

FINITE DEGREE FORMULAS FOR FROBENIUS VOLUMES OF PAIRS

ABSTRACT. In this paper we study the Frobenius volumes, an analogue of local volumes for strongly F -regular singularities over an algebraically closed field of characteristic $p > 0$. We prove the finite degree formula for Frobenius volumes, and show that the Hilbert–Kunz Frobenius volume equals the Frobenius volume divided by $n!$, where n is the dimension of the ambient variety. This gives a positive answer to Problem 49 in Xu–Zhuang’s open problem list in K-stability of Fano varieties.

CONTENTS

1. Introduction	1
2. Notation and conventions	3
3. Comparison of the two Frobenius volumes	4
4. Thresholds of real filtrations on pairs	5
5. A quantitative intersection estimate	9
6. Finite covers and invariant ideals	11
7. Invariant near-minimizers	13
8. Completion and the finite degree formula	15
References	17

1. INTRODUCTION

We work over an algebraically closed field k of characteristic $p > 0$.

The concept of local volumes [Li18] plays a crucial role in K-stability theory for Fano varieties and in singularity theory in characteristic 0; we refer the reader to [LLX20] for a survey. By considering the equivalent definitions of local volumes in [Liu18], it is natural to define the counterparts of local volumes in characteristic $p > 0$, the *Frobenius volume* and the *Hilbert–Kunz Frobenius volume*, in the following way [LLX20, XZ26].

Definition 1.1 (Frobenius volume). Let $(R, \mathfrak{m}, k)$ be an F -finite normal local ring of characteristic $p > 0$ and dimension $n \geq 1$, and let $\Delta \geq 0$ be a $\mathbb{Q}$ -divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular (Definition 2.1). For a nonzero ideal $I \subset R$, write

$$c_{R, \Delta}(I) = \sup\{t \geq 0 : (R, \Delta, I^t) \text{ is strongly } F\text{-regular}\}.$$

For an $\mathfrak{m}$ -primary ideal I , let $e_R(I)$ and $e_R^{\text{HK}}(I)$ denote its Hilbert–Samuel and Hilbert–Kunz multiplicities. Define

$$\text{Fvol}(R, \Delta) = \inf_{I \text{ } \mathfrak{m}\text{-primary}} e_R(I) c_{R, \Delta}(I)^n, \quad \text{Fvol}_{\text{HK}}(R, \Delta) = \inf_{I \text{ } \mathfrak{m}\text{-primary}} e_R^{\text{HK}}(I) c_{R, \Delta}(I)^n.$$

We call $\text{Fvol}(R, \Delta)$ and $\text{Fvol}_{\text{HK}}(R, \Delta)$ the *Frobenius volume* and the *Hilbert–Kunz Frobenius volume* of (R, Δ) respectively. For a closed point x of a normal k -variety X and a $\mathbb{Q}$ -divisor $\Delta \geq 0$ on X such that $(\mathcal{O}_{X, x}, \Delta)$ is strongly F -regular, we write

$$\text{Fvol}(X \ni x, \Delta) = \text{Fvol}(\mathcal{O}_{X, x}, \Delta) \quad \text{and} \quad \text{Fvol}_{\text{HK}}(X \ni x, \Delta) = \text{Fvol}_{\text{HK}}(\mathcal{O}_{X, x}, \Delta),$$

Date: October 14, 2026.

2020 Mathematics Subject Classification. Primary 13A35; Secondary 14B05.

Key words and phrases. Frobenius volume, strongly F -regular pairs, finite degree formula, test ideals, filtrations, Hilbert–Kunz multiplicity.

where Δ also denotes its restriction to $\text{Spec } \mathcal{O}_{X,x}$. We omit the boundary Δ from the notation when it is zero.

A natural question about (Hilbert–Kunz) Frobenius volumes is the *finite degree formula* [XZ26, Problem 49], as cyclic covers are basic operations in birational geometry. In characteristic 0, such a formula was proved for finite Galois morphisms in [XZ21, Theorem 1.3], and for arbitrary finite crepant morphisms, not necessarily Galois, in [Liu26, Theorem 1.1]. In characteristic $p > 0$, finite degree formulas are known for the F -signature [CRST18, Theorem 3.1] and for the Hilbert–Kunz multiplicity [WY00] (see also [Hun13]). The goal of this paper is to prove the finite degree formula for (Hilbert–Kunz) Frobenius volumes. We state it for local rings.

Theorem 1.2. *Let $(R, \mathfrak{m}_R, k) \subset (S, \mathfrak{m}_S, k)$ be a finite local quasi-étale inclusion of normal local rings of k -varieties at closed points, both of dimension $n \geq 1$. Let $\Delta \geq 0$ be a $\mathbb{Q}$ -divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular, let Δ_S be its pullback to $\text{Spec } S$ (Notation 2.5), and let $\delta = [\text{Frac } S : \text{Frac } R]$. Then (S, Δ_S) is strongly F -regular and*

$$\text{Fvol}(S, \Delta_S) = \delta \text{Fvol}(R, \Delta), \quad \text{Fvol}_{\text{HK}}(S, \Delta_S) = \delta \text{Fvol}_{\text{HK}}(R, \Delta).$$

Taking $\Delta = 0$, we obtain a positive answer to [XZ26, Problem 49].

Corollary 1.3. *Let $f : (Y \ni y) \rightarrow (X \ni x)$ be a finite quasi-étale morphism of germs of positive-dimensional normal k -varieties at closed points, corresponding to a finite local inclusion $\mathcal{O}_{X,x} \subset \mathcal{O}_{Y,y}$. Assume that $X \ni x$ is strongly F -regular, and let $\delta = [\text{Frac } \mathcal{O}_{Y,y} : \text{Frac } \mathcal{O}_{X,x}]$. Then $Y \ni y$ is strongly F -regular and*

$$\text{Fvol}(Y \ni y) = \delta \text{Fvol}(X \ni x), \quad \text{Fvol}_{\text{HK}}(Y \ni y) = \delta \text{Fvol}_{\text{HK}}(X \ni x).$$

Theorem 1.2 is deduced from the following statement for complete local rings.

Theorem 1.4. *Let $(A, \mathfrak{m}_A, k) \subset (B, \mathfrak{m}_B, k)$ be a finite local quasi-étale inclusion of complete Noetherian normal local k -domains of dimension $n \geq 1$ such that $\text{Frac } B / \text{Frac } A$ is Galois with group G and $A = B^G$. Let $d = |G|$, let $\Delta \geq 0$ be a $\mathbb{Q}$ -divisor on $\text{Spec } A$ such that (A, Δ) is strongly F -regular, and let Δ_B be its pullback to $\text{Spec } B$. Then (B, Δ_B) is strongly F -regular, $p \nmid d$, and*

$$\text{Fvol}(B, \Delta_B) = d \text{Fvol}(A, \Delta).$$

We also prove the following theorem which shows that the two Frobenius volumes differ by the factor $n!$. Therefore, they essentially play the same role in applications.

Theorem 1.5. *Let R be the normal local ring of a k -variety at a closed point, of dimension $n \geq 1$, and let $\Delta \geq 0$ be a $\mathbb{Q}$ -divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular. Then*

$$\text{Fvol}_{\text{HK}}(R, \Delta) = \frac{\text{Fvol}(R, \Delta)}{n!}.$$

We now explain the proof of Theorem 1.4. It is not known whether the infimum defining $\text{Fvol}(B, \Delta_B)$ is attained, so we work with linearly bounded filtrations (Definition 2.4) rather than with ideals. The threshold $c_{\Delta_B}(\mathcal{F})$ of such a filtration $\mathcal{F}$ (Theorem 4.3(1)) is convex along intersection geodesics (Theorem 4.7); the proof uses test ideals and Takagi’s summation formula [Tak06]. Let $V = \text{Fvol}(B, \Delta_B)$, and let $\mathcal{F}$ be a filtration with $c_{\Delta_B}(\mathcal{F}) = 1$ and multiplicity $E = E(\mathcal{F})$ close to V . By convexity, the midpoint of $\mathcal{F}$ and $g\mathcal{F}$ has multiplicity at least V for every $g \in G$. The homogeneous measure of Blum, Liu, and Qi [BLQ24] then gives

$$E\left(\bigcap_{g \in G} g\mathcal{F}\right) \leq E + (|G| - 1)\sqrt{E^2 - V^2}$$

(Proposition 5.4). Letting $E \rightarrow V$, we obtain G -invariant $\mathfrak{m}_B$ -primary ideals I with $e_B(I)c_{B, \Delta_B}(I)^n$ arbitrarily close to V (Theorem 7.1). For such I , let $D = \text{lcm}(1, \dots, d)$ and $a = I^D \cap A$. Then $\overline{aB} = \overline{I^D}$ (Proposition 6.3), and $c_{A, \Delta}(a) = c_{B, \Delta_B}(aB)$ by the transformation rule for test ideals under finite maps [ST14]. Hence $de_A(a)c_{A, \Delta}(a)^n = e_B(I)c_{B, \Delta_B}(I)^n$

(Theorem 6.4), which proves Theorem 1.4. For Theorem 1.2, completion preserves both Frobenius volumes (Proposition 8.2), and we apply Theorem 1.4 to $\widehat{R} \subset C$ and $\widehat{S} \subset C$ for a Galois closure C (Proposition 8.3). Theorem 1.5 follows by comparing c_Δ and e^{HK} along the powers I^r of a fixed ideal [WY01].

Structure of the paper. Section 2 fixes the notation, and Section 3 proves Theorem 1.5. Sections 4–7 prove Theorem 1.4, and Section 8 proves Theorem 1.2.

2. NOTATION AND CONVENTIONS

Throughout this paper, $p > 0$ is a prime number, and $q := p^e$ for an integer $e \geq 0$. In this section, $(R, \mathfrak{m})$ is an F -finite normal local domain of characteristic p and dimension $n \geq 1$, and $\Delta \geq 0$ is a $\mathbb{Q}$ -Weil divisor on $\text{Spec } R$. For a Weil divisor D on $\text{Spec } R$, let $R(D) := \Gamma(\text{Spec } R, \mathcal{O}(D))$. Every ring to which we apply the results of [ST14] is an F -finite local ring; such a ring has a dualizing complex by a theorem of Gabber [ST14, Section 2.1], hence a canonical module, and satisfies condition (!) of [ST14, Section 2.5]. We use the results of [ST14] without further comment on this condition.

Definition 2.1. For $e \geq 0$, let

$$\mathcal{C}_e^\Delta(R) := \text{Hom}_R(F_*^e R(\lceil (q-1)\Delta \rceil), R),$$

regarded as a subset of $\text{Hom}_R(F_*^e R, R)$ by restriction along $R \subseteq R(\lceil (q-1)\Delta \rceil)$. Let $I \subset R$ be a nonzero ideal and $t \geq 0$. The triple (R, Δ, I^t) is *strongly F -regular* if for every nonzero $c \in R$, there exist $e \geq 1$, $\phi \in \mathcal{C}_e^\Delta(R)$, and $a \in I^{\lceil t(q-1) \rceil}$ such that $\phi(F_*^e(ca)) = 1$ (cf. [ST14, Definition 6.14]). The pair (R, Δ) is *strongly F -regular* if so is (R, Δ, R^0) , and the ring R is *strongly F -regular* if so is the pair $(R, 0)$. We define

$$c_{R,\Delta}(I) := \sup\{t \geq 0 \mid (R, \Delta, I^t) \text{ is strongly } F\text{-regular}\}.$$

The *big test ideal* $\tau_b(R; \Delta, I^t)$ is the smallest nonzero ideal $J \subset R$ with $\phi(F_*^e(I^{\lceil t(q-1) \rceil} J)) \subset J$ for all $e \geq 0$ and $\phi \in \mathcal{C}_e^\Delta(R)$ [ST14, Definition 6.3].

Since $\mathcal{C}_e^0(R) = \text{Hom}_R(F_*^e R, R)$ and $\mathcal{C}_e^\Delta(R) \subseteq \mathcal{C}_e^0(R)$, if (R, Δ, I^t) is strongly F -regular, then so is $(R, 0, I^t)$. In particular, if (R, Δ) is strongly F -regular, then R is strongly F -regular. In the definition of a strongly F -regular pair one may take $a = 1$, after replacing ϕ by the map $F_*^e x \rightarrow \phi(F_*^e(ax))$, which again lies in $\mathcal{C}_e^\Delta(R)$. By [ST14, Proposition 6.18(iv)], where $\mathcal{C}_e^\Delta(R)$ is denoted by $I_{\Delta,e}$ [ST14, Definition 6.8], the triple (R, Δ, I^t) is strongly F -regular if and only if $\tau_b(R; \Delta, I^t) = R$. If $I \subseteq J$ are nonzero ideals, then $c_{R,\Delta}(I) \leq c_{R,\Delta}(J)$ by definition (cf. [TW04, Proposition 2.2]).

