

CAMPANA'S ABELIANITY CONJECTURE FOR PROJECTIVE ORBIFOLDS

ABSTRACT. We prove that the orbifold fundamental group of a special smooth projective orbifold pair is virtually abelian. This answers a conjecture of Campana and extends a recent result of OpenAI to the orbifold projective case.

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Comparison of differential conventions	5
4. Transfer of specialness	9
5. Auxiliary covers and affine actions	13
6. Divisorial valuations	21
7. Proof of Proposition 1.2	25
8. Fundamental groups	26
References	28

1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

The goal of this paper is to prove the following theorem on the orbifold fundamental group of Campana special orbifold pairs, in the projective case. We refer the reader to Definition 2.6 for the definition of special orbifolds in the sense of Campana.

Theorem 1.1. *Let $(X, B = \sum_{i=1}^r (1 - 1/m_i)D_i)$ be a projective orbifold pair, special in the sense of Campana, such that D_i are the irreducible components of B . Let $D := \text{Supp } B$. Pick $x_0 \in X \setminus D$ and let γ_i be a positively oriented meridian about D_i transported to x_0 . Then the orbifold fundamental group*

$$\pi_1^{\text{orb}}(X, B; x_0) := \pi_1(X \setminus D, x_0) / \langle \gamma_i^{m_i} \mid 1 \leq i \leq r \rangle$$

is virtually abelian, i.e. it contains an abelian subgroup of finite index.

Theorem 1.1 proves the projective case of [Cam11a, Conjecture 13.10(1)] and generalizes [Ope26b, Theorem 1.1] to the projective orbifold case. We remark that [Ope26a, Theorem 1.3] proved Theorem 1.1 for the special case when $r = 1$ and $m_1 = 2$. The key new geometric ingredient is the following proposition.

Proposition 1.2. *Let $(X, B = \sum_{i=1}^r (1 - \frac{1}{m_i})D_i)$ be a projective orbifold pair of dimension $n \geq 1$ and $D := \text{Supp } B$. Then there exist a smooth projective variety Y and a surjective morphism $g : Y \rightarrow X$ satisfying the following.*

- (1) *For each i , there exists an effective Weil divisor C_i on Y such that*

$$(1.1) \quad g^* D_i = m_i C_i.$$

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- (2) *There exists a nonempty open subset $U \subset X \setminus D$ such that $g|_{g^{-1}(U)} : g^{-1}(U) \rightarrow U$ is a holomorphically locally trivial fiber bundle whose fiber F is an abelian variety.*
- (3) *For any prime divisor P on X such that $P \neq D_i$ for every i , there exists a nonempty open subset $P^\circ \subset P$ such that g is smooth at some point of $g^{-1}(x)$ for every $x \in P^\circ$.*
- (4) *For each i , there exists a nonempty open subset $D_i^\circ \subset D_i \setminus \bigcup_{k \neq i} D_k$ such that for every $x \in D_i^\circ$, there exists $y \in g^{-1}(x)$ such that some local lift of g to the root chart at x is a submersion at y .*

Moreover, we may take F with

$$(1.2) \quad \dim F = \sum_{i=1}^r m_i \binom{n+1}{2}.$$

Proposition 1.2 is the analogue of [Ope26a, Proposition A.3], which treats the case when $n = 4$, $r = 1$, and $m_1 = 2$. The main difficulty is to deal with boundary components of arbitrary multiplicities and their intersections. To this end, we replace the involutions on elliptic curves used in [Ope26a] by permutation-translation actions of $\mathbb{Z}/m\mathbb{Z}$ on products of elliptic curves (Lemma 5.3), and we obtain the smooth points of g over the auxiliary branch divisors by extracting suitable divisorial valuations on the quotient (Proposition 6.3) instead of blowing up the fixed loci.

We briefly explain how Theorem 1.1 follows from Proposition 1.2. Let $\varpi : \mathcal{R} \rightarrow X$ be the root orbifold of (X, B) (Definition 2.3). By Proposition 3.2(4), (X, B) is special in the sense of Campana if and only if $\mathcal{R}$ is special in the sense of [Ope26b] (Definition 2.7). By Proposition 4.3, which follows [Ope26a, Proposition 3.1], the latter condition implies that Y is special. Hence $\pi_1(Y)$ is virtually abelian by [Ope26b, Theorem 1.1]. Finally, by Proposition 8.3, $\pi_1^{\text{orb}}(X, B; x_0)$ is a quotient of $\pi_1(Y)$, and Theorem 1.1 follows.

Structure of the paper. In Section 2, we recall the definitions of orbifold fundamental groups, root orbifolds, and special orbifold pairs, and the results from [Cam11a, Ope26b] that we use. In Section 3, we compare specialness of (X, B) with specialness of its root orbifold and prove Proposition 3.2. In Section 4, we prove Proposition 4.3. In Sections 5 and 6, we construct the abelian cover $S \rightarrow X$, the abelian variety F with its group action, the quotient $Q = (S \times F)/A$, and the divisorial valuations on Q that we need. In Section 7, we prove Proposition 1.2. In Section 8, we prove Proposition 8.3 and Theorem 1.1.

2. PRELIMINARIES

In this section, we fix notation and recall the definitions and results that we use in the rest of the paper.

All varieties are defined over $\mathbb{C}$ and are integral. We identify a variety with its associated complex analytic space whenever we use the analytic topology. A *prime divisor* on a variety X is an integral closed subvariety of X of codimension one. For a divisor B on X , we denote by $\text{Supp } B$ the support of B . We denote by $\mathbb{C}(X)$ the function field of X . For a prime divisor E on a normal variety W , we denote by $\mathcal{O}_{W,E}$ the local ring of W at the generic point of E , which is a discrete valuation ring, and by ord_E the corresponding valuation of $\mathbb{C}(W)$. For a positive integer m , we denote by $\mu_m := \{\zeta \in \mathbb{C} \mid \zeta^m = 1\}$ the group of m th roots of unity. Let A be a finite group acting on a variety V . For $a \in A$, we denote by $V^a := \{v \in V \mid av = v\}$ the fixed locus of a with its reduced structure. We also write $\text{Fix}(a)$ for V^a when V is clear from the context. For a subgroup $H \subset A$, we write $V^H := \bigcap_{a \in H} V^a$. A holomorphic map $g : Y \rightarrow X$ between complex manifolds is a *submersion* at a point $y \in Y$ if the differential $dg_y : T_y Y \rightarrow T_{g(y)} X$ is surjective. A morphism $g : Y \rightarrow X$ between smooth varieties is *smooth* at a point $y \in Y$ if g is a submersion at y . Let $g : Y \rightarrow X$ be a morphism between varieties and F a variety. We say that g is a *holomorphically locally trivial fiber bundle* with fiber F if every point of X has an open neighborhood W in the analytic topology such that $g^{-1}(W)$ is biholomorphic to $W \times F$ over W .

Definition 2.1. Let X be a smooth projective variety of dimension n . Let $r \geq 0$ be an integer, $D_1, \dots, D_r$ distinct smooth prime divisors on X such that $D := \sum_{i=1}^r D_i$ is a simple normal crossing divisor, and $m_1, \dots, m_r \geq 2$ integers. The pair (X, B) with

$$B := \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) D_i$$

is called a *projective orbifold pair*. Then $D_1, \dots, D_r$ are the irreducible components of B and $D = \text{Supp } B$. The integer m_i is the *multiplicity* of D_i in (X, B) . In the terminology of [Cam11a, Definition 2.1], (X, B) is a smooth, finite, and integral geometric orbifold. If $r = 0$, then $B = 0$ and $D = 0$.

Definition 2.2. Let (X, B) be a projective orbifold pair as in Definition 2.1 and $x_0 \in X \setminus D$ a point. Fix $1 \leq i \leq r$. Let $x_i \in D_i \setminus \bigcup_{k \neq i} D_k$ be a point. Let $\Delta_i \subset X$ be a closed holomorphic disc such that $\Delta_i \cap D = \{x_i\}$ and Δ_i is transverse to D_i at x_i . Let $\partial\Delta_i$ be the boundary circle of Δ_i with the orientation induced by the complex orientation of Δ_i . Let β_i be a path in $X \setminus D$ from x_0 to a point of $\partial\Delta_i$. The loop $\gamma_i := \beta_i \cdot \partial\Delta_i \cdot \beta_i^{-1}$ in $X \setminus D$ based at x_0 is called a *positively oriented meridian* about D_i at x_0 . The conjugacy class of γ_i in $\pi_1(X \setminus D, x_0)$ does not depend on the choices of x_i , Δ_i , and β_i (cf. Lemma 8.2(3)). The *orbifold fundamental group* of (X, B) at x_0 is

$$\pi_1^{\text{orb}}(X, B; x_0) := \pi_1(X \setminus D, x_0) / \langle \gamma_i^{m_i} \mid 1 \leq i \leq r \rangle$$

[Cam11a, Definition 12.2]. Here $\langle \gamma_i^{m_i} \mid 1 \leq i \leq r \rangle$ denotes the normal subgroup of $\pi_1(X \setminus D, x_0)$ generated by $\gamma_1^{m_1}, \dots, \gamma_r^{m_r}$. A group is *virtually abelian* if it has an abelian subgroup of finite index.

Definition 2.3. Let (X, B) be a projective orbifold pair as in Definition 2.1 and $x \in X$ a point. Reordering the indices, we may assume that $D_1, \dots, D_s$ are the components of D containing x , where $0 \leq s \leq r$. An *adapted coordinate neighborhood* of x is an open neighborhood W of x in the analytic topology with holomorphic coordinates $z_1, \dots, z_n$ centered at x satisfying the following. The set W is a polydisc in these coordinates. We have $D_i \cap W = (z_i = 0)$ for $1 \leq i \leq s$ and $D_i \cap W = \emptyset$ for $i > s$.

Let $\widetilde{W}$ be the set of $(w_1, \dots, w_s, z_{s+1}, \dots, z_n) \in \mathbb{C}^n$ such that $(w_1^{m_1}, \dots, w_s^{m_s}, z_{s+1}, \dots, z_n) \in W$. Then $\widetilde{W}$ is a polydisc. The *root chart* of (X, B) over W is the finite surjective holomorphic map $\varpi_W : \widetilde{W} \rightarrow W$ together with the action of $\mu_{m_1} \times \dots \times \mu_{m_s}$ on $\widetilde{W}$ given by

$$(2.1) \quad \begin{aligned} z_i \circ \varpi_W &= w_i^{m_i} \quad (1 \leq i \leq s), & z_i \circ \varpi_W &= z_i \quad (s < i \leq n), \\ (\zeta_1, \dots, \zeta_s) \cdot w_i &= \zeta_i w_i \quad (1 \leq i \leq s), & (\zeta_1, \dots, \zeta_s) \cdot z_i &= z_i \quad (s < i \leq n). \end{aligned}$$

The map ϖ_W identifies W with the quotient $\widetilde{W} / (\mu_{m_1} \times \dots \times \mu_{m_s})$. We also call $\widetilde{W}$ a *root chart* of (X, B) at x .

