_\lambda$. Give x, y, z auxiliary weights $\ell = (0, 1, 2)$ and use highest weight in each finite (m, j) piece. The initial algebra contains $\mathcal{B}_0$, since the highest parts of G, J are yz, z^2 . Its D -refined generator symbols generate the semigroup in (15.9). Thus

$$N_m \leq \dim(\mathcal{B}_0)_m \leq \dim(\text{gr}_\ell \mathcal{B}_\lambda)_m = N_m.$$

All inequalities are equalities, separately in each retained degree. In particular $\text{gr}_\ell \mathcal{B}_\lambda = \mathcal{B}_0$ for the entire series. For a section of highest degree d , the polynomial family

$$s_t(x, y, z) = t^d s(x, t^{-1}y, t^{-2}z)$$

has its highest form at zero. A compatible basis limits to an independent eigenbasis; all nonzero fibers are obtained by one common automorphism of E .

Step 2. We describe every eigenweight block. For the exponents in (15.9),

$$j = 5r - 2a + 6c, \quad \ell = 2r - a + 2c.$$

At fixed j, ℓ, c this gives

$$r = j - 2\ell - 2c, \quad a = 2j - 5\ell - 2c, \quad 2r - a = \ell - 2c.$$

The allowed integers c are the consecutive interval $c_{\min}, \dots, K$, where

$$c_{\min} = \left\lceil \max \left(0, \frac{j - 2\ell - 2m}{2}, \frac{3j - 7\ell - 3m}{4}, j - 2\ell - 3m \right) \right\rceil.$$

Put

$$A = (j - 3\ell)_+, \quad C = (3\ell - j)_+, \quad q_{\max} = \min(\lfloor \ell/2 \rfloor, 2m), \quad K = q_{\max} - C.$$

The other upper bounds on c are $2m, \ell/2, (2j - 5\ell)/2, (j - 2\ell)/2, 2m + j - 3\ell$. Their integer minimum is exactly K , by separating $j \geq 3\ell$ and $j < 3\ell$.

Every generator has z degree at most two, hence every polynomial in this degree- m block has z degree at most $2m$. Its possible monomials are $x^{j-3\ell+q}y^{\ell-2q}z^q$ with $C \leq q \leq q_{\max}$. Writing $Y = y^2, Z = xz$, the block is therefore contained in

$$(15.16) \quad x^A z^C y^{\ell-2q_{\max}} (Y - Z)^{c_{\min}} \mathbb{C}[Y, Z]_{K-c_{\min}}.$$

The full D refinement proves that its dimension is $K - c_{\min} + 1$, equal to that of this containing space. Thus (15.16) is equality. An eigenbasis divisor is supported on the coordinate curves and curves $C_t = (xz = ty^2)$, $t \in \mathbb{C}^*$, with $C_1 = (D = 0)$ distinguished.

Step 3. The coefficient bounds are uniform over all eigenbases. For $n = K - c_{\min} + 1$, the complete binary forms of degree $n-1$ have maximal order sum $n(n-1)/2$ at every specified point. Products of a vanishing and a nonvanishing linear form give the adapted sequence $0, \dots, n-1$. Thus the maximal average orders within a block are

$$\begin{array}{l|l} x=0 & A + (n-1)/2 \\ z=0 & C + (n-1)/2 \\ y=0 & \ell - 2C - K - c_{\min} \\ C_1 & (K + c_{\min})/2 \\ C_t, t \neq 0, 1 & (K - c_{\min})/2. \end{array}$$

The factor two in the y order is from $Y = y^2$. Generic points have no quotient correction, since E is well formed with isolated singularities.

The means on the polytope of Lemma 15.4 are

$$\bar{r} = 89/88, \quad \bar{a} = \bar{c} = 9/11, \quad \bar{j} = 733/88, \quad \bar{\ell} = 125/44.$$

Moreover $A = (a-r)_+$, $C = (r-a)_+$. Here is a full integral certificate for the remaining means. The (r, a) polygon has vertices $(0, 0), (2, 0), (2, 1), (1, 2)$; its c height is

$$H = 3 - r - L, \quad L = \max(0, 1 - r, 1 - a).$$

In these variables

$$c - c_{\min} = \min(c, t, 3 - r - c), \quad t = \min(1 - r/2, (3 - r - a)/4).$$