Definition 2.2. Let $I \subset R$ be an $\mathfrak{m}$ -primary ideal. We write $I^{[q]} := (a^q \mid a \in I)$ and let $\bar{I}$ be the integral closure of I . For (R, Δ) strongly F -regular, let $\Phi_{R,\Delta}(I) := e_R(I)c_{R,\Delta}(I)^n$, so that $\text{Fvol}(R, \Delta) = \inf_I \Phi_{R,\Delta}(I)$ by Definition 1.1. We omit the subscript R when R is fixed.

Lemma 2.3. Let $K \subseteq \mathfrak{m}$ be a nonzero ideal generated by b elements, and let $e \geq 1$.

- (1) $K^{b(q-1)+1} \subseteq K^{[q]} \subseteq \mathfrak{m}^{[q]}$, and $\phi(F_*^e \mathfrak{m}^{[q]}) \subseteq \mathfrak{m}$ for every $\phi \in \text{Hom}_R(F_*^e R, R)$.
- (2) $c_{R,\Delta}(K) \leq b$. In particular, if (R, Δ) is strongly F -regular, then $\text{Fvol}(R, \Delta) \leq \Phi_{R,\Delta}(\mathfrak{m}) < \infty$.

Proof. Every monomial of degree $b(q-1)+1$ in b generators of K has an exponent at least q , which gives the first inclusion in (1), and the second inclusion holds as $K \subseteq \mathfrak{m}$. For $\phi \in \text{Hom}_R(F_*^e R, R)$, $z \in \mathfrak{m}$, and $u \in R$, we have $\phi(F_*^e(z^q u)) = z\phi(F_*^e u) \in \mathfrak{m}$. This proves (1). For (2), let $t > b$. Then $\lceil t(q-1) \rceil \geq b(q-1)+1$, so by (1), $\phi(F_*^e(ca)) \in \mathfrak{m}$ for all $\phi \in \mathcal{C}_e^\Delta(R)$, $c \in R$, and $a \in K^{\lceil t(q-1) \rceil}$. Thus (R, Δ, K^t) is not strongly F -regular, and $c_{R,\Delta}(K) \leq b$. The last assertion follows from $\text{Fvol}(R, \Delta) \leq \Phi_{R,\Delta}(\mathfrak{m}) = e_R(\mathfrak{m})c_{R,\Delta}(\mathfrak{m})^n$. $\square$

Definition 2.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension n . A *linearly bounded filtration* of R (cf. [BLQ24, Section 2.1]) is a family $\mathcal{F} = (\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}}$ of ideals of R such that

- (1) $\mathcal{F}_\lambda = R$ for $\lambda \leq 0$ and $\mathcal{F}_\lambda$ is $\mathfrak{m}$ -primary for $\lambda > 0$,
- (2) $\mathcal{F}_\lambda \supset \mathcal{F}_\mu$ for $\lambda \leq \mu$ and $\mathcal{F}_\lambda \mathcal{F}_\mu \subset \mathcal{F}_{\lambda+\mu}$ for $\lambda, \mu \geq 0$,
- (3) $\mathcal{F}_{\lambda-\epsilon} = \mathcal{F}_\lambda$ for every $\lambda > 0$ and $0 < \epsilon \ll 1$, and
- (4) there exists $b > 0$ such that $\mathcal{F}_\lambda \subset \mathfrak{m}^{\lceil b\lambda \rceil}$ for every $\lambda > 0$.

The *multiplicity* of $\mathcal{F}$ is

$$E_R(\mathcal{F}) := \lim_{j \rightarrow +\infty} \frac{n! \ell_R(R/\mathcal{F}_j)}{j^n}$$

whenever this limit exists. For an automorphism g of R , let $(g\mathcal{F})_\lambda := g(\mathcal{F}_\lambda)$. For filtrations $\mathcal{F}^i$ of R , let $(\bigcap_i \mathcal{F}^i)_\lambda := \bigcap_i \mathcal{F}_\lambda^i$.

Notation 2.5. A finite morphism of normal integral schemes is *quasi-étale* if it is étale in codimension one. A *finite local inclusion* is an inclusion $R \subset S$ of normal local domains with S finite over R . Its *degree* is $[\text{Frac } S : \text{Frac } R]$. It is *quasi-étale* if so is $\text{Spec } S \rightarrow \text{Spec } R$, and *Galois* if $\text{Frac } S/\text{Frac } R$ is a Galois extension. The *divisorial pullback* Δ_S of Δ is its pullback along $\text{Spec } S \rightarrow \text{Spec } R$ [ST14, Section 2.2]; the coefficient of Δ_S at a prime divisor of $\text{Spec } S$ is the coefficient of Δ at its image times the ramification index. If $R \subset S$ is quasi-étale, then all ramification indices are one, so $\Delta_S \geq 0$ has the same coefficients as Δ . We use the germ notation $\text{Fvol}(X \ni x, \Delta)$ and $\text{Fvol}_{\text{HK}}(X \ni x, \Delta)$ of Section 1.

3. COMPARISON OF THE TWO FROBENIUS VOLUMES

In this section, we prove Theorem 1.5. We need the following lemma.

Lemma 3.1. *Let R be an F -finite normal local domain and Δ an effective rational Weil divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular. Let I be a nonzero proper ideal of R and $r \geq 1$ an integer. Then*

$$c_{R, \Delta}(I^r) = \frac{c_{R, \Delta}(I)}{r}.$$

Proof. Fix $t \geq 0$. For $q = p^e$, put $A_q = \lceil rt(q-1) \rceil$ and $B_q = r \lceil t(q-1) \rceil$, so that $A_q \leq B_q \leq A_q + r$. Suppose first that $(R, \Delta, (I^r)^t)$ is strongly F -regular, and let $0 \neq c \in R$. By Definition 2.1, there exist $e \geq 1$, $\phi \in \mathcal{C}_e^\Delta(R)$, and $a \in (I^r)^{\lceil t(q-1) \rceil} = I^{B_q}$ such that $\phi(F_*^e(ca)) = 1$. Since $I^{B_q} \subseteq I^{A_q} = I^{\lceil rt(q-1) \rceil}$, the same e , ϕ , and a show that (R, Δ, I^{rt}) is strongly F -regular. Conversely, suppose that (R, Δ, I^{rt}) is strongly F -regular, and choose $0 \neq b \in I^r$. Let $0 \neq c \in R$. Since R is a domain, $cb \neq 0$, so there exist $e \geq 1$, $\phi \in \mathcal{C}_e^\Delta(R)$, and $a \in I^{A_q}$ such that $\phi(F_*^e(cba)) = 1$. Since $ba \in I^{A_q+r} \subseteq I^{B_q} = (I^r)^{\lceil t(q-1) \rceil}$, the triple $(R, \Delta, (I^r)^t)$ is strongly F -regular. Thus $(R, \Delta, (I^r)^t)$ is strongly F -regular if and only if (R, Δ, I^{rt}) is strongly F -regular. Taking the supremum over t , we obtain $c_{R, \Delta}(I^r) = c_{R, \Delta}(I)/r$. $\square$

We also need the following special case of a theorem of Watanabe and Yoshida.

Proposition 3.2 ([WY01, Theorem 1.1]). *Let $(R, \mathfrak{m}, \kappa)$ be a Cohen–Macaulay Noetherian local ring of characteristic $p > 0$ and dimension $n \geq 1$ such that κ is infinite. Then for every $\mathfrak{m}$ -primary ideal I and every integer $r \geq 1$, we have*

$$\frac{e_R(I)}{n!} \leq \frac{e_R^{\text{HK}}(I^r)}{r^n} \leq \frac{e_R(I)}{r^n} \binom{r+n-1}{n}.$$

In particular, $\lim_{r \rightarrow \infty} e_R^{\text{HK}}(I^r)/r^n = e_R(I)/n!$.

Proof of Theorem 1.5. Since k is perfect and R is essentially of finite type over k , R is F -finite. The residue field of R is k , which is infinite. Since R is a localization of a finitely generated k -algebra, R is a homomorphic image of a regular local ring. The ring R is strongly F -regular

(Definition 2.1), hence Cohen–Macaulay by [HH89, Theorem 3.1(d) and Corollary 2.5]. Thus Lemma 3.1 and Proposition 3.2 apply to R .

Put $V = \text{Fvol}(R, \Delta)$ and $W = \text{Fvol}_{\text{HK}}(R, \Delta)$. For every $\mathfrak{m}$ -primary ideal I , Proposition 3.2 with $r = 1$ gives

$$e_R^{\text{HK}}(I)c_{R,\Delta}(I)^n \geq \frac{e_R(I)c_{R,\Delta}(I)^n}{n!} = \frac{\Phi_{R,\Delta}(I)}{n!} \geq \frac{V}{n!}.$$

Taking the infimum over I , we obtain $W \geq V/n!$.

For the reverse inequality, fix an $\mathfrak{m}$ -primary ideal I . By Lemma 2.3, $c_{R,\Delta}(I) < \infty$. For every integer $r \geq 1$, the ideal I^r is $\mathfrak{m}$ -primary, so Lemma 3.1 gives

$$W \leq e_R^{\text{HK}}(I^r)c_{R,\Delta}(I^r)^n = \frac{e_R^{\text{HK}}(I^r)}{r^n}c_{R,\Delta}(I)^n.$$

Letting $r \rightarrow \infty$ and using Proposition 3.2, we obtain $W \leq e_R(I)c_{R,\Delta}(I)^n/n! = \Phi_{R,\Delta}(I)/n!$. Taking the infimum over I , we obtain $W \leq V/n!$. The theorem follows. $\square$

4. THRESHOLDS OF REAL FILTRATIONS ON PAIRS

In this section, we extend c_Δ from ideals to linearly bounded filtrations (Theorem 4.3) and prove that it is convex along intersection geodesics (Theorem 4.7).

Let R be an F -finite normal local domain of characteristic $p > 0$ and Δ an effective rational Weil divisor on $\text{Spec } R$. Let $c \in R$ be a big sharp test element of (R, Δ) [BSTZ10, Definition 3.1]; such elements exist [ST21, Remark 2.11]. For nonzero ideals $J_1, \dots, J_s$ and real numbers $u_1, \dots, u_s \geq 0$, the test ideal of [ST21, Definition 2.10] is

$$(4.1) \quad \tau_b \left(R; \Delta, \prod_i J_i^{u_i} \right) = \sum_{e \geq 0} \sum_{\phi \in \mathcal{C}_e^{\Delta}(R)} \phi \left(F_*^e \left(c \prod_i J_i^{\lceil u_i p^e \rceil} \right) \right)$$

by [ST21, Remark 2.11]. For $s = 1$, the right side of (4.1) equals the sum with the exponents $\lceil u_1(p^e - 1) \rceil$ and a big sharp test element of the triple in [BSTZ10, Definition-Proposition 3.3], by [ST21, Remark 2.11], and that sum is the big test ideal $\tau_b(R; \Delta, J_1^{u_1})$ of Definition 2.1 (see [ST14, Proposition 6.9] and Theorem 8.1). A nonzero multiple hc of c is again a big sharp test element of (R, Δ) , because the defining condition in [BSTZ10, Definition 3.1] is preserved under multiplication by h (cf. [ST14, Remark 6.6]). Hence (4.1) also holds with hc in place of c . We fix such a c .

Lemma 4.1. *Let R be an F -finite normal local domain of characteristic $p > 0$, Δ an effective rational Weil divisor, and $J_1, \dots, J_s$ nonzero ideals, where $s \geq 1$.*

(1) *For every real number $u > 0$,*

$$\tau_b(R; \Delta, (J_1 + \dots + J_s)^u) = \sum_{\substack{u_i \geq 0 \\ \sum_i u_i = u}} \tau_b \left(R; \Delta, \prod_i J_i^{u_i} \right).$$

(2) *Let B be a nonzero ideal, $r_1, \dots, r_s$ positive integers with $J_i^{r_i} \subseteq B$, and $u_1, \dots, u_s \geq 0$ real numbers. Put $v = \sum_i u_i/r_i$. Then*

$$\tau_b \left(R; \Delta, \prod_i J_i^{u_i} \right) \subseteq \tau_b(R; \Delta, B^v).$$

For $\Delta = 0$ and $s = 2$, Lemma 4.1(1) is [Tak06, Theorem 3.1], where τ_b is denoted by $\tilde{\tau}$ [Tak06, Definition 1.4]. See also [Tak06, Proposition 4.9].