The *root orbifold* $\mathcal{R}$ of (X, B) is the orbifold with underlying space X whose orbifold charts are the root charts of (X, B) , and $\varpi : \mathcal{R} \rightarrow X$ is the natural morphism given by ϖ_W on each root chart (cf. [Cam11a, Remark 2.9(2)]). An *orbifold line bundle* on $\mathcal{R}$ is a $\mu_{m_1} \times \dots \times \mu_{m_s}$ -equivariant holomorphic line bundle on each root chart $\widetilde{W}$, compatible with the identifications on overlapping charts (cf. [Ope26a, Section 1]). An *orbifold vector bundle* on $\mathcal{R}$ and a *morphism of orbifold bundles* on $\mathcal{R}$ are defined in the same way. A *section* of an orbifold line bundle L on $\mathcal{R}$ is a $\mu_{m_1} \times \dots \times \mu_{m_s}$ -invariant holomorphic section of L on each root chart, compatible on overlapping charts. We denote by $H^0(\mathcal{R}, L)$ the space of sections of L . We denote by $\mathcal{O}_{\mathcal{R}}$ the orbifold line bundle given by the trivial line bundle on each root chart. We denote by $\Omega_{\mathcal{R}}^p$ the orbifold vector bundle given by $\Omega_{\widetilde{W}}^p$ on each root chart $\widetilde{W}$ with the action induced by (2.1). For $1 \leq i \leq r$, the *root divisor* R_i is the orbifold divisor on $\mathcal{R}$ given by $(w_i = 0)$ on each root chart at a point of D_i and by 0 on each root chart at a point outside D_i . By (2.1),

$$(2.2) \quad \varpi^* D_i = m_i R_i.$$

For integers $b_1, \dots, b_r$, we denote by $\mathcal{O}_{\mathcal{R}}(\sum_{i=1}^r b_i R_i)$ the orbifold line bundle given by $\mathcal{O}_{\widetilde{W}}(\sum_{i=1}^s b_i(w_i = 0))$ on each root chart $\widetilde{W}$ at a point of $D_1 \cap \dots \cap D_s$.

Let $g : Y \rightarrow X$ be a morphism from a variety Y and $y \in Y$ a point, and let $x := g(y)$. Let $\varpi_W : \widetilde{W} \rightarrow W$ be a root chart of (X, B) at x and $\widetilde{V} \subset g^{-1}(W)$ a connected open neighborhood of y in the analytic topology. A *local lift* of g to the root chart at x is a holomorphic map $\tilde{g} : \widetilde{V} \rightarrow \widetilde{W}$ such that $\varpi_W \circ \tilde{g} = g|_{\widetilde{V}}$. Two local lifts of g on the same neighborhood $\widetilde{V}$ differ by the action of an element of $\mu_{m_1} \times \dots \times \mu_{m_s}$ in (2.1).

Definition 2.4. Let M be a connected compact complex manifold and L a line bundle on M . For a positive integer k , we write $L^k := L^{\otimes k}$. If $H^0(M, L^k) = 0$ for every positive integer k , then we define $\kappa(M, L) := -\infty$. Otherwise, for every positive integer k with $H^0(M, L^k) \neq 0$, let $\phi_k : M \dashrightarrow \mathbb{P}^{N_k}$ be the meromorphic map defined by a basis of $H^0(M, L^k)$, where $N_k := \dim H^0(M, L^k) - 1$. We define $\kappa(M, L)$ to be the maximum of the dimensions of the closures of the images of these maps ϕ_k . We call $\kappa(M, L)$ the *Itaka dimension* of L .

Let (X, B) be a projective orbifold pair as in Definition 2.1 and L an orbifold line bundle on $\mathcal{R}$. We define $\kappa(\mathcal{R}, L)$ in the same way, where $H^0(\mathcal{R}, L^k)$ is the space of sections of L^k in the sense of Definition 2.3 and $\phi_k : X \dashrightarrow \mathbb{P}^{N_k}$ is the meromorphic map defined by a basis of $H^0(\mathcal{R}, L^k)$. Here $H^0(\mathcal{R}, L^k)$ is finite dimensional, since it is the space of global sections of a coherent analytic sheaf on the compact space X (see Proposition 3.2(3)). This map is well defined because the quotient of two sections of L^k is a meromorphic function on X .

Let $A = \bigoplus_{q \geq 0} A_q$ be a graded $\mathbb{C}$ -algebra which is an integral domain such that $\dim_{\mathbb{C}} A_q < +\infty$ for every q . If $A_q = 0$ for every $q > 0$, then we define $\kappa(A) := -\infty$. Otherwise, we define $\kappa(A) := \text{trdeg}_{\mathbb{C}} \text{Frac}(A) - 1$. We also call $\kappa(A)$ the *Itaka dimension* of A . Proposition 3.2 shows that these conventions agree for the line bundles considered in this paper.

Definition 2.5. Let (X, B) be a projective orbifold pair as in Definition 2.1, $1 \leq p \leq n$ an integer, and $q \geq 0$ an integer. We denote by

$$C_{p,q} := S^q \Omega^p(X|B)$$

the subsheaf of $\text{Sym}^q \Omega_X^p(\log D)$ defined as follows [Cam11a, Section 2.5]. Let W be an adapted coordinate neighborhood of a point of X with coordinates $z_1, \dots, z_n$ as in Definition 2.3. Then $C_{p,q}|_W$ is the free $\mathcal{O}_W$ -module with basis

$$(2.3) \quad \left(\prod_{i=1}^s z_i^{-\lfloor t_i(1 - \frac{1}{m_i}) \rfloor} \right) dz_{I_1} \odot \dots \odot dz_{I_q}.$$

Here $(I_1, \dots, I_q)$ runs over the non-decreasing sequences, in the lexicographic order, of subsets of $\{1, \dots, n\}$ with p elements. For $I = \{j_1 < \dots < j_p\}$, we write $dz_I := dz_{j_1} \wedge \dots \wedge dz_{j_p}$. The symbol $\odot$ denotes the symmetric product, and $t_i := \#\{a \mid i \in I_a\}$. In particular, $C_{p,0} = \mathcal{O}_X$ and $C_{p,1} = \Omega_X^p$. The symmetric product induces morphisms $C_{p,q} \otimes C_{p,q'} \rightarrow C_{p,q+q'}$.

We denote by $\Omega_{\mathbb{C}(X)/\mathbb{C}}^p := \bigwedge^p \Omega_{\mathbb{C}(X)/\mathbb{C}}^1$ the $\mathbb{C}(X)$ -vector space of rational p -forms on X . Let $\ell \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ be a one-dimensional $\mathbb{C}(X)$ -subspace. We define $\mathcal{F} := \ell \cap \Omega_X^p$ to be the subsheaf of Ω_X^p of sections which lie in ℓ . For every integer $q \geq 0$, we define

$$(2.4) \quad \mathcal{F}_{q,B} := \ell^{\odot q} \cap C_{p,q}, \quad A_B(\mathcal{F}) := \bigoplus_{q \geq 0} H^0(X, \mathcal{F}_{q,B}).$$

Here $\ell^{\odot q} \subset \text{Sym}_{\mathbb{C}(X)}^q \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ is the q th symmetric power of ℓ , and $\ell^{\odot q} \cap C_{p,q}$ is the subsheaf of $C_{p,q}$ of sections which lie in $\ell^{\odot q}$. Then $A_B(\mathcal{F})$ is a graded $\mathbb{C}$ -algebra which is an integral domain. We define

$$\kappa(X|B, \mathcal{F}) := \kappa(A_B(\mathcal{F})).$$

More generally, let $\mathcal{G} \subset \Omega_X^p$ be a coherent subsheaf of rank one. We let $\ell \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ be the $\mathbb{C}(X)$ -subspace generated by $\mathcal{G}$ and define $\kappa(X|B, \mathcal{G}) := \kappa(X|B, \ell \cap \Omega_X^p)$. The sheaf $\mathcal{F}_{q,B}$ is the

saturation in $C_{p,q}$ of the image of $\mathcal{G}^{\otimes q}$. Thus $\kappa(X|B, \mathcal{G})$ is the Δ -dimension of $\mathcal{G}$ in [Cam11a, Section 3.1 and Definition 3.2] with $\Delta = B$.

Definition 2.6. Let (X, B) be a projective orbifold pair as in Definition 2.1. A *Bogomolov sheaf* of (X, B) is a coherent subsheaf $\mathcal{F} \subset \Omega_X^p$ of rank one for some integer $1 \leq p \leq n$ such that $\kappa(X|B, \mathcal{F}) = p$ [Cam11a, Definition 9.8]. We say that (X, B) is *special* (in the sense of Campana) if (X, B) has no Bogomolov sheaf. By [Cam11a, Theorem 9.9], this is equivalent to Campana's original definition by fibrations of general type [Cam11a, Definition 8.1]. In the latter, the multiplicities of the orbifold base of a fibration are defined by the infimum convention [Cam11a, Definition 4.2].

Definition 2.7. Let M be a connected compact Kähler manifold. We say that M is *special* if there do not exist an integer $1 \leq p \leq \dim M$, a line bundle L on M , and a nonzero morphism of sheaves $L \rightarrow \Omega_M^p$ such that $\kappa(M, L) = p$ [Ope26b, Section 1]. In particular, a point is special. Let (X, B) be a projective orbifold pair as in Definition 2.1. We say that $\mathcal{R}$ is *special* if there do not exist an integer $1 \leq p \leq n$, an orbifold line bundle L on $\mathcal{R}$, and a nonzero morphism of orbifold bundles $L \rightarrow \Omega_{\mathcal{R}}^p$ such that $\kappa(\mathcal{R}, L) = p$. If $B = 0$, then $\mathcal{R} = X$, and X is special if and only if $(X, 0)$ is special in the sense of Definition 2.6 (see Proposition 3.2(4)).

Theorem 2.8 ([Cam11a, Theorem 3.12]). *Let Z be a connected compact Kähler manifold, D a simple normal crossing divisor on Z , $1 \leq a \leq \dim Z$ an integer, and $\mathcal{H} \subset \Omega_Z^a(\log D)$ a coherent subsheaf of rank one. For every integer $q \geq 0$, let $\mathcal{H}^{[q]}$ be the saturation of the image of $\mathcal{H}^{\otimes q}$ in $\text{Sym}^q \Omega_Z^a(\log D)$, that is, the largest coherent subsheaf of rank one of $\text{Sym}^q \Omega_Z^a(\log D)$ containing this image. Then*

$$\kappa(Z, \mathcal{H}) := \kappa \left(\bigoplus_{q \geq 0} H^0 \left(Z, \mathcal{H}^{[q]} \right) \right) \leq a.$$

Theorem 2.9 ([Ser56, Section 12, Theorems 1–3]). *Let X be a projective variety. Then analytification induces an equivalence between the category of coherent algebraic sheaves on X and the category of coherent analytic sheaves on X . Moreover, it induces isomorphisms on all cohomology groups.*

Theorem 2.10 ([Wlo05, Theorem 1.0.2], see also [BM97, Theorem 1.6] and [Kol05, arXiv version 3, Theorem 27]). *Let Z be a projective variety. Then there exists a projective birational morphism $\tilde{Z} \rightarrow Z$ from a smooth projective variety $\tilde{Z}$ which is an isomorphism over the smooth locus of Z .*

Theorem 2.11 ([Ope26b, Theorem 1.1]). *Let M be a connected compact Kähler manifold. If M is special in the sense of Definition 2.7, then $\pi_1(M)$ is virtually abelian.*

3. COMPARISON OF DIFFERENTIAL CONVENTIONS

In this section, we prove Proposition 3.2.