We obtain $0 \leq t \leq (3 - r)/2$ and $H \geq t$. Integrating in c gives

$$\begin{cases} t(3 - r) - t^2/2 - tL, & L \geq t, \\ t(3 - r) - t^2 - L^2/2, & L \leq t. \end{cases}$$

The full cell decomposition and its integrals are

(r, a) vertices	t	L	condition	integral
$(1, 0), (2, 0), (2, 1)$	$1 - r/2$	$1 - a$	$L \geq t$	$5/96$
$(2, 1), (1, 2), (1, 1)$	$(3 - r - a)/4$	0	$L \leq t$	$13/192$
$(0, 0), (1/2, 1/2), (1, 2)$	$(3 - r - a)/4$	$1 - r$	$L \geq t$	$47/256$
$(1/2, 1/2), (1, 1), (1, 2)$	$(3 - r - a)/4$	$1 - r$	$L \leq t$	$11/96$
$(0, 0), (1, 0), (2, 1), (1/2, 1/2)$	$(3 - r - a)/4$	$1 - a$	$L \geq t$	$307/768$
$(2, 1), (1, 1), (1/2, 1/2)$	$(3 - r - a)/4$	$1 - a$	$L \leq t$	$19/192$

The sum is $11/12$, so $\overline{c - c_{\min}} = 1/4$. Splitting the polygon just described by $a = r$ and the three choices in L gives

$$\int (a - r)_+ H = \frac{47}{96}, \quad \int (r - a)_+ H = \frac{115}{96};$$

the normalized means are $47/352, 115/352$. These integrands are explicit quadratic polynomials on the resulting polygons.

Consequently every normalized eigenbasis divisor B_m satisfies

$$(15.17) \quad \begin{aligned} \text{coeff}_{x=0} B_m &\leq 135/352 + O(1/m), & \text{coeff}_{y=0} B_m &\leq 97/176 + O(1/m), \\ \text{coeff}_{z=0} B_m &\leq 203/352 + O(1/m), & \text{coeff}_{C_1} B_m &\leq 9/11 + O(1/m), \\ \text{coeff}_{C_t} B_m &\leq 1/4 + O(1/m) & & (t \neq 0, 1). \end{aligned}$$

For example the y mean is $125/44 - 2(115/352) - 2(9/11) = 97/176$. The exact lattice description and bounded floor errors give uniform $O(1/m)$ errors: only $O(m^2)$ cells meet a facet or piece-change wall, and each weight is $O(m)$. The formula is independent of the chosen nonzero root, so uniformity includes roots varying with m or the basis.

On $x = 1$, let ν_x be the weight- $(1, 2)$ order in (y, z) . Every auxiliary eigenvector of degree ℓ has order ℓ , so $\nu_x(B_m) = 125/44 + O(1/m)$. On the index-five cover of $z = 1$, let ν_z have weights $(2, 1)$ in (x, y) . Its order is $2j - 5\ell$, hence

$$(15.18) \quad \nu_x(B_m) = 125/44 + O(1/m), \quad \nu_z(B_m) = 27/11 + O(1/m).$$

Both are orders on smooth covers of discrepancy three. At $z = 1$, dividing both order and discrepancy by five gives the normalized divisor downstairs and the unchanged ratio.

Step 4. These bounds resolve every surface center. Fix a rational $0 < b < 132/125$. For all sufficiently large m , uniformly in the eigenbasis, every component coefficient of bB_m is less than one and

$$b\nu_x(B_m) < 3, \quad b\nu_z(B_m) < 3.$$

The curves C_t are smooth on orbifold charts and two distinct ones meet only at $P_x = [1 : 0 : 0]$ and $P_z = [0 : 0 : 1]$. Away from coordinate points their support is a union of disjoint smooth components. At $P_y = [0 : 1 : 0]$, the index-three cover has only the two coordinate axes through its origin; no C_t passes there. It is a klt simple normal crossing pair.

At P_x , the branches are $y = 0, z = 0, z = ty^2$. The weight- $(1, 2)$ blowup separates them. On its y chart, $z = y^2v$, the strict transforms meet the exceptional at the distinct points $v = 0, t$. On the other chart's index-two cover, the strict transform of $y = 0$ and the exceptional are the coordinate axes. The crepant exceptional coefficient is $b\nu_x(B_m) - 2 < 1$. All coefficients are below one with simple normal crossing support on these smooth covers, proving klt at P_x .