Proof. Fix $q = p^e$. For (1), write

$$(J_1 + \dots + J_s)^{\lceil uq \rceil} = \sum_{\substack{k_i \in \mathbb{N} \\ \sum_i k_i = \lceil uq \rceil}} \prod_i J_i^{k_i}.$$

Given (k_i) , put $\theta = \lceil uq \rceil - uq \in [0, 1)$, choose j with $k_j > 0$, and put $u_j = (k_j - \theta)/q$ and $u_i = k_i/q$ for $i \neq j$. Then $u_i \geq 0$, $\sum_i u_i = u$, and $\lceil u_i q \rceil = k_i$ for every i . Hence the degree e terms of (4.1) for the left side of (1) lie in the right side of (1). Conversely, if $u_i \geq 0$ and $\sum_i u_i = u$, then $\sum_i \lceil u_i q \rceil \geq \lceil uq \rceil$, so $\prod_i J_i^{\lceil u_i q \rceil} \subseteq (J_1 + \dots + J_s)^{\lceil uq \rceil}$. This proves (1) by (4.1).

For (2), put $N_q = \sum_i \lceil \lceil u_i q \rceil / r_i \rceil$. Then $\prod_i J_i^{\lceil u_i q \rceil} \subseteq B^{N_q}$ and $N_q > qv - s$. Choose a nonzero $h \in B^{s+1}$. Then $h \prod_i J_i^{\lceil u_i q \rceil} \subseteq B^{\lceil vq \rceil}$ for every q . Computing the left side of (2) by (4.1) with hc in place of c , and the right side with c , we obtain (2). $\square$

Lemma 4.2. *Let $(R, \mathfrak{m})$ be an F -finite normal local domain of characteristic $p > 0$ and dimension $n \geq 1$, and Δ an effective rational Weil divisor such that (R, Δ) is strongly F -regular. Let $(A_j)_{j \geq 0}$ be a decreasing sequence of ideals such that $A_0 = R$, A_j is $\mathfrak{m}$ -primary for $j \geq 1$, and $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. Assume that $A_j \subseteq \mathfrak{m}^{\lceil bj \rceil}$ for some $b > 0$ and every $j \geq 1$. Then*

$$0 < \lim_{j \rightarrow \infty} jc_{\Delta}(A_j) = \sup_{j \geq 1} jc_{\Delta}(A_j) < \infty.$$

Proof. Let h_0 be the number of elements of a generating set of $\mathfrak{m}$. We first show that $0 < c_{\Delta}(I) \leq h_0$ for every nonzero ideal $I \subseteq \mathfrak{m}$. Choose a nonzero $h \in I$. Since (R, Δ) is strongly F -regular, there exist $e \geq 1$ and $\phi \in \mathcal{C}_e^{\Delta}(R)$ with $\phi(F_*^e(ch)) = 1$ (Definition 2.1 and the remark after it). For $u = 1/p^e$, the degree e term of (4.1) gives $1 \in \tau_b(R; \Delta, I^u)$, so $c_{\Delta}(I) \geq u > 0$ by [ST14, Proposition 6.18(iv)]. By monotonicity and Lemma 2.3, $c_{\Delta}(I) \leq c_{\Delta}(\mathfrak{m}) \leq h_0$.

Fix $j \geq 1$ and $r \geq 1$, and put $k = \lceil r/j \rceil$. Since $A_j^k \subseteq A_{kj} \subseteq A_r \subseteq \mathfrak{m}^{\lceil br \rceil}$, Lemma 3.1 gives

$$\frac{r}{\lceil r/j \rceil} c_{\Delta}(A_j) \leq rc_{\Delta}(A_r) \leq \frac{rc_{\Delta}(\mathfrak{m})}{\lceil br \rceil} \leq \frac{c_{\Delta}(\mathfrak{m})}{b}.$$

Letting $r \rightarrow \infty$, $\liminf_r rc_{\Delta}(A_r) \geq jc_{\Delta}(A_j)$ for every j , so $\liminf_r rc_{\Delta}(A_r) \geq \sup_j jc_{\Delta}(A_j) \geq \limsup_r rc_{\Delta}(A_r)$. Thus the limit exists and equals the supremum, which lies between $c_{\Delta}(A_1) > 0$ and $c_{\Delta}(\mathfrak{m})/b < \infty$. $\square$

Theorem 4.3. *Let $(R, \mathfrak{m})$ be an excellent F -finite normal local domain of dimension $n \geq 1$, Δ an effective rational Weil divisor such that (R, Δ) is strongly F -regular, and $\mathcal{F}$ a linearly bounded filtration.*

(1) *The number*

$$c_{\Delta}(\mathcal{F}) := \lim_{j \rightarrow \infty} jc_{\Delta}(\mathcal{F}_j) = \sup_{j \geq 1} jc_{\Delta}(\mathcal{F}_j) = \sup_{\lambda > 0} \lambda c_{\Delta}(\mathcal{F}_{\lambda}) = \lim_{\lambda \rightarrow \infty} \lambda c_{\Delta}(\mathcal{F}_{\lambda})$$

is well defined, finite, and positive.

(2) *The multiplicity $E(\mathcal{F}) = \lim_{j \rightarrow \infty} e(\mathcal{F}_j)/j^n$ is finite and positive, and $\text{Fvol}(R, \Delta) \leq E(\mathcal{F})c_{\Delta}(\mathcal{F})^n$.*

(3) *For $s > 0$, the family $\mathcal{F}_{\lambda}^{(s)} := \mathcal{F}_{s\lambda}$ is a linearly bounded filtration with $c_{\Delta}(\mathcal{F}^{(s)}) = c_{\Delta}(\mathcal{F})/s$ and $E(\mathcal{F}^{(s)}) = s^n E(\mathcal{F})$.*

(4) *For every $\mathfrak{m}$ -primary ideal I , the family $\mathcal{F}_{\lambda}^I := I^{\lceil c_{\Delta}(I)\lambda \rceil}$ for $\lambda > 0$ and $\mathcal{F}_{\lambda}^I := R$ for $\lambda \leq 0$ is a linearly bounded filtration with $c_{\Delta}(\mathcal{F}^I) = 1$ and $E(\mathcal{F}^I) = \Phi_{\Delta}(I)$.*

(5) *We have*

$$\text{Fvol}(R, \Delta) = \inf_{\mathcal{F}} E(\mathcal{F})c_{\Delta}(\mathcal{F})^n = \inf_{\mathcal{F}: c_{\Delta}(\mathcal{F})=1} E(\mathcal{F}),$$

where the infima run over all linearly bounded filtrations.

Proof. Let $\alpha = \sup_{j \geq 1} jc_{\Delta}(\mathcal{F}_j)$, and let $b > 0$ be a constant with $\mathcal{F}_{\lambda} \subseteq \mathfrak{m}^{\lceil b\lambda \rceil}$ for all $\lambda > 0$. By Lemma 4.2, $0 < \alpha < \infty$ and $\alpha = \lim_{j \rightarrow \infty} jc_{\Delta}(\mathcal{F}_j)$.

Step 1. In this step, we prove (1) and the formula $c_{\Delta}(\mathcal{F}^{(s)}) = c_{\Delta}(\mathcal{F})/s$ in (3).

Let $\lambda > 0$ and $\rho \geq 1$, and put $k = \lceil \rho/\lambda \rceil$. Since $\mathcal{F}_{\lambda}^k \subseteq \mathcal{F}_{k\lambda} \subseteq \mathcal{F}_{\rho}$, Lemma 3.1 gives $\rho c_{\Delta}(\mathcal{F}_{\rho}) \geq \rho c_{\Delta}(\mathcal{F}_{\lambda})/\lceil \rho/\lambda \rceil$. Letting $\rho \rightarrow \infty$ through the integers gives $\alpha \geq \lambda c_{\Delta}(\mathcal{F}_{\lambda})$, so $\sup_{\lambda > 0} \lambda c_{\Delta}(\mathcal{F}_{\lambda}) = \alpha$.

Letting $\rho \rightarrow \infty$ through the reals gives $\liminf_{\rho} \rho c_{\Delta}(\mathcal{F}_{\rho}) \geq \lambda c_{\Delta}(\mathcal{F}_{\lambda})$ for every $\lambda > 0$, hence $\liminf_{\rho} \rho c_{\Delta}(\mathcal{F}_{\rho}) \geq \alpha \geq \limsup_{\rho} \rho c_{\Delta}(\mathcal{F}_{\rho})$. Thus $\lim_{\rho \rightarrow \infty} \rho c_{\Delta}(\mathcal{F}_{\rho}) = \alpha$. The family $\mathcal{F}^{(s)}$ is decreasing, multiplicative, and left continuous with $\mathcal{F}_{\lambda}^{(s)} \subseteq \mathfrak{m}^{\lceil bs\lambda \rceil}$, so it is a linearly bounded filtration, and $c_{\Delta}(\mathcal{F}^{(s)}) = \sup_{\lambda > 0} \lambda c_{\Delta}(\mathcal{F}_{s\lambda}) = \alpha/s$.

Step 2. In this step, we prove (2) and the formula $E(\mathcal{F}^{(s)}) = s^n E(\mathcal{F})$ in (3).

The completion of R is normal [Sta26, Tag 0C23], so R is analytically unramified, and [Cut14, Theorem 6.5] applied to $(\mathcal{F}_j)_{j \geq 0}$ shows that both limits in

$$E(\mathcal{F}) = \lim_{j \rightarrow \infty} \frac{n! \ell_R(R/\mathcal{F}_j)}{j^n} = \lim_{j \rightarrow \infty} \frac{e(\mathcal{F}_j)}{j^n}$$

exist and are equal. Since $\mathcal{F}_1^j \subseteq \mathcal{F}_j \subseteq \mathfrak{m}^{\lceil bj \rceil}$, the Hilbert–Samuel polynomials of $\mathcal{F}_1$ and $\mathfrak{m}$ [Sta26, Tags 00K8 and 00L8] give $0 < b^n e(\mathfrak{m}) \leq E(\mathcal{F}) \leq e(\mathcal{F}_1) < \infty$. By definition of $\text{Fvol}(R, \Delta)$,

$$\text{Fvol}(R, \Delta) \leq e(\mathcal{F}_j) c_{\Delta}(\mathcal{F}_j)^n = \frac{e(\mathcal{F}_j)}{j^n} (j c_{\Delta}(\mathcal{F}_j))^n$$

for every $j \geq 1$, and letting $j \rightarrow \infty$ gives $\text{Fvol}(R, \Delta) \leq E(\mathcal{F}) c_{\Delta}(\mathcal{F})^n$. Let $s > 0$. Since $\mathcal{F}_{\lceil sj \rceil} \subseteq \mathcal{F}_{sj} \subseteq \mathcal{F}_{\lfloor sj \rfloor}$, the quantity $n! \ell_R(R/\mathcal{F}_{sj})/j^n$ lies between $n! \ell_R(R/\mathcal{F}_{\lceil sj \rceil})/j^n$ and $n! \ell_R(R/\mathcal{F}_{\lfloor sj \rfloor})/j^n$, which tend to $s^n E(\mathcal{F})$. Thus $E(\mathcal{F}^{(s)}) = s^n E(\mathcal{F})$.

Step 3. In this step, we prove (4) and (5).

Let I be an $\mathfrak{m}$ -primary ideal and $a = c_{\Delta}(I)$. By Lemma 3.1, $j c_{\Delta}(I^j) = a$ for $j \geq 1$, so Lemma 4.2 applied to $(I^j)_{j \geq 0}$ gives $0 < a < \infty$. The family $\mathcal{F}^I$ is decreasing and left continuous, $\lceil a\lambda \rceil + \lceil a\mu \rceil \geq \lceil a(\lambda + \mu) \rceil$ gives $\mathcal{F}_{\lambda}^I \mathcal{F}_{\mu}^I \subseteq \mathcal{F}_{\lambda+\mu}^I$, and $\mathcal{F}_{\lambda}^I \subseteq \mathfrak{m}^{\lceil a\lambda \rceil}$. Thus $\mathcal{F}^I$ is a linearly bounded filtration. By Lemma 3.1, Step 2, and $e(I^k) = k^n e(I)$,

$$c_{\Delta}(\mathcal{F}^I) = \lim_{j \rightarrow \infty} \frac{j a}{\lceil a j \rceil} = 1, \quad E(\mathcal{F}^I) = \lim_{j \rightarrow \infty} \frac{\lceil a j \rceil^n e(I)}{j^n} = a^n e(I) = \Phi_{\Delta}(I).$$

Let V_1 and V_2 be the two infima in (5). Then $V_1 \leq V_2$, since the second infimum runs over fewer filtrations. By Step 2, $\text{Fvol}(R, \Delta) \leq V_1$. By the filtrations $\mathcal{F}^I$, $V_2 \leq \inf_I \Phi_{\Delta}(I) = \text{Fvol}(R, \Delta)$. The theorem follows. $\square$

Definition 4.4. Let $\mathcal{F}, \mathcal{H}$ be linearly bounded filtrations and $0 < t < 1$. The *intersection geodesic* of $\mathcal{F}$ and $\mathcal{H}$ is the family

$$\mathcal{K}_{\lambda, t} := \sum_{\substack{\mu, \nu \in \mathbb{R} \\ (1-t)\mu + t\nu = \lambda}} (\mathcal{F}_{\mu} \cap \mathcal{H}_{\nu}) \quad (\lambda > 0), \quad \mathcal{K}_{\lambda, t} := R \quad (\lambda \leq 0).$$

We set $\mathcal{K}_{\bullet, 0} := \mathcal{F}$ and $\mathcal{K}_{\bullet, 1} := \mathcal{H}$. The *midpoint* of $\mathcal{F}$ and $\mathcal{H}$ is $\mathcal{K}_{\bullet, 1/2}$.