Lemma 3.1. *Let W be a connected complex manifold, $\mathcal{M}(W)$ its field of meromorphic functions, $\mathcal{E}$ a holomorphic vector bundle on W , and ℓ a one-dimensional $\mathcal{M}(W)$ -subspace of the space of meromorphic sections of $\mathcal{E}$, generated by v . Let $\ell \cap \mathcal{E} \subset \mathcal{E}$ be the subsheaf whose sections over an open subset $W' \subset W$ are the holomorphic sections s of $\mathcal{E}$ on W' with $s = fv|_{W'}$ for a meromorphic function f on W' . For every integer $q \geq 1$, let $\ell^{\otimes q}$ and $\ell^{\odot q}$ be the subspaces generated by $v^{\otimes q}$ and $v^{\odot q}$. Then:*

- (1) $\ell \cap \mathcal{E}$ is invertible. A section of $\ell \cap \mathcal{E}$ near $x \in W$ generates $(\ell \cap \mathcal{E})_x$ if and only if its coordinates in a frame of $\mathcal{E}$ at x have no common prime factor in $\mathcal{O}_{W,x}$.
- (2) $(\ell \cap \mathcal{E})^{\otimes q} = \ell^{\otimes q} \cap \mathcal{E}^{\otimes q}$ and $\text{Sym}^q(\ell \cap \mathcal{E}) = \ell^{\odot q} \cap \text{Sym}^q \mathcal{E}$ for every integer $q \geq 1$.
- (3) If a finite group acts on W and on $\mathcal{E}$ compatibly and ℓ is stable under this action, then so is $\ell \cap \mathcal{E}$.

The invertibility in (1) and the assertion on tensor powers in (2) are [Ope26a, Lemma 2.2].

Proof. Let $x \in W$, $T := \mathcal{O}_{W,x}$, and η a generator of $(\ell \cap \mathcal{E})_x$ with coordinates $\eta_1, \dots, \eta_b \in T$. If a prime π of the factorial ring T divides every η_j , then $\eta/\pi \in (\ell \cap \mathcal{E})_x = T\eta$, so π is a unit. Conversely, let $u \in (\ell \cap \mathcal{E})_x$ have coordinates $u_1, \dots, u_b$ with no common prime factor, and $\eta = au$ with $a \in \text{Frac}(T)$. For every height one prime P of T , some u_j is a unit in T_P and $au_j = \eta_j \in T$, so $a \in T_P$. Since T is normal, $a \in T$, so u generates $(\ell \cap \mathcal{E})_x$. This proves (1). The coordinates of $\eta^{\odot q}$ in the monomial frame of $\text{Sym}^q \mathcal{E}$ include $\eta_1^q, \dots, \eta_b^q$, so $\eta^{\odot q}$ generates $(\ell^{\odot q} \cap \text{Sym}^q \mathcal{E})_x$ by (1). This proves (2). For (3), an element g of the group preserves holomorphic sections of $\mathcal{E}$ and satisfies $gv = cv$ for a meromorphic function $c \neq 0$, so $g(\ell \cap \mathcal{E}) = \ell \cap \mathcal{E}$. $\square$

Proposition 3.2. *Let (X, B) be a projective orbifold pair as in Definition 2.1, $1 \leq p \leq n$ an integer, $\ell \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ a one-dimensional $\mathbb{C}(X)$ -subspace, and $\mathcal{F} := \ell \cap \Omega_X^p$, $\mathcal{F}_{q,B}$, and $A_B(\mathcal{F})$ as in Definition 2.5. Let $M := \text{lcm}(m_1, \dots, m_r)$ ($M := 1$ if $r = 0$). For an orbifold vector bundle $\mathcal{G}$ on $\mathcal{R}$, let $\varpi_* \mathcal{G}$ be the sheaf on X whose sections over an analytic open subset $W \subset X$ are the sections of $\mathcal{G}$ on the root charts over W . For a line bundle $\mathcal{A}$ on X , let $\varpi^* \mathcal{A}$ be the orbifold line bundle on $\mathcal{R}$ given by $\varpi_W^* \mathcal{A}$ on each root chart, with the action fixing the pullbacks of the sections of $\mathcal{A}$. For a root chart $\varpi_W : \widetilde{W} \rightarrow W$, let $\varpi_W^* \ell$ be the subspace of meromorphic p -forms on $\widetilde{W}$ generated by the pullback of ℓ , and $\varpi_W^* \ell \cap \Omega_{\widetilde{W}}^p$ the subsheaf of Lemma 3.1. Then:*

(1) *The subsheaves $\varpi_W^* \ell \cap \Omega_{\widetilde{W}}^p$ form an orbifold line bundle $M_\ell \subset \Omega_{\mathcal{R}}^p$, and for every integer $q \geq 0$,*

$$(3.1) \quad \varpi_* \text{Sym}^q \Omega_{\mathcal{R}}^p = C_{p,q} \quad \text{and} \quad \varpi_* M_\ell^q = \mathcal{F}_{q,B},$$

where a section of $C_{p,q}$ over W is identified with its pullback to the root charts over W . Moreover, $\bigoplus_{q \geq 0} H^0(\mathcal{R}, M_\ell^q) = A_B(\mathcal{F})$ as graded $\mathbb{C}$ -algebras, and

$$(3.2) \quad \kappa(\mathcal{R}, M_\ell) = \kappa(X|B, \mathcal{F}).$$

(2) *$\mathcal{F}$ is invertible. For $1 \leq i \leq r$, let $\epsilon_i := 1$ if the composition $\mathcal{F} \rightarrow \Omega_X^p \rightarrow \Omega_{D_i}^p$ is zero at the generic point of D_i , and $\epsilon_i := 0$ otherwise. Then*

$$(3.3) \quad M_\ell = \varpi^* \mathcal{F} \otimes \mathcal{O}_{\mathcal{R}} \left(\sum_{i=1}^r \epsilon_i (m_i - 1) R_i \right),$$

$$(3.4) \quad \mathcal{F}_{q,B} = \mathcal{F}^q \otimes \mathcal{O}_X \left(\sum_{i=1}^r \epsilon_i \left[q \left(1 - \frac{1}{m_i} \right) \right] D_i \right)$$

for every integer $q \geq 0$, and

$$(3.5) \quad M_\ell^M = \varpi^* \mathcal{A}, \quad \mathcal{A} := \mathcal{F}^M \otimes \mathcal{O}_X \left(\sum_{i=1}^r \epsilon_i M \left(1 - \frac{1}{m_i} \right) D_i \right).$$

(3) *For every orbifold line bundle L on $\mathcal{R}$, there exists a line bundle $\mathcal{A}_L$ on X with $L^M = \varpi^* \mathcal{A}_L$ such that the pullback identifies $H^0(X, \mathcal{A}_L^k)$ with $H^0(\mathcal{R}, L^{kM})$ for every integer $k \geq 0$. Moreover,*

$$(3.6) \quad \kappa(\mathcal{R}, L) = \kappa \left(\bigoplus_{q \geq 0} H^0(\mathcal{R}, L^q) \right).$$

If $j : L \rightarrow \Omega_{\mathcal{R}}^p$ is a nonzero morphism of orbifold bundles, then there exists a unique one-dimensional $\mathbb{C}(X)$ -subspace $\ell' \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ such that j factors through an injective morphism $L \rightarrow M_{\ell'}$, and

$$(3.7) \quad \kappa(\mathcal{R}, L) \leq \kappa(\mathcal{R}, M_{\ell'}) = \kappa(X|B, \ell' \cap \Omega_X^p) \leq p.$$

- (4) (X, B) is special if and only if $\mathcal{R}$ is special. Moreover, $\mathcal{R}$ is special if and only if $\kappa(\mathcal{R}, M_{\ell'}) \leq p' - 1$ for every integer $1 \leq p' \leq n$ and every one-dimensional $\mathbb{C}(X)$ -subspace $\ell' \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^{p'}$.

Proof. We use the notation of Definitions 2.3 and 2.5 and identify coherent algebraic and analytic sheaves on X by Theorem 2.9. Let $x \in X$, $D_1, \dots, D_s$ the components of D containing x , $\varpi_W : \widetilde{W} \rightarrow W$ a root chart at x with coordinates $z_1, \dots, z_n$ on W and $w_1, \dots, w_s, z_{s+1}, \dots, z_n$ on $\widetilde{W}$, $0 \in \widetilde{W}$ the point over x , $\mu := \mu_{m_1} \times \dots \times \mu_{m_s}$, $w_j := z_j$ for $j > s$, and $dw_I := dw_{j_1} \wedge \dots \wedge dw_{j_p}$ for $I = \{j_1 < \dots < j_p\} \subset \{1, \dots, n\}$. By (2.1), $\zeta = (\zeta_1, \dots, \zeta_s) \in \mu$ multiplies dw_I by $\prod_{j \in I, j \leq s} \zeta_j$, and

$$(3.8) \quad \varpi_W^*(dz_{I_1} \odot \dots \odot dz_{I_q}) = \left(\prod_{i=1}^s m_i^{t_i} w_i^{t_i(m_i-1)} \right) dw_{I_1} \odot \dots \odot dw_{I_q}, \quad t_i := \#\{a \mid i \in I_a\}.$$

Step 1. In this step, we prove (1).

Let v generate ℓ . A holomorphic section $s = f v^{\odot q}$ of $C_{p,q}$ near x with f meromorphic lies in $\mathcal{F}_{q,B}$. Indeed, for a nonzero $g = h v^{\odot q} \in (\mathcal{F}_{q,B})_x$ with h rational and $f/h = a/b$ with a, b holomorphic near x and $b \neq 0$, the section $bs = ag$ lies in $\mathcal{F}_{q,B}$. Since $\ell^{\odot q}$ is a $\mathbb{C}(X)$ -subspace, $C_{p,q}/\mathcal{F}_{q,B}$ is torsion-free, hence locally a subsheaf of a free sheaf, so its analytification is torsion-free by the exactness in Theorem 2.9, and $s \in \mathcal{F}_{q,B}$. The same holds for $\mathcal{F}$ and $\ell \cap \Omega_X^p(\log D)$. Since X is irreducible, it is a connected complex manifold [Har77, Appendix B, Section 1], and Lemma 3.1 applies with $W = X$ to these three sheaves and the subspaces generated by ℓ and $\ell^{\odot q}$ over the meromorphic functions on X .

Let $c_1, \dots, c_s$ be integers, $0 \leq a_i \leq m_i - 1$ the residue of c_i modulo m_i , and b a holomorphic function on $\widetilde{W}$, expanded as a power series in $w_1, \dots, w_s$ with coefficients holomorphic in $z_{s+1}, \dots, z_n$. Then $b(\zeta \cdot w) = \prod_i \zeta_i^{c_i} b(w)$ for every $\zeta \in \mu$ if and only if only monomials $w_1^{e_1} \dots w_s^{e_s}$ with $e_i \equiv c_i \pmod{m_i}$ occur in b , if and only if $b = w_1^{a_1} \dots w_s^{a_s} h(z)$ with h holomorphic on W .

Since $\varpi_W \circ \zeta = \varpi_W$, the subspace $\varpi_W^* \ell$ is μ -stable, so $\varpi_W^* \ell \cap \Omega_W^p$ is an invertible μ -stable subsheaf of Ω_W^p by Lemma 3.1(1) and (3). The identifications of overlapping root charts preserve pulled back forms, so these subsheaves form the orbifold line bundle M_ℓ , and $M_\ell^q = (\varpi_W^* \ell)^{\odot q} \cap \text{Sym}^q \Omega_W^p$ on $\widetilde{W}$ by Lemma 3.1(2).

Let $\theta = \sum_{(I)} b_{(I)} dw_{I_1} \odot \dots \odot dw_{I_q}$ be a holomorphic section of $\text{Sym}^q \Omega_W^p$, where (I) runs over the index sequences of Definition 2.5. Since ζ multiplies the monomial indexed by (I) by $\prod_i \zeta_i^{t_i}$, θ is invariant if and only if $b_{(I)} = w_1^{a_1} \dots w_s^{a_s} h_{(I)}(z)$ for every (I) , where a_i is the residue of $-t_i$ and $h_{(I)}$ is holomorphic on W . Since $t_i(m_i - 1) - a_i = m_i \lfloor t_i(1 - \frac{1}{m_i}) \rfloor$, (3.8) gives

$$w_1^{a_1} \dots w_s^{a_s} dw_{I_1} \odot \dots \odot dw_{I_q} = \left(\prod_{i=1}^s m_i^{-t_i} \right) \varpi_W^* \left(\prod_{i=1}^s z_i^{-\lfloor t_i(1 - \frac{1}{m_i}) \rfloor} dz_{I_1} \odot \dots \odot dz_{I_q} \right).$$

Thus the invariant holomorphic sections of $\text{Sym}^q \Omega_W^p$ are the pullbacks of the sections of $C_{p,q}$ over W by (2.3). This proves the first identity in (3.1).