At P_z , first pass to the index-five cover. Its branches are $y = 0, x = 0, x = ty^2$. The weight- $(2, 1)$ blowup uses these formulas with coefficient $b\nu_z(B_m) - 2 < 1$. The initial finite cover is étale in codimension one, so discrepancies of pulled-back pairs scale by ramification indices and klt descends. Quasi-étale descent also treats the isolated quotient charts of the weighted blowup. This covers all valuations on E . The largest limiting ratio in the check is $(125/44)/3 = 125/132$; all the others are strictly smaller.

Step 5. The complete degeneration controls arbitrary bases and valuations. For a tested v over E , choose in each $W_{m,j}$ a basis compatible with v and the auxiliary highest-weight filtration. Its order is the maximal sum, and its limit is an independent eigenbasis by Step 1. The polynomial families there have nonzero special sections. Multiplying the normalized divisor family by $15mN_m$ makes it Cartier, since $\mathcal{O}_E(15)$ is invertible. Its support is proper over $\mathbb{A}^1$ because E is projective. The product family is a rational Gorenstein flat family of connected normal klt pairs with zero boundary. Thus [BL21, Definition 7 and Proposition 10(2)], or the finite-degree argument of Lemma 2.10, transfers the strict central threshold above b to a nonzero fiber. All nonzero fibers are isomorphic pairs. It follows that

$$bS_m(W; v) \leq A_E(v).$$

The lower bound on m in Step 4 is independent of v and of the basis. Maximality of the filtration sum then makes (E, bB'_m) log canonical for every other basis divisor as well. Let $m \rightarrow \infty$, then $b \rightarrow 132/125$, to obtain (15.14).

Finally choose a complete anticanonical basis compatible with G, E , and write $\pi^*D_m = S_m(E)E + \Theta_m$. The residual divisor is effective, has no E component, and restricts to the actual refined basis divisor. For fixed $b < 132/125$, its restriction is log canonical for all sufficiently large m by Step 5. Plt inversion of adjunction with the actual zero different gives

$$A_X(G) \geq bS_m(G) + (9 - bS_m(E)) \operatorname{ord}_G(E).$$

Passing to the limits and using (15.10) yields

$$S_X(G) \leq \frac{125}{132} A_X(G) + \left(\frac{733}{88} - 9 \frac{125}{132} \right) \operatorname{ord}_G(E) = \frac{125}{132} A_X(G) - \frac{17}{88} \operatorname{ord}_G(E).$$

The discrepancy is positive by smoothness. A positive multiple of $-E$ is the ideal defining the extraction, so its pullback has positive order at every valuation centered at p . This proves (15.15). $\square$

16. THE FAMILY THEOREMS AND THE CLASSIFICATION

We now assemble the center estimates. The selection of a global minimizer will be made once, before its actual center is classified.

Lemma 16.1. *Let X be a smooth prime Fano threefold of degree 18 or 22. Every prime divisor over X with positive-dimensional center has nonnegative β_X . The inequality $\beta_X(G) \geq 0$ also holds for divisors G centered at points with no incident line or with a single reduced incident line.*

Proof. For a prime divisor B on X , write $B \sim aH$, $a \geq 1$. Its discrepancy is one and

$$S_X(B) = \int_0^{1/a} (1 - at)^3 dt = \frac{1}{4a}.$$

For $t > 1/a$ a negative multiple of H is not pseudoeffective, as its intersection with H^2 is negative. A primitive divisorial valuation centered at B is ord_B , because the local ring at its generic point is a discrete valuation ring and its units have order zero.

Curve centers are covered by Proposition 4.2 and Corollaries 4.7 and 5.4, including lines and conics with every contact type. Proposition 6.2 treats empty incidence. Propositions 6.4 and 6.5 treat one reduced line. These point results quantify over centers containing the point; in particular they apply to centers equal to it. $\square$

Proof of Theorem 1.2. Fix a prime divisor G over a smooth degree-eighteen member X . If $\beta_X(G) < 0$, Lemma 16.1 leaves a point center p with at least length-two incidence. The exhaustive partitions are those of Lemma 3.2. Reduced pairs and reduced triples have positive beta by Propositions 7.4 and 8.9. Thus the partition must be 2, 2 + 1 or 3.

For partition 2, Proposition 9.4 applies to the entire sole-support scheme and gives a positive extraction surplus.