Lemma 4.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathcal{F}, \mathcal{H}$ linearly bounded filtrations of R , and $0 < t < 1$. Then $\mathcal{K}_{\bullet, t}$ is a linearly bounded filtration, and for $\lambda > 0$,

$$\mathcal{K}_{\lambda, t} = \sum_{\substack{\mu, \nu \geq 0 \\ (1-t)\mu + t\nu = \lambda}} (\mathcal{F}_{\mu} \cap \mathcal{H}_{\nu}).$$

Proof. Write $a = 1 - t$, $b = t$, and $\mathcal{K} = \mathcal{K}_{\bullet, t}$, and let $b_F, b_H > 0$ be constants with $\mathcal{F}_{\lambda} \subseteq \mathfrak{m}^{\lceil b_F \lambda \rceil}$ and $\mathcal{H}_{\lambda} \subseteq \mathfrak{m}^{\lceil b_H \lambda \rceil}$ for $\lambda > 0$. For $\lambda > 0$, we call (μ, ν) an admissible pair for λ if $\mu, \nu \geq 0$ and $a\mu + b\nu = \lambda$.

Step 1. In this step, we prove the displayed formula, that $\mathcal{K}$ is decreasing and multiplicative, and that $\mathcal{K}_{\lambda}$ is $\mathfrak{m}$ -primary with $\mathcal{K}_{\lambda} \subseteq \mathfrak{m}^{\lceil b_1 \lambda \rceil}$ for $\lambda > 0$, where $b_1 = \min(b_F, b_H)$.

If $\mu < 0$ and $a\mu + b\nu = \lambda > 0$, then $\nu > \lambda/b$, so $\mathcal{F}_{\mu} \cap \mathcal{H}_{\nu} = \mathcal{H}_{\nu} \subseteq \mathcal{F}_0 \cap \mathcal{H}_{\lambda/b}$. The case $\nu < 0$ is symmetric. Thus $\mathcal{K}_{\lambda}$ is the sum of $\mathcal{F}_{\mu} \cap \mathcal{H}_{\nu}$ over the admissible pairs (μ, ν) for λ . If $0 < \lambda \leq \lambda'$ and (μ', ν') is admissible for λ' , then $(\lambda/\lambda')(\mu', \nu')$ is admissible for λ and $\mathcal{F}_{\mu'} \cap \mathcal{H}_{\nu'} \subseteq \mathcal{F}_{\lambda\mu'/\lambda'} \cap \mathcal{H}_{\lambda\nu'/\lambda'}$.

Thus $\mathcal{K}_{\lambda'} \subseteq \mathcal{K}_\lambda$. If (μ, ν) and (μ', ν') are admissible for λ and ρ , then $(\mu + \mu', \nu + \nu')$ is admissible for $\lambda + \rho$ and $(\mathcal{F}_\mu \cap \mathcal{H}_\nu)(\mathcal{F}_{\mu'} \cap \mathcal{H}_{\nu'}) \subseteq \mathcal{F}_{\mu+\mu'} \cap \mathcal{H}_{\nu+\nu'}$. Thus $\mathcal{K}_\lambda \mathcal{K}_\rho \subseteq \mathcal{K}_{\lambda+\rho}$, also when $\lambda = 0$ or $\rho = 0$ since $\mathcal{K}_0 = R$. If (μ, ν) is admissible for λ , then $\max(\mu, \nu) \geq \lambda$, so $\mathcal{F}_\mu \cap \mathcal{H}_\nu \subseteq \mathfrak{m}^{\lceil b_1 \lambda \rceil}$. Since (λ, λ) is admissible for λ , we have $\mathcal{F}_\lambda \cap \mathcal{H}_\lambda \subseteq \mathcal{K}_\lambda \subseteq \mathfrak{m}^{\lceil b_1 \lambda \rceil}$, and the left ideal is $\mathfrak{m}$ -primary.

Step 2. In this step, we prove that $\mathcal{K}$ is left continuous.

Fix $\lambda_0 > 0$, a real number $M > (\lambda_0 + 1)/\min(a, b)$, and an integer $N_0 \geq 1$ with $\mathfrak{m}^{N_0} \subseteq \mathcal{F}_M \cap \mathcal{H}_M$. For $0 < \lambda < \lambda_0 + 1$, every admissible pair (μ, ν) for λ satisfies $\mu, \nu \leq M$, so $\mathcal{F}_\mu, \mathcal{H}_\nu$, and $\mathcal{K}_\lambda$ contain $\mathfrak{m}^{N_0}$. Put $U_\mu = \mathcal{F}_\mu/\mathfrak{m}^{N_0}$ and $V_\nu = \mathcal{H}_\nu/\mathfrak{m}^{N_0}$ for $0 \leq \mu, \nu \leq M$. Since (U_μ) and (V_ν) are chains of submodules of the finite-length module $R/\mathfrak{m}^{N_0}$, they take finitely many values. For a value P of (U_μ) and a value Q of (V_ν) put

$$u_P = \max\{\mu \in [0, M] \mid P \subseteq U_\mu\}, \quad v_Q = \max\{\nu \in [0, M] \mid Q \subseteq V_\nu\}.$$

These maxima exist, as each set is an initial interval of $[0, M]$ which contains its supremum by left continuity. We claim that for $0 < \lambda < \lambda_0 + 1$,

$$(4.2) \quad \mathcal{K}_\lambda/\mathfrak{m}^{N_0} = \sum_{\lambda \leq au_P + bv_Q} (P \cap Q),$$

the sum running over all pairs of values (P, Q) . If (μ, ν) is admissible for λ , then $P = U_\mu$ and $Q = V_\nu$ satisfy $u_P \geq \mu$ and $v_Q \geq \nu$, so $\lambda \leq au_P + bv_Q$, and $(\mathcal{F}_\mu \cap \mathcal{H}_\nu)/\mathfrak{m}^{N_0} = P \cap Q$ since both ideals contain $\mathfrak{m}^{N_0}$. Conversely, if $\lambda \leq au_P + bv_Q$, then $\sigma = \lambda/(au_P + bv_Q) \in (0, 1]$, the pair $(\sigma u_P, \sigma v_Q)$ is admissible for λ , and $P \subseteq U_{u_P} \subseteq U_{\sigma u_P}$ and $Q \subseteq V_{v_Q} \subseteq V_{\sigma v_Q}$, so $P \cap Q \subseteq \mathcal{K}_\lambda/\mathfrak{m}^{N_0}$. This proves (4.2). There are finitely many numbers $au_P + bv_Q$, so for $0 < \epsilon \ll 1$ the right side of (4.2) is the same for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0]$. Since every $\mathcal{K}_\lambda$ contains $\mathfrak{m}^{N_0}$, we have $\mathcal{K}_\lambda = \mathcal{K}_{\lambda_0}$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0]$.

By Steps 1 and 2, $\mathcal{K}$ satisfies the conditions of Definition 2.4. $\square$

Lemma 4.6. *Let $(R, \mathfrak{m})$ be an F -finite normal local domain of characteristic $p > 0$ and dimension $n \geq 1$, and Δ an effective rational Weil divisor such that (R, Δ) is strongly F -regular. Let $(A_i)_{i \geq 0}$ and $(B_i)_{i \geq 0}$ be decreasing sequences of ideals such that $A_0 = B_0 = R$, A_i and B_i are $\mathfrak{m}$ -primary for $i \geq 1$, $A_i A_j \subseteq A_{i+j}$ and $B_i B_j \subseteq B_{i+j}$ for all $i, j \geq 0$, and*

$$A_i \subseteq \mathfrak{m}^{\lceil b_A i \rceil}, \quad B_i \subseteq \mathfrak{m}^{\lceil b_B i \rceil} \quad (i \geq 1)$$

for some $b_A, b_B > 0$. Put

$$\alpha = \sup_{i \geq 1} ic_\Delta(A_i), \quad \beta = \sup_{i \geq 1} ic_\Delta(B_i), \quad D_r = \sum_{i=0}^r (A_i \cap B_{r-i}) \quad (r \geq 0).$$

Then $(D_r)_{r \geq 0}$ satisfies the hypotheses of Lemma 4.2, and

$$\lim_{r \rightarrow \infty} rc_\Delta(D_r) \leq \alpha + \beta.$$

Lemma 4.6 is an analogue of [Tak06, Proposition 4.9], with intersections in place of products.

Proof. For $1 \leq i \leq r+1$ we have $A_i \cap B_{r+1-i} \subseteq A_{i-1} \cap B_{r+1-i}$, and $A_0 \cap B_{r+1} \subseteq A_0 \cap B_r$. Thus $D_{r+1} \subseteq D_r$. Since $(A_i \cap B_{r-i})(A_j \cap B_{s-j}) \subseteq A_{i+j} \cap B_{r+s-i-j}$, we have $D_r D_s \subseteq D_{r+s}$. For $0 \leq i \leq r$, one of i and $r-i$ is at least $r/2$, so $A_i \cap B_{r-i} \subseteq \mathfrak{m}^{\lceil b_0 r \rceil}$ with $b_0 = \min(b_A, b_B)/2$. Hence $A_r \subseteq D_r \subseteq \mathfrak{m}^{\lceil b_0 r \rceil}$ for $r \geq 1$, so D_r is $\mathfrak{m}$ -primary, and $D_0 = R$. Thus $(D_r)_{r \geq 0}$ satisfies the hypotheses of Lemma 4.2.

Fix $r \geq 1$ and a real number $u > (\alpha + \beta)/r$, and put $J_i = A_i \cap B_{r-i} \supseteq A_i B_{r-i} \neq 0$ for $0 \leq i \leq r$. Let $u_0, \dots, u_r \geq 0$ with $\sum_i u_i = u$, and put $x = \sum_i i u_i$ and $y = \sum_i (r-i) u_i$. Then $x + y = ru > \alpha + \beta$, so $x > \alpha$ or $y > \beta$. Put $N = r!$. Then $J_i^{N/i} \subseteq A_N$ for $1 \leq i \leq r$, and

$J_i^{N/(r-i)} \subseteq B_N$ for $0 \leq i \leq r-1$. Since omitting the factor $J_0^{u_0}$ or $J_r^{u_r}$ enlarges the left side of (4.1), Lemma 4.1(2) gives

$$\tau_b \left(R; \Delta, \prod_{i=0}^r J_i^{u_i} \right) \subseteq \tau_b(R; \Delta, A_N^{x/N}) \cap \tau_b(R; \Delta, B_N^{y/N}).$$

By definition of α and β , $c_\Delta(A_N) \leq \alpha/N$ and $c_\Delta(B_N) \leq \beta/N$. Thus $x/N > c_\Delta(A_N)$ or $y/N > c_\Delta(B_N)$, and the corresponding ideal on the right is proper by [ST14, Proposition 6.18(iv)]. Hence $\tau_b(R; \Delta, \prod_i J_i^{u_i}) \subseteq \mathfrak{m}$. Since $D_r = J_0 + \cdots + J_r$, Lemma 4.1(1) gives $\tau_b(R; \Delta, D_r^u) \subseteq \mathfrak{m}$, so (R, Δ, D_r^u) is not strongly F -regular by [ST14, Proposition 6.18(iv)]. Thus $rc_\Delta(D_r) \leq \alpha + \beta$ for every $r \geq 1$, and the lemma follows. $\square$

Theorem 4.7. *Let $(R, \mathfrak{m})$ be an excellent F -finite normal local domain of dimension $n \geq 1$ and Δ an effective rational Weil divisor such that (R, Δ) is strongly F -regular. Let $\mathcal{F}, \mathcal{H}$ be linearly bounded filtrations and $0 < t < 1$. Then $\mathcal{K}_{\bullet, t}$ is a linearly bounded filtration and*

$$c_\Delta(\mathcal{K}_{\bullet, t}) \leq (1-t)c_\Delta(\mathcal{F}) + tc_\Delta(\mathcal{H}).$$