Let $\sigma = \sum_{(I)} \sigma_{(I)} dz_{I_1} \odot \dots \odot dz_{I_q}$ be a section of $C_{p,q}$ over W and $c_{(I)}$ the coefficients of $v^{\odot q}$. If $\sigma \in \mathcal{F}_{q,B}$, then $\varpi_W^* \sigma$ is an invariant section of M_ℓ^q . Conversely, an invariant section of M_ℓ^q on $\widetilde{W}$ is $\varpi_W^* \sigma$ for such a σ by the first identity, and equals $f(\varpi_W^* v)^{\odot q}$ with f meromorphic. Comparing the coefficients of $dw_{I_1} \odot \dots \odot dw_{I_q}$ for an (I) with $c_{(I)} \neq 0$ in (3.8) gives $f = \varpi_W^*(\sigma_{(I)}/c_{(I)})$, so $\sigma = (\sigma_{(I)}/c_{(I)})v^{\odot q} \in \mathcal{F}_{q,B}$. This proves the second identity in (3.1). Both identities respect symmetric products, so $\bigoplus_{q \geq 0} H^0(\mathcal{R}, M_\ell^q) = A_B(\mathcal{F})$, and $\kappa(\mathcal{R}, M_\ell)$ is the maximal dimension of the closures of the images of the rational maps defined by bases of $H^0(X, \mathcal{F}_{q,B})$, which is $\kappa(X|B, \mathcal{F})$ by [Cam11a, Lemma 3.1]. This proves (3.2).

Step 2. In this step, we prove (2).

By Lemma 3.1(1), $\mathcal{F}$ is invertible, and the pullbacks of its sections define an injective morphism $\varpi^*\mathcal{F} \rightarrow M_\ell$ of orbifold line bundles which is an isomorphism over $X \setminus D$. Let x be a general point of D_i , so that $s = 1$, $z_1 = w_1^{m_i}$, $R_i = (w_1 = 0)$, and $\mu = \mu_{m_i}$, and let $v_x = \sum_{|I|=p} f_I dz_I$ generate $\mathcal{F}$ near x , so that $\min_I \text{ord}_{D_i}(f_I) = 0$. By (3.8), the coefficient of dw_I in $\varpi_W^* v_x$ has order $m_i \text{ord}_{D_i}(f_I) + m_i - 1$ along R_i if $1 \in I$, and $m_i \text{ord}_{D_i}(f_I)$ otherwise. Let δ be the minimum of these orders, u a generator of M_ℓ at a point of R_i , and $\varpi_W^* v_x = gu$ with g holomorphic. By Lemma 3.1(1), the coefficients of u have no common prime factor, so $\text{ord}_{R_i}(g) = \delta$. If $\text{ord}_{D_i}(f_I) = 0$ for some $I \not\ni 1$, then $\delta = 0$, and otherwise some $I \ni 1$ has $\text{ord}_{D_i}(f_I) = 0$ and $\delta = m_i - 1$. Since $v_x|_{D_i} = \sum_{I \not\ni 1} f_I|_{D_i} dz_I|_{D_i}$, these cases are $\epsilon_i = 0$ and $\epsilon_i = 1$. Hence $\varpi^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{R}}(\sum_i \epsilon_i(m_i - 1)R_i) \rightarrow M_\ell$ is an isomorphism outside a closed subset of codimension at least two of every root chart, so its determinant is a unit. This proves (3.3).

Let $b_1, \dots, b_r$ be integers. The sections of $\mathcal{O}_{\mathcal{R}}(\sum_i b_i R_i)$ on $\widetilde{W}$ are the functions $f w_1^{-b_1} \dots w_s^{-b_s}$ with f holomorphic, and such a function is invariant if and only if $f(\zeta \cdot w) = \prod_i \zeta_i^{b_i} f(w)$ for every $\zeta \in \mu$, that is, $f = w_1^{a_1} \dots w_s^{a_s} h(z)$ with $a_i \equiv b_i \pmod{m_i}$ by **Step 1**. Then $f w_1^{-b_1} \dots w_s^{-b_s} = \varpi_W^*(z_1^{-\lfloor b_1/m_1 \rfloor} \dots z_s^{-\lfloor b_s/m_s \rfloor} h)$, hence

$$(3.9) \quad \varpi_* \mathcal{O}_{\mathcal{R}}\left(\sum_{i=1}^r b_i R_i\right) = \mathcal{O}_X\left(\sum_{i=1}^r \left\lfloor \frac{b_i}{m_i} \right\rfloor D_i\right).$$

For $x \in X$ and a generator v_x of $\mathcal{F}$ near x , $(\varpi_W^* v_x)^{\odot q}$ is an invariant frame of $\varpi^*\mathcal{F}^q$, so (3.1), (3.3), and (3.9) give

$$\mathcal{F}_{q,B} = \varpi_* M_\ell^q = \mathcal{F}^q \otimes \varpi_* \mathcal{O}_{\mathcal{R}}\left(\sum_{i=1}^r q \epsilon_i(m_i - 1) R_i\right) = \mathcal{F}^q \otimes \mathcal{O}_X\left(\sum_{i=1}^r \epsilon_i \left\lfloor q \left(1 - \frac{1}{m_i}\right) \right\rfloor D_i\right),$$

which is (3.4). Finally, $M(1 - \frac{1}{m_i})$ is an integer, so $\mathcal{O}_{\mathcal{R}}(M \epsilon_i(m_i - 1) R_i) = \varpi^* \mathcal{O}_X(\epsilon_i M(1 - \frac{1}{m_i}) D_i)$ by (2.2), and (3.5) follows from (3.3).

Step 3. In this step, we prove (3.6) and the first assertion of (3).

Let L be an orbifold line bundle on $\mathcal{R}$, χ the character of μ on the fiber of L over 0, and e a frame of L near 0. Replacing e by $\frac{1}{|\mu|} \sum_{\zeta \in \mu} \chi(\zeta)^{-1} \zeta \cdot e$, which takes the value $e(0)$ at 0, and shrinking W , we may assume that $\zeta \cdot e = \chi(\zeta)e$ for every $\zeta \in \mu$. Write $\chi(\zeta) = \prod_i \zeta_i^{c_i}$ and let a_i be the residue of c_i modulo m_i . A section $f e$ of L on $\widetilde{W}$ is invariant if and only if $f(\zeta \cdot w) = \chi(\zeta) f(w)$ for every ζ , that is, $f = w_1^{a_1} \dots w_s^{a_s} h(z)$ with h holomorphic on W by **Step 1**. Hence $\varpi_* L$ is invertible, generated over W by $w_1^{a_1} \dots w_s^{a_s} e$, and so is every $\varpi_* L^d$. Since the exponent of μ divides M , $\chi^M = 1$, so $\mathcal{A}_L := \varpi_* L^M$ is a line bundle on X generated over W by the invariant frame e^M , and $L^M = \varpi^* \mathcal{A}_L$. By **Step 1**, the sections of $L^{kM} = \varpi^* \mathcal{A}_L^k$ on $\widetilde{W}$ are the products of e^{kM} with the holomorphic functions on W , which identifies $H^0(X, \mathcal{A}_L^k)$ with $H^0(\mathcal{R}, L^{kM})$ compatibly with products.

Let $\mathcal{S} := \bigoplus_{q \geq 0} H^0(\mathcal{R}, L^q)$ and $\mathcal{S}^{(M)} := \bigoplus_{k \geq 0} H^0(\mathcal{R}, L^{kM}) = \bigoplus_{k \geq 0} H^0(X, \mathcal{A}_L^k)$. By the identity theorem, a nonzero section of L^q is nonzero on every root chart, so $\mathcal{S}$ is an integral domain, and $\mathcal{S}_q = H^0(X, \varpi_* L^q)$ is finite-dimensional since $\varpi_* L^q$ is invertible [Har77, Theorem II.5.19]. Every homogeneous $s \in \mathcal{S}$ satisfies $s^M \in \mathcal{S}^{(M)}$, so $\text{Frac}(\mathcal{S})$ is algebraic over $\text{Frac}(\mathcal{S}^{(M)})$, and $\mathcal{S}_q = 0$ for every $q > 0$ if and only if $\mathcal{S}_k^{(M)} = 0$ for every $k > 0$. Thus $\kappa(\mathcal{S}) = \kappa(\mathcal{S}^{(M)})$, and (3.6) follows from

$$(3.10) \quad \kappa(\mathcal{R}, L) = \kappa(X, \mathcal{A}_L) = \kappa(\mathcal{S}^{(M)}).$$

If $H^0(\mathcal{R}, L^q) = 0$ for every $q > 0$, then all terms of (3.10) are $-\infty$. Otherwise, for $H^0(\mathcal{R}, L^q) \neq 0$, the composition of the map ϕ_q of Definition 2.4 with the Veronese embedding ν of degree M is defined by the monomials of degree M in a basis of $H^0(\mathcal{R}, L^q)$, which span a nonzero subspace of $H^0(\mathcal{R}, L^{qM}) = H^0(X, \mathcal{A}_L^q)$, hence is the composition of ϕ_{qM} with a linear projection. Since ν is a closed embedding, the closure of $\phi_q(X)$ has dimension at most that of the closure of $\phi_{qM}(X)$,

and ϕ_{kM} is defined by a basis of $H^0(X, \mathcal{A}_L^k)$, so $\kappa(\mathcal{R}, L) = \kappa(X, \mathcal{A}_L)$. Finally, let $K_k \subset \mathbb{C}(X)$ be the field generated by the quotients of the nonzero elements of $H^0(X, \mathcal{A}_L^k)$, and $K := \bigcup_{k \geq 1} K_k$, a field since $K_k \subset K_{jk}$. For $t_0, \dots, t_c \in H^0(X, \mathcal{A}_L^k)$ with $t_0 \neq 0$ and $T_0, \dots, T_c$ the homogeneous coordinates of $\mathbb{P}^c$, the closure of the image of the rational map $X \dashrightarrow \mathbb{P}^c$ defined by $t_0, \dots, t_c$ meets $(T_0 \neq 0)$ in an affine variety with coordinate ring $\mathbb{C}[t_1/t_0, \dots, t_c/t_0] \subset \mathbb{C}(X)$, so this closure has dimension $\text{trdeg}_{\mathbb{C}} \mathbb{C}(t_1/t_0, \dots, t_c/t_0)$ [Har77, Proposition I.1.7 and Theorem I.1.8A], and $\kappa(X, \mathcal{A}_L) = \text{trdeg}_{\mathbb{C}} K$. For $0 \neq t \in H^0(X, \mathcal{A}_L^d)$ and an indeterminate u , sending $s \in H^0(X, \mathcal{A}_L^{jd})$ to $(s/t^j)u^j$ embeds $\bigoplus_{j \geq 0} H^0(X, \mathcal{A}_L^{jd})$, over which $\mathcal{S}^{(M)}$ is integral, into $K[u]$. The image contains u , and the quotients s/t^j generate $K = \bigcup_{j \geq 1} K_{jd}$, so the fraction field of the image is $K(u)$ and $\kappa(\mathcal{S}^{(M)}) = \text{trdeg}_{\mathbb{C}} K(u) - 1 = \text{trdeg}_{\mathbb{C}} K$.