For partition 2 + 1, the simple line is ordinary and the double line is special. The actual Artin incidence dictionary of Lemma 7.1 gives widths 2, 3, 4 in the ordinary-line link. Width is not identified with the local length of the point fiber. At width two, Proposition 10.3 applies in every root position. At width three, Lemma 10.2 puts the entire cluster off the negative section, which is the other case of that proposition. Both give $\delta_p(X) \geq 3360/3323 > 1$. At width four, Proposition 10.5 gives

$$S_X(G) \leq \frac{71}{72}A_X(G) - \frac{95}{72}\text{ord}_G(E) < A_X(G).$$

For partition 3, Lemma 11.1 gives the exhaustive two coefficient cases. Proposition 11.4 gives $S_X(G) \leq 47A_X(G)/48 - 5\text{ord}_G(E)/8$ in the first case. Proposition 11.6 gives $S_X(G) \leq 8A_X(G)/9 - 5\text{ord}_G(E)/12$ in the second. These are inequalities for every later divisorial valuation, with no discrepancy or monomiality restriction.

Every partition contradicts $\beta_X(G) < 0$. Thus $A_X(G) \geq S_X(G)$ for every prime divisor over X , and Theorem 2.3 proves K-semistability. $\square$

Proof of Theorem 1.3. Suppose that a smooth degree-twenty-two member X is unstable. Proposition 2.4 chooses a prime divisor G on this same X , special and computing the global threshold, with $A_X(G)/S_X(G) = \delta(X) < 1$. We keep this divisor throughout the proof.

Lemma 16.1 makes its center a point p of partition 1 + 1, 1 + 1 + 1, 2, 2 + 1 or 3. For the first partition, Proposition 13.2 gives $\delta_p(X) \geq 88/87 > 1$. For the second, Proposition 13.4 gives positive beta at every point-centered divisor. For partitions 2 and 2 + 1, Proposition 14.3 gives

nonnegative beta without a width, contact or later-valuation restriction. Its repeated special-contact alternative proves semistability of the whole member and therefore also contradicts the assumed instability.

For partition 3, Lemma 15.2 gives exactly one or two conics, including reducible and nonreduced conics. The one-conic alternative would make the whole member semistable by Proposition 15.3. Thus the fixed minimizer G is centered at a two-conic point. Proposition 15.5 gives

$$\beta_X(G) \geq \frac{7}{132}A_X(G) + \frac{17}{88}\text{ord}_G(E) > 0,$$

again a contradiction.

Every center of the fixed global minimizer G has been excluded. Therefore X is K-semistable. No replacement of a divisor at a prescribed center enters this argument. $\square$

Proposition 16.2 (Known families). *Every smooth Fano threefold of Picard number one outside the prime anticanonical degrees 18 and 22 is K-semistable.*

Proof. Write $-K_X = rA$ with A the ample Picard generator. The classification recalled in [KPS18, Section 1.1, Tables 1 and 2] gives

r	A^3
4	1
3	2
2	1, 2, 3, 4, 5
1	2, 4, 6, 8, 10, 12, 14, 16, 18, 22.

The Picard group is torsion-free and equals $\mathbb{Z}[A]$. At index one and degrees four and ten, the two geometric types in that table belong to one deformation family; both types are included.

The all-member result [AZ21, Theorem C and Theorem 5.1] gives uniform K-stability for index two and $A^3 \leq 4$, and index one and $A^3 \leq 16$. Its degree is A^3 , including the index-two cases. The theorem imposes no genericity or very-ampleness restriction beyond its smooth Picard-one Fano hypotheses.

The three remaining higher-index varieties are the unique V_5 , the smooth quadric and $\mathbb{P}^3$. For V_5 , [CS09, Theorems 1.10 and 1.17] give equivariant alpha invariant $5/6 > 3/4$ and a Kähler–Einstein metric. The metric-to-K-polystability implication is [Ber16, Theorem 1.1]. The quadric and projective space are K-polystable homogeneous spaces, as recorded in [ACC⁺23, Section 3.2, p. 89]. All these varieties are therefore K-semistable. The classification leaves only the two stated prime degrees. $\square$

Proof of Theorem 1.1. Let X be any smooth complex Fano threefold of Picard number one. Proposition 16.2 applies unless X is prime of anticanonical degree 18 or 22. In these two cases Theorems 1.2 and 1.3, respectively, prove K-semistability for every smooth member. These alternatives exhaust the classification. $\square$

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