Proof. Write $a = 1-t$, $b = t$, and $\mathcal{K} = \mathcal{K}_{\bullet, t}$. By Lemma 4.5, $\mathcal{K}$ is a linearly bounded filtration. Put $A_i = \mathcal{F}_i^{(1/a)} = \mathcal{F}_{i/a}$ and $B_j = \mathcal{H}_j^{(1/b)} = \mathcal{H}_{j/b}$ for integers $i, j \geq 0$, and $D_r = \sum_{i=0}^r (A_i \cap B_{r-i})$. By Theorem 4.3(1) and (3), (A_i) and (B_j) satisfy the hypotheses of Lemma 4.6, with

$$\alpha = \sup_{i \geq 1} ic_\Delta(A_i) = c_\Delta(\mathcal{F}^{(1/a)}) = ac_\Delta(\mathcal{F}), \quad \beta = \sup_{j \geq 1} jc_\Delta(B_j) = bc_\Delta(\mathcal{H}).$$

Each summand $\mathcal{F}_{i/a} \cap \mathcal{H}_{(r-i)/b}$ of D_r is a summand of $\mathcal{K}_r$, so $D_r \subseteq \mathcal{K}_r$. Conversely, let $r \geq 1$ and $\mu, \nu \geq 0$ with $a\mu + b\nu = r$, so that $\mathcal{K}_r$ is the sum of the ideals $\mathcal{F}_\mu \cap \mathcal{H}_\nu$ over such pairs by Lemma 4.5, and put $i = \min\{[a\mu], r-1\}$ and $j = r-1-i$. Then $i \leq a\mu$. If $[a\mu] \leq r-1$, then $j = r-1-[a\mu] \leq r-a\mu = b\nu$. Otherwise $a\mu = r$, $i = r-1$, and $j = 0 = b\nu$. Hence $\mathcal{F}_\mu \cap \mathcal{H}_\nu \subseteq A_i \cap B_j \subseteq D_{r-1}$. Thus $D_r \subseteq \mathcal{K}_r \subseteq D_{r-1}$ for $r \geq 1$, and for $r \geq 2$,

$$rc_\Delta(D_r) \leq rc_\Delta(\mathcal{K}_r) \leq \frac{r}{r-1}(r-1)c_\Delta(D_{r-1}).$$

By Lemma 4.2, both outer terms tend to $\lim_{r \rightarrow \infty} rc_\Delta(D_r)$. Since $\mathcal{K}$ is a linearly bounded filtration, the middle term tends to $c_\Delta(\mathcal{K})$ by Theorem 4.3(1). By Lemma 4.6,

$$c_\Delta(\mathcal{K}) = \lim_{r \rightarrow \infty} rc_\Delta(D_r) \leq \alpha + \beta = (1-t)c_\Delta(\mathcal{F}) + tc_\Delta(\mathcal{H}).$$

This finishes the proof. $\square$

5. A QUANTITATIVE INTERSECTION ESTIMATE

A Noetherian local ring is *analytically irreducible* if its completion is a domain.

Theorem 5.1 ([BLQ24, Definition 4.1, Propositions 4.6, 4.7 and 4.9, and Lemma 4.11]). *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain containing a field, of dimension $n \geq 1$, let $\mathcal{F}, \mathcal{H}$ be linearly bounded filtrations, and let $\mathcal{K}_{\bullet, t}$ be their intersection geodesics (Definition 4.4 and Lemma 4.5). Then there exist a positive Radon measure μ on $\mathbb{R}_{\geq 0}^2$ and a real number $D \geq 1$ satisfying the following.*

- (1) μ has no mass on the axes, $\text{supp } \mu \subseteq \{x/D \leq y \leq Dx\}$, and $\mu(sU) = s^n \mu(U)$ for all Borel $U \subseteq \mathbb{R}_{\geq 0}^2$ and $s > 0$.
- (2) For $0 \leq t \leq 1$ and $a, b > 0$,

$$\mu\{(1-t)x + ty < a\} = a^n E(\mathcal{K}_{\bullet, t}), \quad \mu(\{x < a\} \cup \{y < b\}) = E((\mathcal{F}_{a\lambda} \cap \mathcal{H}_{b\lambda})_\lambda).$$

- (3) The measure $\rho(U) = \mu\{(x, y) \in \mathbb{R}_{\geq 0}^2 : x + y < 1, x/(x+y) \in U\}$ on $(0, 1)$ is finite with compact support, and for $0 \leq t \leq 1$,

$$E(\mathcal{K}_{\bullet, t}) = \int_{(0,1)} ((1-t)z + t(1-z))^{-n} d\rho(z).$$

Lemma 5.2. *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain containing a field, of dimension $n \geq 1$, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be linearly bounded filtrations. Then $\mathcal{A} = \bigcap_{i=1}^k \mathcal{B}_i$ is a linearly bounded filtration, $E(\mathcal{A})$ exists, and $0 < E(\mathcal{A}) < \infty$.*

Proof. For $\lambda > 0$, $\mathcal{A}_\lambda$ is a finite intersection of $\mathfrak{m}$ -primary ideals, hence $\mathfrak{m}$ -primary. For $\lambda, \eta \geq 0$, $\mathcal{A}_\lambda \mathcal{A}_\eta \subseteq \bigcap_{i=1}^k (\mathcal{B}_i)_\lambda (\mathcal{B}_i)_\eta \subseteq \mathcal{A}_{\lambda+\eta}$. If $(\mathcal{B}_i)_{\lambda-\epsilon} = (\mathcal{B}_i)_\lambda$ for every i , then $\mathcal{A}_{\lambda-\epsilon} = \mathcal{A}_\lambda$. If $(\mathcal{B}_1)_\lambda \subseteq \mathfrak{m}^{[\alpha\lambda]}$ for all $\lambda > 0$, then $\mathcal{A}_\lambda \subseteq \mathfrak{m}^{[\alpha\lambda]}$. Thus $\mathcal{A}$ is linearly bounded. Choose an integer $\ell \geq 1$ with $\mathfrak{m}^\ell \subseteq \mathcal{A}_1$. Then

$$\mathfrak{m}^{\ell j} \subseteq (\mathcal{A}_1)^j \subseteq \mathcal{A}_j \subseteq \mathfrak{m}^{[\alpha j]}$$

for every integer $j \geq 1$. Since R is analytically irreducible, hence analytically unramified, $E(\mathcal{A})$ exists by [Cut14, Theorem 6.5]. By the Hilbert–Samuel polynomial [Sta26, Tags 00K8 and 00L8], $\alpha^n e_R(\mathfrak{m}) \leq E(\mathcal{A}) \leq \ell^n e_R(\mathfrak{m})$. $\square$

Theorem 5.3. *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain containing a field, of dimension $n \geq 1$, and let $\mathcal{F}, \mathcal{H}$ be linearly bounded filtrations. Let $A = E(\mathcal{F})$, $B = E(\mathcal{H})$, and $P = E(\mathcal{K}_{\bullet, 1/2})$. Then*

$$(5.1) \quad P \leq \sqrt{AB} \quad \text{and} \quad \max\{A, B\} \leq E(\mathcal{F} \cap \mathcal{H}) \leq \frac{A + B + \sqrt{(A + B)^2 - 4P^2}}{2}.$$

In particular, if $A = B = E$, then

$$(5.2) \quad 0 \leq E(\mathcal{F} \cap \mathcal{H}) - E \leq \sqrt{E^2 - P^2} \leq \sqrt{2E(E - P)}.$$

Proof. By Lemma 5.2, A , B , and $E(\mathcal{F} \cap \mathcal{H})$ are positive and finite. Let μ and ρ be as in Theorem 5.1 and $C_\rho \subset (0, 1)$ the support of ρ . On $\mathbb{R}_{>0}^2$ write $r = x + y$ and $z = x/(x + y)$. For a Borel set $U \subseteq (0, 1)$ and $s > 0$, $\{z \in U, 0 < r < s\} = s\{z \in U, 0 < r < 1\}$, so Theorem 5.1(1) gives

$$(5.3) \quad \mu\{z \in U, 0 < r < s\} = s^n \rho(U).$$

With $U = (0, 1) \setminus C_\rho$ and $s \rightarrow \infty$, (5.3) shows that μ is concentrated on $\{z \in C_\rho\}$. We claim that for every positive continuous function T on C_ρ ,

$$(5.4) \quad \mu\{z \in C_\rho, 0 < r < T(z)\} = \int_{C_\rho} T(z)^n d\rho(z).$$

For a positive Borel simple function T , (5.4) is (5.3) summed over the level sets of T . In general, choose positive Borel simple functions $T_k \leq T \leq T'_k$ converging uniformly to T . Since ρ is finite, $\int T_k^n d\rho$ and $\int T'_k^n d\rho$ converge to $\int T^n d\rho$, and the claim follows.

On $\mathbb{R}_{>0}^2$, the condition $x < 1$ or $y < 1$ reads $r < \max\{z^{-1}, (1 - z)^{-1}\}$, and the right-hand side is positive and continuous on C_ρ . Since μ has no mass on the axes and is concentrated on $\{z \in C_\rho\}$, (5.4) and Theorem 5.1(2) with $a = b = 1$ give

$$(5.5) \quad E(\mathcal{F} \cap \mathcal{H}) = \int_{(0,1)} \max\{z^{-n}, (1 - z)^{-n}\} d\rho(z).$$

Write $a(z) = z^{-n}$ and $b(z) = (1 - z)^{-n}$, and integrate over $(0, 1)$ against ρ . Theorem 5.1(3) at $t = 0, 1, 1/2$ gives

$$(5.6) \quad A = \int a d\rho, \quad B = \int b d\rho, \quad P = 2^n \rho((0, 1)).$$

Since $A > 0$, $\rho \neq 0$ and $P > 0$. Let $Q = \int \sqrt{ab} d\rho$. Since $z(1 - z) \leq 1/4$, $\sqrt{ab} \geq 2^n$, so $Q \geq P$. By the Cauchy–Schwarz inequality,

$$(5.7) \quad P \leq Q \leq \left(\int a d\rho \right)^{1/2} \left(\int b d\rho \right)^{1/2} = \sqrt{AB}.$$

Let $V = \int |a - b| d\rho < \infty$. Since $|a - b| = |\sqrt{a} - \sqrt{b}|(\sqrt{a} + \sqrt{b})$, the Cauchy–Schwarz inequality and (5.7) give

$$V^2 \leq \int (\sqrt{a} - \sqrt{b})^2 d\rho \int (\sqrt{a} + \sqrt{b})^2 d\rho = (A + B)^2 - 4Q^2 \leq (A + B)^2 - 4P^2.$$

Since $\max\{a, b\} = (a + b + |a - b|)/2$ and $\max\{a, b\} \geq a, b$, (5.5) and (5.6) give

$$\max\{A, B\} \leq E(\mathcal{F} \cap \mathcal{H}) = \frac{A + B + V}{2} \leq \frac{A + B + \sqrt{(A + B)^2 - 4P^2}}{2}.$$

This proves (5.1). If $A = B = E$, the upper bound is $E + \sqrt{E^2 - P^2}$, and $0 < P \leq E$ gives $E^2 - P^2 = (E - P)(E + P) \leq 2E(E - P)$. This proves (5.2). $\square$

Proposition 5.4. *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain containing a field, of dimension $n \geq 1$, let G be a finite group of local automorphisms of R , and let $\mathcal{F}$ be a linearly bounded filtration. Let $E = E(\mathcal{F})$ and $\mathcal{L} = \bigcap_{g \in G} g\mathcal{F}$, and for $g \in G$ let P_g be the multiplicity of the midpoint (Definition 4.4) of $\mathcal{F}$ and $g\mathcal{F}$. Then $\mathcal{L}$ is a linearly bounded filtration, $h\mathcal{L} = \mathcal{L}$ for every $h \in G$, and*

$$(5.8) \quad 0 \leq E(\mathcal{L}) - E \leq \sum_{g \neq 1} \sqrt{E^2 - P_g^2}.$$

If $0 \leq v \leq E$ and $P_g \geq v$ for every $g \neq 1$, then $E(\mathcal{L}) \leq E + (|G| - 1)\sqrt{E^2 - v^2}$.