Step 4. In this step, we prove the last assertion of (3).

Let $j : L \rightarrow \Omega_{\mathcal{R}}^p$ be a nonzero morphism of orbifold bundles, injective on every root chart by the identity theorem, so that invariant sections give an injective morphism $\varpi_* L \rightarrow \varpi_* \Omega_{\mathcal{R}}^p = \Omega_X^p$ by (3.1), with $\varpi_* L$ invertible by **Step 3**. Let $\ell' \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^p$ be the $\mathbb{C}(X)$ -subspace generated by its generic stalk. In the notation of **Step 3**, $j(w_1^{a_1} \cdots w_s^{a_s} e)$ is an invariant holomorphic p -form on $\widetilde{W}$, hence the pullback of a section of $\varpi_* L$, which lies in ℓ' . Thus $j(e) \in \varpi_W^* \ell'$ and $j(L) \subset M_{\ell'}$ on $\widetilde{W}$, so j factors through an injective morphism $L \rightarrow M_{\ell'}$. If j factors through $M_{\ell''}$, then $\varpi_* L \subset \varpi_* M_{\ell''} = \ell'' \cap \Omega_X^p$ by (3.1), so $\ell'' = \ell'$. The injective morphisms $L^q \rightarrow M_{\ell'}^q$ identify $H^0(\mathcal{R}, L^q)$ with subspaces of $H^0(\mathcal{R}, M_{\ell'}^q)$, so each ϕ_q of L is a linear projection of that of $M_{\ell'}$, and $\kappa(\mathcal{R}, L) \leq \kappa(\mathcal{R}, M_{\ell'}) = \kappa(X|B, \ell' \cap \Omega_X^p)$ by (3.2).

Let $\mathcal{F}' := \ell' \cap \Omega_X^p$ and $\mathcal{H} := \ell' \cap \Omega_X^p(\log D)$. By Lemma 3.1(1) and (2), $\mathcal{H}$ is invertible and $\text{Sym}^q \mathcal{H} = \ell'^{\odot q} \cap \text{Sym}^q \Omega_X^p(\log D)$, which is the saturation $\mathcal{H}^{[q]}$ of Theorem 2.8, since every coherent subsheaf of rank one of $\text{Sym}^q \Omega_X^p(\log D)$ containing the image of $\mathcal{H}^{\otimes q}$ lies in $\ell'^{\odot q}$. Since $C_{p,q} \subset \text{Sym}^q \Omega_X^p(\log D)$, we have $\mathcal{F}'_{q,B} = \ell'^{\odot q} \cap C_{p,q} \subset \mathcal{H}^{[q]}$, so $A_B(\mathcal{F}')$ is a graded subalgebra of $\bigoplus_{q \geq 0} H^0(X, \mathcal{H}^{[q]})$, and

$$\kappa(X|B, \mathcal{F}') \leq \kappa(X, \mathcal{H}) \leq p$$

by Theorem 2.8, as X is a connected compact Kähler manifold and D is a simple normal crossing divisor. This proves (3.7).

Step 5. In this step, we prove (4).

Let $1 \leq p' \leq n$ be an integer and $\ell' \subset \Omega_{\mathbb{C}(X)/\mathbb{C}}^{p'}$ a one-dimensional $\mathbb{C}(X)$ -subspace. By (3.7) applied to the inclusion $M_{\ell'} \rightarrow \Omega_{\mathcal{R}}^{p'}$, $\kappa(\mathcal{R}, M_{\ell'}) = \kappa(X|B, \ell' \cap \Omega_X^{p'}) \leq p'$. By Definition 2.5, every coherent subsheaf $\mathcal{G} \subset \Omega_X^{p'}$ of rank one generating ℓ' , such as $\ell' \cap \Omega_X^{p'}$, has $\kappa(X|B, \mathcal{G}) = \kappa(X|B, \ell' \cap \Omega_X^{p'})$. Thus, by Definition 2.6, (X, B) is not special if and only if $\kappa(\mathcal{R}, M_{\ell'}) = p'$ for some p' and ℓ' , and then the inclusion $M_{\ell'} \rightarrow \Omega_{\mathcal{R}}^{p'}$ shows that $\mathcal{R}$ is not special (Definition 2.7). Conversely, if $\mathcal{R}$ is not special, then there exist p' , an orbifold line bundle L , and a nonzero morphism $j : L \rightarrow \Omega_{\mathcal{R}}^{p'}$ with $\kappa(\mathcal{R}, L) = p'$, and (3.7) gives $\kappa(\mathcal{R}, M_{\ell'}) = p'$ for the ℓ' given by (3). This proves (4). $\square$

4. TRANSFER OF SPECIALNESS

In this section, we prove Proposition 4.3.

Lemma 4.1 ([Ope26a, Lemma 2.1]). *Let Z be a smooth projective variety, $1 \leq p \leq \dim Z$ an integer, L a line bundle on Z , and $i : L \rightarrow \Omega_Z^p$ a nonzero morphism. Let $\ell \subset \Omega_{\mathbb{C}(Z)/\mathbb{C}}^p$ be the one-dimensional $\mathbb{C}(Z)$ -subspace generated by $i(L)$. Let q be a positive integer, $c \geq 0$ an integer, and $s_0, \dots, s_c \in H^0(Z, L^q)$ sections such that $s_0 \neq 0$. Then*

$$(4.1) \quad d(s_a/s_0) \wedge \omega = 0 \quad \text{in } \Omega_{\mathbb{C}(Z)/\mathbb{C}}^{p+1}$$

for every $0 \leq a \leq c$ and every $\omega \in \ell$. In particular, $\kappa(Z, L) \leq p$. If $\kappa(Z, L) = p$, then there exist a positive integer q and $s_0, \dots, s_p \in H^0(Z, L^q)$ with $s_0 \neq 0$ such that the form

$$(4.2) \quad d(s_1/s_0) \wedge \cdots \wedge d(s_p/s_0) \in \Omega_{\mathbb{C}(Z)/\mathbb{C}}^p$$

is nonzero and generates ℓ .

Lemma 4.2 ([Ope26a, Lemma 2.4]). *Let $g : Y \rightarrow X$ be a dominant morphism of smooth projective varieties and $U_0 \subset X$ a nonempty open subset such that $g^{-1}(x)$ is connected for every $x \in U_0$. Let $u \in \mathbb{C}(Y)$ be a rational function. Assume that there exists a nonempty open subset $U_1 \subset U_0$ such that for every $x \in U_1$, the function u is constant on the set of points of $g^{-1}(x)$ at which it is defined. Then $u = g^*v$ for some $v \in \mathbb{C}(X)$.*

Proposition 4.3. *Let (X, B) be a projective orbifold pair as in Definition 2.1 and $g : Y \rightarrow X$ a surjective morphism from a smooth projective variety Y . Assume the following.*

- (1) *For each i , there exists an effective Weil divisor C_i on Y such that $g^*D_i = m_i C_i$.*
- (2) *There exists a nonempty open subset $U \subset X \setminus D$ such that g is smooth at every point of $g^{-1}(U)$, the fiber $g^{-1}(x)$ is connected for every $x \in U$, and the cotangent bundle $\Omega_{g^{-1}(x)}^1$ is trivial for every $x \in U$.*
- (3) *For any prime divisor P on X such that $P \neq D_i$ for every i , there exists a nonempty open subset $P^\circ \subset P$ such that g is smooth at some point of $g^{-1}(x)$ for every $x \in P^\circ$.*
- (4) *For each i , there exists a nonempty open subset $D_i^\circ \subset D_i \setminus \bigcup_{k \neq i} D_k$ such that for every $x \in D_i^\circ$, there exists $y \in g^{-1}(x)$ such that some local lift of g to the root chart at x is a submersion at y .*

If $\mathcal{R}$ is special, then Y is special.

Proposition 4.3 generalizes [Ope26a, Proposition 3.1] to arbitrary multiplicities and several boundary components.

Proof. Suppose that Y is not special. By Definition 2.7, there exist $1 \leq p \leq \dim Y$, a line bundle L on Y , and a nonzero morphism $i : L \rightarrow \Omega_Y^p$ with $\kappa(Y, L) = p$. We construct an orbifold line bundle $\mathcal{K}$ on $\mathcal{R}$ with a nonzero morphism $\mathcal{K} \rightarrow \Omega_{\mathcal{R}}^p$ and $\kappa(\mathcal{R}, \mathcal{K}) = p$. Throughout, W is an adapted coordinate neighborhood of a point $x_0 \in X$ and $\varpi_W : \widetilde{W} \rightarrow W$ the root chart over W , with the notation of Definition 2.3.

Step 1. In this step, we construct local lifts of g and pullbacks of orbifold bundles to Y .

Let $y \in Y$, $x_0 := g(y)$, and c_i a local equation at y of the Cartier divisor C_i [Har77, Proposition II.6.11 and Remark II.6.11.1A] for $1 \leq i \leq s$. By (1), $u_i := (z_i \circ g)/c_i^{m_i}$ is holomorphic and nowhere zero near y . Let $\widetilde{V} \subset g^{-1}(W)$ be a polydisc neighborhood of y on which every u_i has a holomorphic m_i th root u_i^{1/m_i} . By (2.1),

$$\tilde{g} := \left(u_1^{1/m_1} c_1, \dots, u_s^{1/m_s} c_s, z_{s+1} \circ g, \dots, z_n \circ g \right) : \widetilde{V} \rightarrow \widetilde{W}$$

is a local lift of g . Two local lifts of g on a connected open subset $\widetilde{V}' \subset g^{-1}(W)$ differ by a unique element of $\mu_{m_1} \times \cdots \times \mu_{m_s}$, as $(w_i \circ \tilde{g}')^{m_i} = z_i \circ g \neq 0$ on $\widetilde{V}'$ for every local lift $\tilde{g}'$ by the surjectivity of g . We write $f : Y \rightarrow \mathcal{R}$ for the collection of local lifts, so that $\varpi \circ f = g$.

For an orbifold line bundle M on $\mathcal{R}$, the pullbacks $\tilde{g}^*(M|_{\widetilde{W}})$ glue along these unique elements, which satisfy the cocycle condition, by the equivariant structure of M to a line bundle f^*M on Y . Likewise, a section ς of M and a morphism $M \rightarrow \Omega_{\mathcal{R}}^p$ pull back to a section $f^*\varsigma$ of f^*M and a morphism $f^*M \rightarrow \Omega_Y^p$, the latter given on $\widetilde{V}$ by $\tilde{g}^*(M|_{\widetilde{W}}) \rightarrow \tilde{g}^*\Omega_{\widetilde{W}}^p \rightarrow \Omega_Y^p$. Over $X \setminus D$, the local lifts are the restrictions of g , so $f^*M = g^*M$, $f^*\varsigma = g^*\varsigma$, and $f^*M \rightarrow \Omega_Y^p$ is $g^*M \rightarrow g^*\Omega_X^p \rightarrow \Omega_Y^p$.

Step 2. In this step, we descend ratios of sections of a power of L to X .

By Lemma 4.1, there exist a positive integer q and $s_0, \dots, s_p \in H^0(Y, L^q)$ with $s_0 \neq 0$ such that (4.2) is nonzero and generates ℓ as in Lemma 4.1. Let $u_a := s_a/s_0 \in \mathbb{C}(Y)$ for $0 \leq a \leq p$,

so that (4.2) is the form $du_1 \wedge \cdots \wedge du_p$. The functions $u_1, \dots, u_p$ are algebraically independent over $\mathbb{C}$. Otherwise, let $G \in \mathbb{C}[T_1, \dots, T_p]$ be a nonzero polynomial of minimal degree such that $G(u_1, \dots, u_p) = 0$. Then $\sum_{j=1}^p \frac{\partial G}{\partial T_j}(u_1, \dots, u_p) du_j = 0$ in $\Omega_{\mathbb{C}(Y)/\mathbb{C}}^1$ with some nonzero coefficient, by the minimality of $\deg G$ and since $\mathbb{C}$ has characteristic zero, so $du_1 \wedge \cdots \wedge du_p = 0$, a contradiction. By [Har77, Exercise II.3.19], there exists a nonempty open subset $U_1 \subset U$ such that $i \in H^0(Y, L^{-1} \otimes \Omega_Y^p)$ and s_0 do not vanish identically on $F := g^{-1}(x)$ for every $x \in U_1$. Fix $x \in U_1$. By (2), F is a connected compact complex manifold, hence irreducible, and

$$(4.3) \quad 0 \rightarrow T_{X,x}^* \otimes \mathcal{O}_F \rightarrow \Omega_Y^1|_F \rightarrow \Omega_F^1 \rightarrow 0$$

is an exact sequence of vector bundles on F . For $0 \leq k \leq p$, let $\mathcal{G}^k \subset \Omega_Y^p|_F$ be the image of $\bigwedge^k(T_{X,x}^* \otimes \mathcal{O}_F) \otimes \bigwedge^{p-k}(\Omega_Y^1|_F)$ under the wedge product via (4.3), and $\mathcal{G}^{p+1} := 0$. These are subbundles with

$$\mathcal{G}^k / \mathcal{G}^{k+1} \cong \bigwedge^k T_{X,x}^* \otimes \bigwedge^{p-k} \Omega_F^1 \quad (0 \leq k \leq p),$$

which are trivial by (2). Let k be the largest integer such that $i|_F$ factors through $\mathcal{G}^k$. Then some coordinate of the nonzero morphism $L|_F \rightarrow \mathcal{G}^k / \mathcal{G}^{k+1} \cong \mathcal{O}_F^{\oplus N}$ is a nonzero section η of $L^{-1}|_F$. Since F is irreducible, compact, and connected, the holomorphic functions $\eta^q s_a|_F$ are constant and $\eta^q s_0|_F \neq 0$, so $u_a|_F = (\eta^q s_a|_F) / (\eta^q s_0|_F)$ is defined and constant on F . By Lemma 4.2, there exist $v_a \in \mathbb{C}(X)$ with

$$(4.4) \quad u_a = g^* v_a \quad (0 \leq a \leq p).$$

Since $g^* : \mathbb{C}(X) \rightarrow \mathbb{C}(Y)$ is injective, $v_1, \dots, v_p$ are algebraically independent over $\mathbb{C}$. Let

$$(4.5) \quad \sigma := dv_1 \wedge \cdots \wedge dv_p \in \Omega_{\mathbb{C}(X)/\mathbb{C}}^p.$$

Then $g^* \sigma = du_1 \wedge \cdots \wedge du_p \neq 0$ generates ℓ , and $p \leq n = \dim_{\mathbb{C}(X)} \Omega_{\mathbb{C}(X)/\mathbb{C}}^1$ [Har77, Theorems I.4.8A and II.8.6A].

Step 3. In this step, we descend the coefficients of $i^{\otimes q}(s_a)$ to X .

Let $\theta_a := i^{\otimes q}(s_a) \in H^0(Y, (\Omega_Y^p)^{\otimes q})$ for $0 \leq a \leq p$. Since $g^* \sigma$ generates ℓ , there exist $h_a \in \mathbb{C}(Y)$ with

$$(4.6) \quad \theta_a = h_a (g^* \sigma)^{\otimes q} \quad (0 \leq a \leq p).$$

Since i is generically injective and $s_a = u_a s_0$, we have $h_0 \neq 0$ and $h_a = u_a h_0$. Let $U' \subset U$ be a nonempty open subset on which σ is a nowhere vanishing regular section of Ω_X^p . Since g is a submersion on $g^{-1}(U')$, $g^* \sigma$ vanishes nowhere there, so h_a is holomorphic on $g^{-1}(U')$ by (4.6), hence constant on the compact connected fiber $g^{-1}(x)$ for every $x \in U'$. By Lemma 4.2, there exist $b_a \in \mathbb{C}(X)$ with

$$(4.7) \quad h_a = g^* b_a \quad (0 \leq a \leq p).$$

Since g^* is injective, $b_0 \neq 0$ and $b_a = v_a b_0$ by (4.4).

Step 4. In this step, we construct $\mathcal{K}$ and sections $T_0, \dots, T_p$ of $\mathcal{K}^q$.

Let $\mathcal{E} := (\Omega_X^p)^{\otimes q}$ and $\Lambda_a := b_a \sigma^{\otimes q}$ for $0 \leq a \leq p$. Let $\tau := \varpi_W^* \sigma$, an invariant meromorphic p -form on $\widetilde{W}$, and ℓ_τ the space of its meromorphic multiples. By Lemma 3.1(1)–(3), $\mathcal{K}_{\widetilde{W}} := \ell_\tau \cap \Omega_{\widetilde{W}}^p$ is an invariant invertible subsheaf of $\Omega_{\widetilde{W}}^p$ with $\mathcal{K}_{\widetilde{W}}^q = \ell_\tau^{\otimes q} \cap (\Omega_{\widetilde{W}}^p)^{\otimes q}$. These subsheaves are compatible on overlapping root charts, as τ is defined by ϖ_W , and define an orbifold line bundle $\mathcal{K}$ on $\mathcal{R}$ with a nonzero morphism $\mathcal{K} \rightarrow \Omega_{\mathcal{R}}^p$. Let

$$(4.8) \quad T_a := \varpi_W^* \Lambda_a = (\varpi_W^* b_a) \tau^{\otimes q} \quad (0 \leq a \leq p),$$

an invariant meromorphic section of $(\Omega_{\widetilde{W}}^p)^{\otimes q}$ compatible on overlapping root charts. By (4.6) and (4.7), $\theta_a = g^* \Lambda_a$, and if T_a is holomorphic on every root chart, then $T_a \in H^0(\mathcal{R}, \mathcal{K}^q)$. Fix $0 \leq a \leq p$.

For a prime divisor P on X , let $\text{ord}_P \Lambda_a$ be the minimum order along P of the coordinates of Λ_a in a frame of $\mathcal{E}$ at the generic point of P . We claim that $\text{ord}_P \Lambda_a \geq 0$ if $P \neq D_1, \dots, D_r$. Otherwise, let $e := -\text{ord}_P \Lambda_a > 0$ and ζ a local equation of P at its generic point. Shrinking P° in (3), we may assume that ζ and $\Lambda' := \zeta^e \Lambda_a$ are regular on an open neighborhood X' of P° and Λ' vanishes nowhere on P° . By (3) and the implicit function theorem, g has a holomorphic section $t : W_x \rightarrow Y$ on an analytic open neighborhood W_x of some $x \in P^\circ$. Pulling back $(\zeta \circ g)^e \theta_a = g^* \Lambda'$, an equality of holomorphic sections on $g^{-1}(X')$, by t gives $\zeta^e t^* \theta_a = \Lambda'$ on $W_x \cap X'$, whose left side vanishes at x while $\Lambda'(x) \neq 0$, a contradiction.

Let $e_i := \max\{0, -\text{ord}_{D_i} \Lambda_a\}$ for $1 \leq i \leq r$ and $E := \sum_{i=1}^r e_i D_i$. By the claim, Λ_a is a regular section of $\mathcal{E} \otimes \mathcal{O}_X(E)$ [Har77, Proposition II.6.3A], so by (2.2)

$$\Theta := \prod_{j=1}^s w_j^{m_j e_j} T_a$$

is a holomorphic section of $(\Omega_{\widetilde{W}}^p)^{\otimes q}$, with coordinates Θ_J in the coordinate frame. Since the Taylor series of Θ_J at the origin converges on the polydisc $\widetilde{W}$, the order of Θ_J along $(w_i = 0)$ is constant along $(w_i = 0)$. Hence it suffices to show that for $1 \leq i \leq s$ with $e_i > 0$, every Θ_J is divisible by $w_i^{m_i e_i}$ on some open subset $\widetilde{W}^\circ \subset \widetilde{W}$ meeting $(w_i = 0)$.

Fix such an i and $x \in W \cap D_i^\circ$, nonempty as D_i° is dense in D_i . Let $W' \subset W \setminus \bigcup_{k \neq i} D_k$ be an adapted coordinate neighborhood of x with coordinates z_i and $z_j - z_j(x)$ for $j \neq i$, and $\varpi_{W'} : \widetilde{W}' \rightarrow W'$ the root chart over W' , with coordinate w'_i over z_i . Then $\rho := (w_i, (z_j - z_j(x))) \circ \varpi_W)_{j \neq i} : \varpi_W^{-1}(W') \rightarrow \widetilde{W}'$ satisfies $\varpi_{W'} \circ \rho = \varpi_W$. Since $w_j \neq 0$ on $\varpi_W^{-1}(W')$ for $j \leq s$, $j \neq i$, ρ is the quotient by the free action of $\prod_{j \leq s, j \neq i} \mu_{m_j}$, hence a covering map, so it is biholomorphic on every connected component of $\varpi_W^{-1}(W')$ as $\widetilde{W}'$ is simply connected. Since root charts at x are locally identified by biholomorphisms commuting with ϖ , (4) gives $y \in g^{-1}(x)$ and a local lift $\tilde{g}' : \tilde{V} \rightarrow \widetilde{W}'$ of g which is a submersion at y . Then $\tilde{g} := \rho^{-1} \circ \tilde{g}' : \tilde{V} \rightarrow \widetilde{W}$, for the inverse of ρ on one component, is a local lift of g which is a submersion at y .

Let $\tilde{x} := \tilde{g}(y)$, so that $w_i(\tilde{x})^{m_i} = z_i(x) = 0$. By the implicit function theorem, $\tilde{g}$ has a holomorphic section $\tilde{t} : \widetilde{W}^\circ \rightarrow \tilde{V}$ on an open neighborhood $\widetilde{W}^\circ \subset \widetilde{W}$ of $\tilde{x}$. Since $\varpi_W \circ \tilde{g} = g$, we have $\tilde{g}^* T_a = \tilde{g}^* \varpi_W^* \Lambda_a = g^* \Lambda_a = \theta_a$ and $\tilde{g}^* \Theta = \prod_{j=1}^s (w_j \circ \tilde{g})^{m_j e_j} \theta_a$ on $\tilde{V} \setminus g^{-1}(D)$, which is dense in $\tilde{V}$ by the surjectivity of g , hence on $\tilde{V}$ by continuity. Pulling back by $\tilde{t}$ gives $\Theta = \prod_{j=1}^s w_j^{m_j e_j} \tilde{t}^* \theta_a$, so every Θ_J is divisible by $w_i^{m_i e_i}$ on $\widetilde{W}^\circ$, which meets $(w_i = 0)$ at $\tilde{x}$. Thus T_a is holomorphic on $\widetilde{W}$.

Step 5. In this step, we compare $\kappa(\mathcal{R}, \mathcal{K})$ with $\kappa(Y, f^* \mathcal{K})$ and conclude.

By **Step 4**, $T_0, \dots, T_p \in H^0(\mathcal{R}, \mathcal{K}^q)$, and by (4.8) and (4.7), $T_0 \neq 0$ and $T_a/T_0 = \varpi^* v_a$. Since $1, v_1, \dots, v_p$ are linearly independent over $\mathbb{C}$, we extend $T_0, \dots, T_p$ to a basis of $H^0(\mathcal{R}, \mathcal{K}^q)$. The map $X \dashrightarrow \mathbb{P}^{N_q}$ defined by this basis (Definition 2.4) followed by the projection to the first $p+1$ coordinates is $x \rightarrow (1 : v_1(x) : \dots : v_p(x))$, whose image is dense in $\mathbb{P}^p$ because $v_1, \dots, v_p$ are algebraically independent, so $\kappa(\mathcal{R}, \mathcal{K}) \geq p$.