Proof. Let $g \in G$. Since $g(\mathfrak{m}) = \mathfrak{m}$, g maps each $\mathfrak{m}^k$ onto itself and a composition series of R/I to one of $R/g(I)$. Hence $g\mathcal{F}$ is linearly bounded, $\ell_R(R/g(\mathcal{F}_\lambda)) = \ell_R(R/\mathcal{F}_\lambda)$ for all λ , and $E(g\mathcal{F}) = E$. By Lemma 5.2, $\mathcal{L}$ is linearly bounded and $0 < E(\mathcal{L}) < \infty$. For $h \in G$, since $g \rightarrow hg$ permutes G , $h(\mathcal{L}_\lambda) = \bigcap_{g \in G} (hg)(\mathcal{F}_\lambda) = \mathcal{L}_\lambda$.

Fix $\lambda > 0$. The R -linear map

$$\mathcal{F}_\lambda / \mathcal{L}_\lambda \rightarrow \bigoplus_{g \neq 1} \mathcal{F}_\lambda / (\mathcal{F}_\lambda \cap g(\mathcal{F}_\lambda)), \quad [u] \rightarrow ([u])_{g \neq 1},$$

is injective, since $\mathcal{F}_\lambda \cap \bigcap_{g \neq 1} g(\mathcal{F}_\lambda) = \mathcal{L}_\lambda$. Taking lengths gives

$$\ell_R(R/\mathcal{L}_\lambda) - \ell_R(R/\mathcal{F}_\lambda) \leq \sum_{g \neq 1} (\ell_R(R/(\mathcal{F}_\lambda \cap g(\mathcal{F}_\lambda))) - \ell_R(R/\mathcal{F}_\lambda)).$$

Set $\lambda = j$, multiply by $n!/j^n$, and let $j \rightarrow \infty$. By Lemma 5.2 and (5.2) applied to $\mathcal{F}$ and $g\mathcal{F}$,

$$E(\mathcal{L}) - E \leq \sum_{g \neq 1} (E(\mathcal{F} \cap g\mathcal{F}) - E) \leq \sum_{g \neq 1} \sqrt{E^2 - P_g^2}.$$

Since $\mathcal{L}_\lambda \subseteq \mathcal{F}_\lambda$ for all λ , $E(\mathcal{L}) \geq E$. This proves (5.8). Finally, if $0 \leq v \leq P_g$ for every $g \neq 1$, then $P_g \leq E$ by (5.1), so $\sqrt{E^2 - P_g^2} \leq \sqrt{E^2 - v^2}$. Summing over $g \neq 1$ gives the last assertion. $\square$

6. FINITE COVERS AND INVARIANT IDEALS

In this section, we prove Theorem 6.4.

Theorem 6.1 ([ST14, Setting 6.27, Corollaries 6.30, 6.31, and 7.7]). *Let $R \subset S$ be a finite separable inclusion of F -finite normal local domains. Let $f : \text{Spec } S \rightarrow \text{Spec } R$ be the induced morphism and Ram_f its ramification divisor. Let Δ be an effective rational Weil divisor on $\text{Spec } R$ and $\Delta_S := f^*\Delta - \text{Ram}_f$. Assume that $\Delta_S \geq 0$. Then the following hold.*

- (1) *If $R \subset S$ is quasi-étale and R is strongly F -regular, then $\text{Tr}_{\text{Frac } S / \text{Frac } R}(S) = R$.*
- (2) *If $\text{Tr}_{\text{Frac } S / \text{Frac } R}(S) = R$, then for every nonzero ideal $I \subset R$ and every $t \geq 0$,*

$$\tau_b(S; \Delta_S, (IS)^t) \cap R = \tau_b(R; \Delta, I^t),$$

and (R, Δ, I^t) is strongly F -regular if and only if $(S, \Delta_S, (IS)^t)$ is strongly F -regular.

Proposition 6.2 ([HS06, Theorem 11.2.7]). *Let $R \subset S$ be a finite local inclusion of Noetherian local domains of the same dimension $n \geq 1$ such that $R/\mathfrak{m}_R \rightarrow S/\mathfrak{m}_S$ is an isomorphism, and let $d := [\text{Frac } S : \text{Frac } R]$. Then $e_S(IS) = d e_R(I)$ for every $\mathfrak{m}_R$ -primary ideal I .*

Proof. This is [HS06, Theorem 11.2.7] for the local ring S , whose residue field is that of R . $\square$

Proposition 6.3. *Let $R \subset S$ be a finite local inclusion of normal domains such that $\text{Frac } S/\text{Frac } R$ is a Galois extension with group G and $R = S^G$. Let $d := |G|$ and $D := \text{lcm}(1, \dots, d)$. Then for every G -stable $\mathfrak{m}_S$ -primary ideal I , the ideal $a := I^D \cap R$ is $\mathfrak{m}_R$ -primary and*

$$\overline{aS} = \overline{I^D}.$$

Proof. Since S is the integral closure of R in $\text{Frac } S$, every $g \in G$ maps S onto itself. Since $R \subset S$ is local, $\sqrt{a} = \sqrt{I^D} \cap R = \mathfrak{m}_S \cap R = \mathfrak{m}_R$, so a is $\mathfrak{m}_R$ -primary. Let z be an indeterminate, $U := S[IZ]$, and $V := S[az^D] = \bigoplus_{j \geq 0} (aS)^j z^{Dj} \subset U$.

Let $x \in I$ and write

$$\prod_{g \in G} (T - g(x)) = T^d + b_1 T^{d-1} + \dots + b_d.$$

Since G permutes the factors and $g(x) \in I$ for every $g \in G$, we have $b_i \in I^i \cap S^G = I^i \cap R$, so $b_i^{D/i} \in I^D \cap R = a$ as i divides D . Hence $(b_i z^i)^{D/i} = b_i^{D/i} z^D \in V$, so $b_i z^i$ is integral over V , and

$$(xz)^d + (b_1 z)(xz)^{d-1} + \dots + b_d z^d = 0$$

shows that xz is integral over $V[b_1 z, \dots, b_d z^d]$, hence over V . Thus $U = V[xz \mid x \in I]$ is integral over V .

Let $y \in I^D$. Then $yz^D \in U$ satisfies $(yz^D)^\ell + v_1(yz^D)^{\ell-1} + \dots + v_\ell = 0$ for some $\ell \geq 1$ and $v_1, \dots, v_\ell \in V$. Let $\gamma_i z^{Di}$ be the part of v_i of degree Di , so $\gamma_i \in (aS)^i$. Taking the parts of degree $D\ell$ gives $y^\ell + \gamma_1 y^{\ell-1} + \dots + \gamma_\ell = 0$. Thus $aS \subset I^D \subset \overline{aS}$, and $\overline{aS} = \overline{I^D}$. $\square$

Theorem 6.4. *Let k be an algebraically closed field of characteristic $p > 0$ and $(A, \mathfrak{m}_A, k) \subset (B, \mathfrak{m}_B, k)$ a finite local quasi-étale Galois inclusion of complete Noetherian normal local k -domains of dimension $n \geq 1$. Let $G := \text{Gal}(\text{Frac } B/\text{Frac } A)$ and assume that $A = B^G$. Let $d := |G|$ and $D := \text{lcm}(1, \dots, d)$. Let Δ be an effective rational Weil divisor on $\text{Spec } A$ such that (A, Δ) is strongly F -regular, and let Δ_B be its divisorial pullback. Then A and B are F -finite, (B, Δ_B) is strongly F -regular, $p \nmid d$, and the following hold.*

(1) $c_{A, \Delta}(a) = c_{B, \Delta_B}(aB)$ for every nonzero ideal $a \subset A$.

(2) For every G -stable $\mathfrak{m}_B$ -primary ideal I , the ideal $a := I^D \cap A$ is $\mathfrak{m}_A$ -primary and

$$e_A(a) = \frac{D^n}{d} e_B(I), \quad c_{A, \Delta}(a) = \frac{c_{B, \Delta_B}(I)}{D}, \quad d \Phi_{A, \Delta}(a) = \Phi_{B, \Delta_B}(I).$$

(3) We have

$$\inf_{\substack{I \text{ } \mathfrak{m}_B\text{-primary} \\ gI=I \text{ } (g \in G)}} \Phi_{B, \Delta_B}(I) = d \text{Fvol}(A, \Delta).$$

Proof. We first verify the hypotheses of Theorem 6.1 for $A \subset B$. We then compare aB with I^D for a G -stable $\mathfrak{m}_B$ -primary ideal I and $a := I^D \cap A$. Let $\pi : \text{Spec } B \rightarrow \text{Spec } A$ be the induced morphism.

Step 1. In this step, we prove (1), $p \nmid d$, and the strong F -regularity of (B, Δ_B) .

By [Sta26, Tag 032A], A and B are quotients of power series rings $P = k'[[x_1, \dots, x_s]]$ over fields $k' \simeq k$. Since k' is perfect, $P = \bigoplus_{\eta \in \{0, \dots, p-1\}^s} x^\eta P^\eta$ is F -finite, hence so are A and B . Since π is étale at every prime of height at most one, $\text{Frac } B/\text{Frac } A$ is separable, $\text{Ram}_\pi = 0$, and $\pi^* \Delta - \text{Ram}_\pi = \pi^* \Delta = \Delta_B \geq 0$.

The ring A is strongly F -regular (Definition 2.1). By Theorem 6.1(1), $\text{Tr}_{\text{Frac } B/\text{Frac } A}(B) = A$. Since B is the integral closure of A in $\text{Frac } B$, every $g \in G$ maps B onto itself and fixes $\mathfrak{m}_B$. As

g fixes A and A surjects onto $B/\mathfrak{m}_B$, g acts trivially on $B/\mathfrak{m}_B = k$. Hence $\mathrm{Tr}_{\mathrm{Frac} B/\mathrm{Frac} A}(b) = \sum_{g \in G} g(b) \equiv db \pmod{\mathfrak{m}_B}$ for $b \in B$. Choosing $b_0 \in B$ with $\mathrm{Tr}_{\mathrm{Frac} B/\mathrm{Frac} A}(b_0) = 1$ gives $db_0 = 1$ in k , so $p \nmid d$.

By Theorem 6.1(2), for every nonzero ideal $a \subset A$ and $t \geq 0$, (A, Δ, a^t) is strongly F -regular if and only if $(B, \Delta_B, (aB)^t)$ is. Taking $a = A$ and $t = 0$ shows that (B, Δ_B) is strongly F -regular, and taking the supremum over t gives (1).

Step 2. In this step, we fix a G -stable $\mathfrak{m}_B$ -primary ideal I , let $a := I^D \cap A$, and prove $e_B(aB) = D^n e_B(I)$.

By Proposition 6.3, a is $\mathfrak{m}_A$ -primary, and $J := aB$ and $L := I^D$ satisfy $J \subset L \subset \bar{J}$. Since $A \subset B$ is finite and local, J is $\mathfrak{m}_B$ -primary. Let z be an indeterminate. If $y \in L$ satisfies $y^\ell + \gamma_1 y^{\ell-1} + \cdots + \gamma_\ell = 0$ with $\gamma_i \in J^i$, then $(yz)^\ell + \gamma_1 z(yz)^{\ell-1} + \cdots + \gamma_\ell z^\ell = 0$ with $\gamma_i z^i \in B[Jz]$. Since $L \subset \bar{J}$ is finitely generated, $B[Lz]$ is a finite $B[Jz]$ -module. Choose homogeneous generators $v_\nu z^{r_\nu}$ of this module and let $h := \max_\nu r_\nu$. Taking the parts of degree $r + h$ gives $L^{r+h} = \sum_\nu J^{r+h-r_\nu} v_\nu \subset J^r$ for every $r \geq 1$. Hence

$$(6.1) \quad L^{r+h} \subset J^r \subset L^r \quad (r \geq 1),$$

and $\ell_B(B/L^r) \leq \ell_B(B/J^r) \leq \ell_B(B/L^{r+h})$. Dividing by $r^n/n!$ and letting $r \rightarrow \infty$ gives $e_B(J) = e_B(L)$. Since $\ell_B(B/L^r) = \ell_B(B/I^{Dr})$, we have $e_B(L) = D^n e_B(I)$, so

$$(6.2) \quad e_B(aB) = D^n e_B(I).$$

Step 3. In this step, we prove $c_{B, \Delta_B}(aB) = c_{B, \Delta_B}(I)/D$.

By (6.1) and monotonicity (Definition 2.1), $c_{B, \Delta_B}(L^{r+h}) \leq c_{B, \Delta_B}(J^r) \leq c_{B, \Delta_B}(L^r)$ for every $r \geq 1$. Since (B, Δ_B) is strongly F -regular by **Step 1** and J, L are nonzero proper ideals, Lemma 3.1 gives

$$\frac{r}{r+h} c_{B, \Delta_B}(L) \leq c_{B, \Delta_B}(J) \leq c_{B, \Delta_B}(L) \quad (r \geq 1).$$

Since $c_{B, \Delta_B}(L) < \infty$ by Lemma 2.3, letting $r \rightarrow \infty$ gives $c_{B, \Delta_B}(J) = c_{B, \Delta_B}(L)$, and Lemma 3.1 gives

$$(6.3) \quad c_{B, \Delta_B}(aB) = c_{B, \Delta_B}(I^D) = \frac{c_{B, \Delta_B}(I)}{D}.$$

Step 4. We conclude the proof in this step.