By **Step 1** and Theorem 2.9, $f^* \mathcal{K}$ is an algebraic line bundle on Y with an algebraic morphism $f^* \mathcal{K} \rightarrow \Omega_Y^p$. On $U' \subset X \setminus D$, $\mathcal{K} = \mathcal{O}_X \sigma$ as σ vanishes nowhere there, so $f^* \mathcal{K} \rightarrow \Omega_Y^p$ sends the generator $g^* \sigma$ of $f^* \mathcal{K}|_{g^{-1}(U')}$ to $g^* \sigma$, which vanishes nowhere by **Step 3**, so $f^* \mathcal{K} \rightarrow \Omega_Y^p$ is nonzero and $\kappa(Y, f^* \mathcal{K}) \leq p$ by Lemma 4.1.

Let k be a positive integer with $H^0(\mathcal{R}, \mathcal{K}^k) \neq 0$. Since a nonzero section ς of $\mathcal{K}^k$ is nonzero at some point of $X \setminus D$, where $f^* \varsigma = g^* \varsigma$ and g is surjective, $f^* : H^0(\mathcal{R}, \mathcal{K}^k) \rightarrow H^0(Y, (f^* \mathcal{K})^k)$ is injective. Let $\phi : X \dashrightarrow \mathbb{P}^N$ and $\phi' : Y \dashrightarrow \mathbb{P}^{N'}$ be defined by a basis $\varsigma_0, \dots, \varsigma_N$ of $H^0(\mathcal{R}, \mathcal{K}^k)$ and by a basis of $H^0(Y, (f^* \mathcal{K})^k)$ extending $f^* \varsigma_0, \dots, f^* \varsigma_N$. Since g is surjective, ϕ' followed by the projection to the first $N+1$ coordinates is $\phi \circ g$, which has the same image as ϕ , so

$$\kappa(Y, f^*K) \geq \kappa(\mathcal{R}, K) \text{ and}$$

$$p \leq \kappa(\mathcal{R}, K) \leq \kappa(Y, f^*K) \leq p.$$

Since $1 \leq p \leq n$ by **Step 2**, this contradicts the assumption that $\mathcal{R}$ is special. The proposition follows. $\square$

5. AUXILIARY COVERS AND AFFINE ACTIONS

The goal of this section is to construct the cover $\pi : S \rightarrow X$ and the quotient $Q = (S \times F)/A$ used in the proof of Proposition 1.2.

5.1. Quotients by finite groups.

Lemma 5.1. *Let W be a normal projective variety and A a finite group acting on W . Then there exist a normal projective variety W/A and a finite surjective morphism $p : W \rightarrow W/A$ satisfying the following.*

- (1) *The fibers of p are the A -orbits, $\mathcal{O}_{W/A} = (p_*\mathcal{O}_W)^A$, and every A -invariant morphism from W to a variety factors uniquely through p . For every A -invariant affine open subset $Z \subset W$, $p(Z)$ is an affine open subset of W/A with $\mathcal{O}_{W/A}(p(Z)) = \mathcal{O}_W(Z)^A$. In particular, $\mathbb{C}(W/A) = \mathbb{C}(W)^A$.*
- (2) *Let $w \in W$, $H \subset A$ the stabilizer of w , and $\mathcal{N}$ an H -invariant analytic neighborhood of w with $a\mathcal{N} \cap \mathcal{N} = \emptyset$ for every $a \in A \setminus H$. Then $p(\mathcal{N})$ is an analytic neighborhood of $p(w)$, the fibers of $p|_{\mathcal{N}}$ are the H -orbits in $\mathcal{N}$, and every H -invariant holomorphic function on $\mathcal{N}$ is of the form $\phi \circ p|_{\mathcal{N}}$ for a unique holomorphic function ϕ on $p(\mathcal{N})$. In particular, if H is trivial, then $p|_{\mathcal{N}} : \mathcal{N} \rightarrow p(\mathcal{N})$ is a biholomorphism.*

Proof. Let L_0 be an ample line bundle on W and $L := \bigotimes_{a \in A} a^*L_0$. Then L is ample, and A acts on $R := \bigoplus_{k \geq 0} H^0(W, L^k)$ by permuting the factors of L . Let $k > 0$ be such that L^k is very ample. For every finite subset $\Sigma \subset W$, there exists a section of L^k vanishing at no point of $A\Sigma$, and the product of its A -translates is an A -invariant section of $L^{k|A|}$ vanishing at no point of $A\Sigma$. Hence every A -orbit lies in an A -invariant affine open subset of W . By [SGA1, Exposé V, Proposition 1.1, Corollaires 1.2 and 1.5, and Proposition 1.8], there exists a finite surjective morphism $p : W \rightarrow W/A$ to a scheme such that the fibers of p are the A -orbits, $\mathcal{O}_{W/A} = (p_*\mathcal{O}_W)^A$, every A -invariant morphism from W factors uniquely through p , and $p(Z) = \text{Spec } \mathcal{O}_W(Z)^A$ is open for every A -invariant affine open subset $Z \subset W$. Since $\mathcal{O}_W(Z)$ is a normal domain, so is $\mathcal{O}_W(Z)^A$, and $\text{Frac}(\mathcal{O}_W(Z)^A) = \text{Frac}(\mathcal{O}_W(Z))^A$. Thus W/A is a normal variety with $\mathbb{C}(W/A) = \mathbb{C}(W)^A$. The graded $\mathbb{C}$ -algebra R is finitely generated with $W = \text{Proj } R$ (cf. [Har77, Exercise II.5.14]), and R^A is finitely generated since R is integral over R^A . The affine open subsets $D_+(f)$ for homogeneous $f \in R^A$ of positive degree are A -invariant and cover W , and $\mathcal{O}_W(D_+(f))^A = (R_{(f)}^A)^A = (R^A)_{(f)}$. Hence $W/A = \text{Proj } R^A$ is projective. This proves (1).

Let w , H , and $\mathcal{N}$ be as in (2). Since p is finite, it is proper, so p is a closed map in the analytic topology [SGA1, Exposé XII, Proposition 3.2(v)]. Hence $p(\mathcal{N}) = (W/A) \setminus p(W \setminus A\mathcal{N})$ is open, and $p^{-1}(p(\mathcal{N})) = A\mathcal{N} = \coprod_{a \in A/H} a\mathcal{N}$ by (1), so the fibers of $p|_{\mathcal{N}}$ are the H -orbits in $\mathcal{N}$. The same holds for every H -invariant open subset $\mathcal{N}' \subset \mathcal{N}$. By (1), [SGA1, Exposé XII, Théorème 4.2] applied to the proper morphism p and the coherent sheaf $\mathcal{O}_W$, and the exactness of analytification, the holomorphic functions on $p(\mathcal{N}')$ are the A -invariant holomorphic functions on $A\mathcal{N}'$, that is, the H -invariant holomorphic functions on $\mathcal{N}'$. This gives ϕ . If H is trivial, then $p|_{\mathcal{N}}$ is a bijective holomorphic map inducing an isomorphism of the sheaves of holomorphic functions, hence a biholomorphism. This proves (2). $\square$

Lemma 5.2. *Let F be a smooth variety, H a finite group acting on F , and $f \in F^H$ a point. Then there exist holomorphic coordinates centered at f on an H -invariant analytic neighborhood of f in which every $h \in H$ acts by the linear map $dh_f : T_f F \rightarrow T_f F$. In particular, F^H is smooth near f and $T_f(F^H) = (T_f F)^H$.*

Proof. Let $d := \dim F$, $\phi : \mathcal{N}_0 \rightarrow \mathbb{C}^d$ holomorphic coordinates centered at f on an analytic neighborhood $\mathcal{N}_0$ of f , and $\mathcal{N} := \bigcap_{h \in H} h\mathcal{N}_0$. We identify $T_f F$ with $\mathbb{C}^d$ by $d\phi_f$ and write $\rho(h) := dh_f$, so that $\rho(hh') = \rho(h)\rho(h')$ for $h, h' \in H$. Let

$$\Phi := \frac{1}{|H|} \sum_{h \in H} \rho(h)^{-1} \circ \phi \circ h : \mathcal{N} \rightarrow \mathbb{C}^d.$$

The substitution $h \rightarrow hh'$ gives $\Phi \circ h' = \rho(h') \circ \Phi$ for $h' \in H$, and $d\Phi_f = \frac{1}{|H|} \sum_{h \in H} \rho(h)^{-1} \circ d\phi_f \circ dh_f = d\phi_f$ is invertible. By the inverse function theorem, Φ restricts to holomorphic coordinates centered at f on an H -invariant analytic neighborhood of f , in which h' acts by $\rho(h')$ and F^H is the linear subspace $(\mathbb{C}^d)^H$. $\square$

5.2. Affine actions of arbitrary order.

Lemma 5.3. *Let $m, N \geq 2$ be integers, $\mathcal{E}$ an elliptic curve, and $t \in \mathcal{E}$ a point of order m . We index the coordinates of $\mathcal{E}^m$ by $k \in \mathbb{Z}/m\mathbb{Z}$ and define automorphisms θ, ϑ of the variety $\mathcal{E}^m$ by*

$$(5.1) \quad \theta(x)_k := x_{k-1}, \quad c := (t, \dots, t) \in \mathcal{E}^m, \quad \vartheta(x) := \theta^{-1}(x) + c.$$

Let

$$(5.2) \quad F_{m,N} := \prod_{1 \leq i < j \leq N} \mathcal{E}_{ij}^m, \quad A_{m,N} := (\mathbb{Z}/m\mathbb{Z})^N,$$

where each $\mathcal{E}_{ij}^m$ is a copy of $\mathcal{E}^m$, and let $e_1, \dots, e_N$ be the standard generators of $A_{m,N}$. For $1 \leq i < j \leq N$, let e_i act on $\mathcal{E}_{ij}^m$ by θ , e_j by ϑ , and e_k trivially for $k \neq i, j$. Then the following hold.

- (1) This defines a faithful action of $A_{m,N}$ on the abelian variety $F_{m,N}$ of dimension $m \binom{N}{2}$ by automorphisms of varieties, and for all integers u, v ,

$$(5.3) \quad \theta^u \vartheta^v(x) = \theta^{u-v}(x) + vc.$$

- (2) For each i , the fixed locus $F_{m,N}^{e_i}$ is a translate of an abelian subvariety isomorphic to $\mathcal{E}^d$, where

$$(5.4) \quad d := (N-1) + m \binom{N-1}{2},$$

so it is nonempty, smooth, connected, and of codimension $(m-1)(N-1)$. At every point of $F_{m,N}^{e_i}$, the eigenvalue 1 of the differential of e_i has multiplicity d and every other m th root of unity has multiplicity $N-1$. For $i \neq j$, $F_{m,N}^{e_i} \cap F_{m,N}^{e_j} = \emptyset$.

- (3) Let $\tau := e_1 + \dots + e_N$ and $I \subset \{1, \dots, N\}$ with $|I| \leq N-2$. Every $h \in H_I := \langle \tau, e_i \mid i \in I \rangle$ can be written uniquely as

$$(5.5) \quad h = a\tau + \sum_{i \in I} b_i e_i, \quad a, b_i \in \mathbb{Z}/m\mathbb{Z}.$$

If $a \neq 0$, then h has no fixed point on $F_{m,N}$. Consequently, the stabilizer in H_I of every point of $F_{m,N}$ is contained in $\langle e_i \mid i \in I \rangle$, the kernel of the character $\chi_I : H_I \rightarrow \mathbb{C}^*$, $h \rightarrow \exp(2\pi\sqrt{-1}a/m)$.