Proposition 6.2 applied to $A \subset B$ gives $e_B(aB) = d e_A(a)$. Together with (6.2), (1), and (6.3), this proves the first two identities in (2), and their product gives

$$d \Phi_{A, \Delta}(a) = D^n e_B(I) \left(\frac{c_{B, \Delta_B}(I)}{D} \right)^n = \Phi_{B, \Delta_B}(I).$$

This proves (2). By (2), $\Phi_{B, \Delta_B}(I) \geq d \mathrm{Fvol}(A, \Delta)$ for every G -stable $\mathfrak{m}_B$ -primary ideal I . Conversely, for every $\mathfrak{m}_A$ -primary ideal $\mathfrak{b}$, the ideal $\mathfrak{b}B$ is G -stable and $\mathfrak{m}_B$ -primary, and (1) and Proposition 6.2 give $\Phi_{B, \Delta_B}(\mathfrak{b}B) = d \Phi_{A, \Delta}(\mathfrak{b})$. Taking the infimum over $\mathfrak{b}$ proves (3). $\square$

7. INVARIANT NEAR-MINIMIZERS

In this section, we prove Theorem 7.1 and deduce Theorem 1.4.

Theorem 7.1. *Let $(R, \mathfrak{m})$ be an excellent F -finite normal local domain of dimension $n \geq 1$ and Δ an effective rational Weil divisor on $\mathrm{Spec} R$ such that (R, Δ) is strongly F -regular. Let G be a finite group of local automorphisms of R with $g\Delta = \Delta$ for $g \in G$. Then*

$$(7.1) \quad \mathrm{Fvol}(R, \Delta) = \inf_{\substack{I \text{ } \mathfrak{m}\text{-primary} \\ gI=I \text{ } (g \in G)}} \Phi_{R, \Delta}(I).$$

Moreover, let $V := \text{Fvol}(R, \Delta)$, I an $\mathfrak{m}$ -primary ideal, $E := \Phi_{R, \Delta}(I)$, and

$$\mathcal{L}_\lambda := \bigcap_{g \in G} g \left(I^{\lceil c_\Delta(I)^\lambda \rceil} \right) \quad (\lambda > 0).$$

Then for every $\epsilon > 0$ there exists an integer r_0 such that

$$(7.2) \quad \Phi_{R, \Delta}(\mathcal{L}_r) \leq E + (|G| - 1)\sqrt{E^2 - V^2} + \epsilon$$

for every integer $r \geq r_0$.

Proof. Let $\mathcal{F} := \mathcal{F}^I$, so that $\mathcal{L} = \bigcap_{g \in G} g\mathcal{F}$. We first show that thresholds and multiplicities are G -invariant. Let $g \in G$. Since $g\Delta = \Delta$, g preserves $R(\lceil (q-1)\Delta \rceil)$, so for every $\phi \in \mathcal{C}_e^\Delta(R)$ the map

$$\phi^g(F_*^e x) := g \left(\phi \left(F_*^e(g^{-1}(x)) \right) \right)$$

belongs to $\mathcal{C}_e^\Delta(R)$. For a nonzero ideal J , $t \geq 0$, and a nonzero $c \in R$, if $\phi(F_*^e(g^{-1}(c)a)) = 1$ with $a \in J^{\lceil t(q-1) \rceil}$, then $\phi^g(F_*^e(cg(a))) = 1$ with $g(a) \in (gJ)^{\lceil t(q-1) \rceil}$. By Definition 2.1, $(R, \Delta, (gJ)^t)$ is strongly F -regular whenever (R, Δ, J^t) is. Hence $c_\Delta(gJ) \geq c_\Delta(J)$, and the same inequality for g^{-1} gives equality. Moreover, g maps a composition series of R/J^j to one of $R/(gJ)^j$, so $e_R(gJ) = e_R(J)$ for $\mathfrak{m}$ -primary J . Let $\mathcal{H}$ be a linearly bounded filtration. Since $g\mathfrak{m} = \mathfrak{m}$, so is $g\mathcal{H}$, and Theorem 4.3(1) and (2) give $c_\Delta(g\mathcal{H}) = c_\Delta(\mathcal{H})$ and $E(g\mathcal{H}) = E(\mathcal{H})$.

We now prove (7.2). By Theorem 4.3(4), $c_\Delta(\mathcal{F}) = 1$ and $E(\mathcal{F}) = E$, and $0 \leq V \leq E$ by the definition of V . For $g \in G \setminus \{1\}$, let $\mathcal{K}^g$ be the midpoint of $\mathcal{F}$ and $g\mathcal{F}$ (Definition 4.4) and $P_g := E(\mathcal{K}^g)$. Since $c_\Delta(g\mathcal{F}) = 1$, Theorem 4.7 gives $c_\Delta(\mathcal{K}^g) \leq 1$. Thus $V \leq P_g c_\Delta(\mathcal{K}^g)^n \leq P_g$ by Theorem 4.3(2).

The completion of R is normal [Sta26, Tag 0C23] and local, hence a domain, so R is analytically irreducible. By Proposition 5.4 with $v := V$, $\mathcal{L}$ is an invariant linearly bounded filtration with $E(\mathcal{L}) \leq E + (|G| - 1)\sqrt{E^2 - V^2}$.

Since $1 \in G$, we have $\mathcal{L}_\lambda \subseteq \mathcal{F}_\lambda$. By Definition 2.1, c_Δ is monotone under inclusion. Thus Theorem 4.3(1) gives $rc_\Delta(\mathcal{L}_r) \leq c_\Delta(\mathcal{L}) \leq c_\Delta(\mathcal{F}) = 1$ for $r \geq 1$. Each $\mathcal{L}_r$ is an invariant $\mathfrak{m}$ -primary ideal, and

$$\Phi_{R, \Delta}(\mathcal{L}_r) = \frac{e_R(\mathcal{L}_r)}{r^n} (rc_\Delta(\mathcal{L}_r))^n \leq \frac{e_R(\mathcal{L}_r)}{r^n}.$$

By Theorem 4.3(2), the right-hand side converges to $E(\mathcal{L})$, which proves (7.2).

Finally, the right-hand side of (7.1) is at least V by definition. For each positive integer s , choose an $\mathfrak{m}$ -primary ideal I_s with $E_s := \Phi_{R, \Delta}(I_s) < V + \frac{1}{s}$. By (7.2) with $\epsilon = \frac{1}{s}$, there exists an invariant $\mathfrak{m}$ -primary ideal J_s with

$$V \leq \Phi_{R, \Delta}(J_s) \leq E_s + (|G| - 1)\sqrt{E_s^2 - V^2} + \frac{1}{s}.$$

Since $E_s \rightarrow V$, we have $\Phi_{R, \Delta}(J_s) \rightarrow V$, which proves (7.1). $\square$

Proof of Theorem 1.4. By Theorem 6.4, B is F -finite, (B, Δ_B) is strongly F -regular, $p \nmid d$, and

$$\inf_{\substack{I \text{ } \mathfrak{m}_B\text{-primary} \\ gI=I \text{ } (g \in G)}} \Phi_{B, \Delta_B}(I) = d \text{Fvol}(A, \Delta).$$

Since B is the integral closure of A in $\text{Frac } B$, G acts on B by local automorphisms. Since $A \subset B$ is quasi-étale, the coefficient of Δ_B at a prime divisor of $\text{Spec } B$ is the coefficient of Δ at its image (Notation 2.5). As G permutes the prime divisors of $\text{Spec } B$ over each prime divisor of $\text{Spec } A$, we have $g\Delta_B = \Delta_B$ for $g \in G$. Since B is excellent [Sta26, Tag 07QW], Theorem 7.1 applies to (B, Δ_B) and G , and the left-hand side of the display equals $\text{Fvol}(B, \Delta_B)$. $\square$

8. COMPLETION AND THE FINITE DEGREE FORMULA

In this section, we prove Theorem 1.2 and Corollary 1.3.

Theorem 8.1 ([ST14, Lemma 6.5, Remarks 6.6–6.7, and Proposition 6.9]). *Let R be an F -finite normal local domain, Δ an effective rational Weil divisor, $I \subset R$ a nonzero ideal, and $t \geq 0$. Let $0 \neq \psi \in \mathcal{C}_{e_0}^\Delta(R)$ for some $e_0 > 0$, and let $0 \neq u \in I$ be such that R_u is regular, $\Delta|_{\text{Spec } R_u} = 0$, and ψ_u generates $\text{Hom}_{R_u}(F_*^{e_0} R_u, R_u)$ as an $F_*^{e_0} R_u$ -module. Then*

$$(8.1) \quad \tau_b(R; \Delta, I^t) = \sum_{e \geq 0} \sum_{\phi \in \mathcal{C}_e^\Delta(R)} \phi \left(F_*^e \left(u^a I^{\lceil t(p^e - 1) \rceil} \right) \right)$$

for every sufficiently large integer a .

Proposition 8.2. *Let R be the normal local ring of a k -variety at a closed point, and Δ an effective rational Weil divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular. Let $A = \widehat{R}$. For a height-one prime Q of A , let the coefficient of Q in $\widehat{\Delta}$ be the coefficient of $Q \cap R$ in Δ if $Q \cap R$ has height one, and 0 otherwise. Then:*

- (1) $(A, \widehat{\Delta})$ is strongly F -regular.
- (2) $\tau_b(A; \widehat{\Delta}, (IA)^t) = \tau_b(R; \Delta, I^t)A$ for every nonzero ideal $I \subset R$ and every $t \geq 0$.
- (3) $J = (J \cap R)A$ for every $\mathfrak{m}_A$ -primary ideal J of A .
- (4) $\text{Fvol}(A, \widehat{\Delta}) = \text{Fvol}(R, \Delta)$ and $\text{Fvol}_{\text{HK}}(A, \widehat{\Delta}) = \text{Fvol}_{\text{HK}}(R, \Delta)$.

Proof. The ring R is excellent and F -finite, A is a normal local domain, and $R \rightarrow A$ is faithfully flat with geometrically regular fibers [Sta26, Tags 00MC, 07QT, 07QU, 07QW, and 0C23].

Step 1. In this step, we show that $\widehat{\Delta}$ is an effective rational Weil divisor and that

$$(8.2) \quad \mathcal{C}_e^\Delta(R) \otimes_R A = \mathcal{C}_e^{\widehat{\Delta}}(A) \subset \text{Hom}_A(F_*^e A, A)$$

for every $e \geq 1$.

Let Q be a height-one prime of A and $P = Q \cap R$. By going down, P has height at most one. If P has height one, then A_Q/PA_Q is a zero-dimensional localization of the regular fiber of $R \rightarrow A$ over P , so $PA_Q = QA_Q$. Define $\widehat{Z}$ for every integral Weil divisor $Z = \sum_P z_P [P]$ on $\text{Spec } R$ in the same way. Since a nonzero element of A lies in finitely many height-one primes, $\widehat{Z}$ and $\widehat{\Delta}$ have finite support, and $\lceil \alpha \widehat{\Delta} \rceil = \lceil \alpha \widehat{\Delta} \rceil$ for $\alpha \geq 0$.

By flatness, $R(Z) \otimes_R A$ is a submodule of $\text{Frac } A$, and it is reflexive by [Sta26, Tag 087R] applied twice to the biduality isomorphism of $R(Z)$. Its localization at a height-one prime Q is $(QA_Q)^{-z_P}$ if $P = Q \cap R$ has height one, and A_Q if $P = 0$. Thus $R(Z) \otimes_R A = A(\widehat{Z})$ by [Sta26, Tag 0AVB].

Let M be a finite R -module and $q = p^e$. The subgroups $(\mathfrak{m}_R^r)^{[q]}M$ of $F_*^e M$ are cofinal with the subgroups $\mathfrak{m}_R^s M$, so the completion of $F_*^e M$ is $F_*^e(M \otimes_R A)$, and [Sta26, Tag 00MA] gives

$$(8.3) \quad F_*^e M \otimes_R A = F_*^e(M \otimes_R A).$$

For $M = R$, this shows that A is F -finite. For $M = R(\lceil (q-1)\Delta \rceil)$, together with [Sta26, Tag 087R] and $R(Z) \otimes_R A = A(\widehat{Z})$, it identifies $\mathcal{C}_e^\Delta(R) \otimes_R A$ with $\mathcal{C}_e^{\widehat{\Delta}}(A)$. Flat base change preserves the image of the restriction map to $\text{Hom}_R(F_*^e R, R)$, which gives (8.2).

Step 2. In this step, we prove (1) and (2).

Fix a nonzero ideal $I \subset R$ and $t \geq 0$. Choose a nonzero $\psi \in \mathcal{C}_1^\Delta(R)$ and a nonzero $u \in I$ such that R_u is regular, $\Delta|_{\text{Spec } R_u} = 0$, and ψ_u generates $\text{Hom}_{R_u}(F_* R_u, R_u)$ as an $F_* R_u$ -module. Such u exists because the regular locus of R is open and ψ generates $\text{Hom}_{\text{Frac } R}(F_* \text{Frac } R, \text{Frac } R)$. By [Sta26, Tag 0H7S], A_u is regular, and $\widehat{\Delta}|_{\text{Spec } A_u} = 0$ by the definition of $\widehat{\Delta}$. By (8.2) and (8.3), $\psi_A := \psi \otimes 1$ lies in $\mathcal{C}_1^{\widehat{\Delta}}(A)$ and $(\psi_A)_u$ generates $\text{Hom}_{A_u}(F_* A_u, A_u)$. By Theorem 8.1 for R and for A , (8.1) holds for (R, Δ, I^t) and for $(A, \widehat{\Delta}, (IA)^t)$ with $c := u^a$ for some $a \geq 1$.

The degree e summand of (8.1) for R is the image of the evaluation map $\mathcal{C}_e^\Delta(R) \otimes_R F_*^e(cI^{[t(p^e-1)]}) \rightarrow R$, whose base change to A is the evaluation map for A by (8.2) and (8.3). Flat base change preserves images, and summing over e gives (2).

By faithful flatness, an ideal $J \subset R$ satisfies $JA = A$ if and only if $J = R$. A triple is strongly F -regular if and only if its big test ideal is the unit ideal [ST14, Proposition 6.18(iv)]. Thus (2) implies

$$(8.4) \quad c_{A,\hat{\Delta}}(IA) = c_{R,\Delta}(I),$$

and the case $I = R$, $t = 0$ gives (1).

Step 3. In this step, we prove (3) and (4).

Let J be an $\mathfrak{m}_A$ -primary ideal and $v \geq 1$ with $\mathfrak{m}_A^v \subset J$. Since $R/\mathfrak{m}_R^v = A/\mathfrak{m}_A^v$ [Sta26, Tag 031C], the ideal $I := J \cap R$ contains $\mathfrak{m}_R^v$, and IA and J have the same image in $A/\mathfrak{m}_A^v$. Thus $IA = J$, which proves (3). Since $R/I' = A/I'A$ for every $\mathfrak{m}_R$ -primary ideal I' , taking $I' = I^r$ and $I' = I^{[q]}$ gives $e_A(IA) = e_R(I)$ and $e_A^{\text{HK}}(IA) = e_R^{\text{HK}}(I)$ for every $\mathfrak{m}_R$ -primary ideal I . With (8.4), $\Phi_{A,\hat{\Delta}}(IA) = \Phi_{R,\Delta}(I)$ and $e_A^{\text{HK}}(IA)c_{A,\hat{\Delta}}(IA)^n = e_R^{\text{HK}}(I)c_{R,\Delta}(I)^n$. By (3), (4) follows by taking infima. $\square$

Proposition 8.3. *Let $(R, \mathfrak{m}_R, k) \subset (S, \mathfrak{m}_S, k)$ be a finite local quasi-étale inclusion of normal local rings of k -varieties at closed points, of dimension $n \geq 1$, and Δ an effective rational Weil divisor on $\text{Spec } R$ such that (R, Δ) is strongly F -regular. Let $\delta = [\text{Frac } S : \text{Frac } R]$, $A = \hat{R}$, $B = \hat{S}$, Δ_S the divisorial pullback of Δ to $\text{Spec } S$, and $\Delta_A = \hat{\Delta}$ and $\Delta_B = \widehat{\Delta_S}$ as in Proposition 8.2. Then:*

- (1) $B = S \otimes_R A$ is a domain, $A \subset B$ is a finite local quasi-étale inclusion, and $[\text{Frac } B : \text{Frac } A] = \delta$.
- (2) *There exists a complete normal local k -domain $(C, \mathfrak{m}_C, k)$ with finite local quasi-étale Galois inclusions $A \subset C$ and $B \subset C$; the rings A , B , and C are F -finite. For $G = \text{Gal}(\text{Frac } C / \text{Frac } A)$ and $H = \text{Gal}(\text{Frac } C / \text{Frac } B)$, we have $C^G = A$, $C^H = B$, $|G| = \delta|H|$, and $p \nmid |G|$.*
- (3) *The divisorial pullbacks of Δ_A and Δ_B to $\text{Spec } C$ coincide. This divisor Δ_C is effective and G -invariant, and (A, Δ_A) , (B, Δ_B) , and (C, Δ_C) are strongly F -regular.*

Proof. Let $K = \text{Frac } A$ and $L = \text{Frac } B$. We take C to be the integral closure of A in a finite Galois extension of K containing L .

Step 1. In this step, we prove (1) and the strong F -regularity of (A, Δ_A) and (B, Δ_B) .

Since $R \subset S$ is quasi-étale, it is separable with zero ramification divisor, and R is strongly F -regular (Definition 2.1). The rings R and S are F -finite, so (S, Δ_S) is strongly F -regular by Theorem 6.1. By [Sta26, Tag 0C23] and Proposition 8.2(1), A and B are normal local domains with residue field k , and (A, Δ_A) and (B, Δ_B) are strongly F -regular.

Since $S/\mathfrak{m}_R S$ is Artinian local, the $\mathfrak{m}_R S$ -adic and $\mathfrak{m}_S$ -adic topologies on S coincide, so $B = S \otimes_R A$ [Sta26, Tags 00MA and 07N9], and $A \subset B$ is finite and local by flatness. Since $B \otimes_A K = \text{Frac } S \otimes_{\text{Frac } R} K$ is a δ -dimensional K -algebra and a localization of the domain B , it equals L and $[L : K] = \delta$. Let $U_0 \subset \text{Spec } R$ be an open subset containing every point of codimension at most one over which $\text{Spec } S \rightarrow \text{Spec } R$ is étale, and U its preimage in $\text{Spec } A$. By going down, U contains every point of $\text{Spec } A$ of codimension at most one, and $\text{Spec } B \rightarrow \text{Spec } A$ is étale over U . This proves (1).

Step 2. In this step, we construct C as in (2).

Let $V = \text{Spec } B \times_{\text{Spec } A} U$, an integral scheme finite étale over U . By [Sta26, Tag 03SF, Section 58.7], there exists a connected finite étale Galois cover $W \rightarrow U$ factoring through V . Its function field M is a finite Galois extension of K containing L . The integral closure C of A in M is a finite A -module [Sta26, Tags 07QV, 032R, and 032F], a normal domain with $\text{Frac } C = M$

containing B , and a product of complete local rings [Sta26, Tag 0325], hence complete and local with residue field k . Thus $A \subset C$ and $B \subset C$ are finite local inclusions, and $\dim C = n$.

Since W is normal and finite over U with function field M , it is the preimage of U in $\operatorname{Spec} C$, so $A \subset C$ is quasi-étale. Since $B \otimes_A K = L$ is a field, every height-one prime of B contracts to a nonzero prime of A , which has height one by going down, so V contains every height-one point of $\operatorname{Spec} B$. Since $W \rightarrow V$ is finite étale and W is the preimage of V in $\operatorname{Spec} C$, $B \subset C$ is quasi-étale. Both inclusions are Galois because M/K and M/L are.

Step 3. In this step, we prove the first two assertions of (3).

Let Q be a height-one prime of B , $P = Q \cap A$, $\mathfrak{q} = Q \cap S$, and $\mathfrak{p} = \mathfrak{q} \cap R = P \cap R$. By **Step 2** and going down, P has height one and $\mathfrak{p}, \mathfrak{q}$ have height at most one. If $\mathfrak{p} = 0$, then $\mathfrak{q} = 0$ because $S \otimes_R \operatorname{Frac} R = \operatorname{Frac} S$ is a field, and Δ_A, Δ_B have coefficient 0 at P, Q . If $\mathfrak{p}$ has height one, then so does $\mathfrak{q}$, and the coefficients of Δ_A at P and of Δ_B at Q both equal the coefficient of Δ at $\mathfrak{p}$ because $R \subset S$ is quasi-étale. Since $A \subset B$ is quasi-étale, $\Delta_B = f_{B/A}^* \Delta_A$ for the finite morphism $f_{B/A} : \operatorname{Spec} B \rightarrow \operatorname{Spec} A$.

Let $f_{C/A}$ and $f_{C/B}$ be the finite morphisms from $\operatorname{Spec} C$ to $\operatorname{Spec} A$ and $\operatorname{Spec} B$, and $\Delta_C = f_{C/A}^* \Delta_A$. Since $C \otimes_A K = M$ is a field, every height-one prime of C contracts to height-one primes of A and of B by going down, and all ramification indices are one. Thus $\Delta_C = f_{C/B}^* f_{B/A}^* \Delta_A = f_{C/B}^* \Delta_B$ is effective. Every $g \in G$ preserves C and permutes the height-one primes of C over each height-one prime of A , at which Δ_C has the same coefficient, so Δ_C is G -invariant.

Step 4. In this step, we prove (2) and finish the proof of (3).

By the Cohen structure theorem [Sta26, Tag 032A], each of A, B , and C is a quotient of a power series ring over k , which is free over its subring of p -th powers, hence F -finite. Since A is strongly F -regular and $A \subset C$ is quasi-étale, Theorem 6.1 gives $\operatorname{Tr}_{M/K}(C) = A$ and the strong F -regularity of (C, Δ_C) .

Since A and B are normal and C is integral over them, $C^G = C \cap K = A$ and $C^H = C \cap L = B$. Since M/L is Galois with group H , $|G| = [M : L][L : K] = \delta|H|$. Every $g \in G$ preserves $\mathfrak{m}_C$ and induces the identity on $C/\mathfrak{m}_C = k = A/\mathfrak{m}_A$. Thus $\operatorname{Tr}_{M/K}(x) = \sum_{g \in G} g(x) \equiv |G|x \pmod{\mathfrak{m}_C}$ for every $x \in C$. If p divided $|G|$, then $\operatorname{Tr}_{M/K}(C) \subset \mathfrak{m}_C \cap A = \mathfrak{m}_A$, contradicting $\operatorname{Tr}_{M/K}(C) = A$. This proves (2) and (3). $\square$

Proof of Theorem 1.2. Let $A = \widehat{R}$, $B = \widehat{S}$, and $C, G, H, \Delta_A, \Delta_B, \Delta_C$ be as in Proposition 8.3. Since $R \subset S$ is quasi-étale and R is strongly F -regular, (S, Δ_S) is strongly F -regular by Theorem 6.1. By Proposition 8.3(2) and (3), $A \subset C$ and $B \subset C$ satisfy the hypotheses of Theorem 1.4 with the boundaries Δ_A and Δ_B , where $\dim C = n$ because C is finite over A . Theorem 1.4 for $A \subset C$ and for $B \subset C$ gives

$$(8.5) \quad \operatorname{Fvol}(C, \Delta_C) = |G| \operatorname{Fvol}(A, \Delta_A) = |H| \operatorname{Fvol}(B, \Delta_B).$$

The three rings are F -finite by Proposition 8.3(2), so the three Frobenius volumes in (8.5) are finite by Lemma 2.3. Since $|G| = \delta|H|$, (8.5) gives $\operatorname{Fvol}(B, \Delta_B) = \delta \operatorname{Fvol}(A, \Delta_A)$, and Proposition 8.2(4) for (R, Δ) and (S, Δ_S) gives $\operatorname{Fvol}(S, \Delta_S) = \delta \operatorname{Fvol}(R, \Delta)$. By Theorem 1.5 for (R, Δ) and (S, Δ_S) ,

$$\operatorname{Fvol}_{\operatorname{HK}}(S, \Delta_S) = \frac{\operatorname{Fvol}(S, \Delta_S)}{n!} = \delta \frac{\operatorname{Fvol}(R, \Delta)}{n!} = \delta \operatorname{Fvol}_{\operatorname{HK}}(R, \Delta). \quad \square$$

Proof of Corollary 1.3. This is Theorem 1.2 for $\Delta = 0$, $R = \mathcal{O}_{X,x}$, and $S = \mathcal{O}_{Y,y}$. $\square$

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