- (4) Let $r \geq 1$ and $m_1, \dots, m_r \geq 2$ be integers. For each α , let $\mathcal{E}_\alpha$ be an elliptic curve, $t_\alpha \in \mathcal{E}_\alpha$ a point of order m_α , and $F_{m_\alpha,N}$ the abelian variety (5.2) with the action of $A_{m_\alpha,N}$ defined from $\mathcal{E}_\alpha$ and t_α . Let $F := \prod_{\alpha=1}^r F_{m_\alpha,N}$ with the product action of $A := \prod_{\alpha=1}^r A_{m_\alpha,N}$, and $e_{\alpha,1}, \dots, e_{\alpha,N}$ the standard generators of $A_{m_\alpha,N}$. If $a \in A$ has two equal nonzero coordinates in one factor $A_{m_\alpha,N}$, then a has no fixed point on F . For every α and j , the fixed locus $F^{e_{\alpha,j}} = F_{m_\alpha,N}^{e_{\alpha,j}} \times \prod_{\beta \neq \alpha} F_{m_\beta,N}$ is nonempty, smooth, and connected.

Lemma 5.3 generalizes [Ope26a, Lemma A.2] to arbitrary orders.

Proof. Since θ is a group automorphism with $\theta(c) = c$, induction on $v \geq 0$ gives $\vartheta^v(x) = \theta^{-v}(x) + vc$, the identity $\vartheta^{-1}(x) = \theta(x) - c$ gives the same for $v < 0$, and applying θ^u yields (5.3). Thus $\theta\vartheta = \vartheta\theta$ and $\vartheta^m(x) = x + mc = x$, so the assignments in the statement define an action of $A_{m,N}$ on $F_{m,N}$. Suppose that $u = \sum_k u_k e_k$ acts trivially, and let $i < j$. By (5.3), u acts on $\mathcal{E}_{ij}^m$ by $x \rightarrow \theta^{u_i - u_j}(x) + u_j c$. Evaluating at $x = 0$ gives $u_j t = 0$, so $u_j = 0$, and then $\theta^{u_i} = \text{id}$ gives $u_i = 0$. This proves (1).

The fixed locus of θ on $\mathcal{E}^m$ is the diagonal $\Delta_{\mathcal{E}} := \{(x, \dots, x) \mid x \in \mathcal{E}\}$, and the fixed locus of ϑ is $\{(x, x - t, \dots, x - (m-1)t) \mid x \in \mathcal{E}\} = \Delta_{\mathcal{E}} - (0, t, \dots, (m-1)t)$, as $mt = 0$. Hence $F_{m,N}^{e_i}$ is the product of translates of $\Delta_{\mathcal{E}}$ on the $N-1$ factors $\mathcal{E}_{ij}^m$ with $j \neq i$ and of the remaining $\binom{N-1}{2}$ factors, a translate of an abelian subvariety isomorphic to $\mathcal{E}^{N-1} \times (\mathcal{E}^m)^{\binom{N-1}{2}} \cong \mathcal{E}^d$. This gives (5.4) and the codimension $m\binom{N}{2} - d = (m-1)(N-1)$. We identify the tangent space of $\mathcal{E}^m$ at every point with $\mathbb{C}^m$ by translation. The differential of e_i at a point of $F_{m,N}^{e_i}$ is $d\theta$ or $d\theta^{-1}$ on the $N-1$ factors $\mathcal{E}_{ij}^m$ and the identity on the other factors, where $d\theta$ is the cyclic permutation of the coordinates of $\mathbb{C}^m$. For $\lambda \in \mu_m$, the vector $(\lambda^{-k})_{k \in \mathbb{Z}/m\mathbb{Z}}$ is an eigenvector of $d\theta$ with eigenvalue λ , and these m vectors form a basis of $\mathbb{C}^m$ by the Vandermonde determinant, which gives the multiplicities. If $i < j$ and $x \in F_{m,N}^{e_i} \cap F_{m,N}^{e_j}$, then $e_i + e_j$ fixes x , but by (5.3) it acts on $\mathcal{E}_{ij}^m$ by the translation by $c \neq 0$. This proves (2).

Fix $j \notin I$. The coordinate of $h = a\tau + \sum_{i \in I} b_i e_i$ at j is a , which determines a and then the b_i . If $a \neq 0$, then since $|I| \leq N-2$ there exist $j < k$ in $\{1, \dots, N\} \setminus I$, and the coordinates of h at j and k are both a . By (5.3), h acts on $\mathcal{E}_{jk}^m$ by the translation by ac , which has no fixed point as $at \neq 0$. Hence the stabilizers in H_I are contained in $\{a = 0\} = \langle e_i \mid i \in I \rangle = \ker \chi_I$. This proves (3).

Let $a = (a_\alpha)_\alpha \in A$ have coordinates $u \neq 0$ at $j < k$ in $A_{m_\alpha, N}$. By (5.3) for $\mathcal{E}_\alpha$ and t_α , a acts on the factor $\mathcal{E}_{jk}^{m_\alpha}$ of $F_{m_\alpha, N}$ by the translation by uc , which has no fixed point. The assertion on $F^{e_{\alpha,j}}$ follows from (2). This proves (4). $\square$

5.3. Cyclic covers.

Proposition 5.4. *Let (X, B) be a projective orbifold pair as in Definition 2.1 with $n \geq 1$, and $N := n + 1$. Then the following hold.*

(1) *There exist line bundles $\mathcal{H}_1, \dots, \mathcal{H}_r$ on X and nonempty smooth divisors*

$$(5.6) \quad G_{i,j} \in |\mathcal{H}_i^{m_i}(-D_i)|, \quad 1 \leq i \leq r, \quad 1 \leq j \leq N,$$

such that $D + G$ is a reduced simple normal crossing divisor, where $G := \sum_{i,j} G_{i,j}$.

Fix $\mathcal{H}_i$ and $G_{i,j}$ as in (1), and let s_{D_i} and $s_{G_{i,j}}$ be sections of $\mathcal{O}_X(D_i)$ and $\mathcal{H}_i^{m_i}(-D_i)$ with divisors D_i and $G_{i,j}$. Let

$$(5.7) \quad S_{i,j} := \text{Spec}_X \bigoplus_{a=0}^{m_i-1} \mathcal{H}_i^{-a} \rightarrow X, \quad y_{i,j}^{m_i} = s_{D_i} s_{G_{i,j}},$$

be the cyclic cover of degree m_i branched along $D_i + G_{i,j}$, with algebra structure induced by $s_{D_i} s_{G_{i,j}} \in H^0(X, \mathcal{H}_i^{m_i})$, and $y_{i,j}$ the section of the pullback of $\mathcal{H}_i$ to $S_{i,j}$ given by $\mathcal{H}_i^{-1} \hookrightarrow \bigoplus_{a=0}^{m_i-1} \mathcal{H}_i^{-a}$. Let T be the fiber product over X of the $S_{i,j}$, $\pi : S \rightarrow X$ the normalization of T , and

$$(5.8) \quad A := \prod_{i=1}^r (\mathbb{Z}/m_i \mathbb{Z})^N$$

with standard generators $e_{i,j}$. Fix a primitive m_i th root of unity ζ_i for each i , and let $e_{i,j}$ act on T over X by $y_{i,j} \rightarrow \zeta_i y_{i,j}$ and $y_{i',j'} \rightarrow y_{i',j'}$ for $(i', j') \neq (i, j)$. For $x \in X$, let

$$(5.9) \quad K(x) := \{i \mid x \in D_i\}, \quad J_i(x) := \{j \mid x \in G_{i,j}\}, \quad P(x) := \{(i, j) \mid j \in J_i(x)\},$$

and $\tau_i := \sum_{j=1}^N e_{i,j} \in A$.

- (2) S is a smooth projective variety, π is finite and surjective, the action of A on T extends to a faithful action on S , π identifies X with the quotient S/A of Lemma 5.1, and the fibers of π are the A -orbits.
- (3) Let $x \in X$ with $K(x) = \{1, \dots, s\}$, and W an adapted coordinate neighborhood of x with coordinates $z_1, \dots, z_n$ such that $G_{i,j} \cap W = (z_{i,j} = 0)$ for $(i, j) \in P(x)$, where the $z_{i,j}$ are distinct coordinates among $z_{s+1}, \dots, z_n$, and $G_{i,j} \cap W = \emptyset$ for $(i, j) \notin P(x)$. Then, possibly shrinking W , $\pi^{-1}(W)$ is the disjoint union of

$$\prod_{i \in K(x)} m_i^{N-|J_i(x)|-1} \prod_{i \notin K(x)} m_i^{N-|J_i(x)|}$$

open subsets, the charts of S over W , each biholomorphic over W to the polydisc $\widehat{W}$ with coordinates u_i ($i \in K(x)$), $v_{i,j}$ ($(i, j) \in P(x)$), and z_l for the remaining indices l , such that

$$(5.10) \quad z_i \circ \pi = u_i^{m_i} \quad (i \in K(x)), \quad z_{i,j} \circ \pi = v_{i,j}^{m_i} \quad ((i, j) \in P(x)), \quad z_l \circ \pi = z_l \quad (\text{otherwise}).$$

If $x \notin D \cup G$, then $\pi^{-1}(W)$ is the disjoint union of $|A|$ charts, each mapped biholomorphically onto W by π , and A permutes them simply transitively. Moreover, π is étale over $X \setminus (D \cup G)$.

- (4) Let $x \in X$ and $I_x \subset A$ the stabilizer of a point of $\pi^{-1}(x)$. Then I_x does not depend on the point, and

$$(5.11) \quad I_x = \bigoplus_{i \in K(x)} \langle \tau_i \rangle \oplus \bigoplus_{(i,j) \in P(x)} \langle e_{i,j} \rangle.$$

Every $h \in I_x$ can be written uniquely as $h = \sum_{i \in K(x)} a_i \tau_i + \sum_{(i,j) \in P(x)} b_{i,j} e_{i,j}$ with $a_i, b_{i,j} \in \mathbb{Z}/m_i \mathbb{Z}$, and I_x preserves every chart of S over W , on which h acts by

$$(5.12) \quad u_i \rightarrow \zeta_i^{a_i} u_i, \quad v_{i,j} \rightarrow \zeta_i^{b_{i,j}} v_{i,j},$$

and trivially on the remaining coordinates. Moreover, $|J_i(x)| \leq N - 2$ for every $i \in K(x)$.

- (5) For every i and j , $\pi^* D_i = m_i (\pi^{-1}(D_i))_{\text{red}}$ and $\pi^* G_{i,j} = m_i S^{e_{i,j}}$, where $S^{e_{i,j}} = (\pi^{-1}(G_{i,j}))_{\text{red}}$ is a smooth divisor each of whose irreducible components dominates an irreducible component of $G_{i,j}$. If $x \in G_{i,j}$ lies on no other irreducible component of $D + G$, then $I_x = \langle e_{i,j} \rangle$. For every irreducible component T' of $S^{e_{i,j}}$ with maximal ideal $\mathfrak{m}_{T'}$ of $\mathcal{O}_{S,T'}$, we have $\text{ord}_{T'}(\pi^* G_{i,j}) = m_i$, the subgroup of A fixing T' pointwise is $\langle e_{i,j} \rangle$, and $e_{i,j}$ acts on $\mathfrak{m}_{T'}/\mathfrak{m}_{T'}^2$ by multiplication by ζ_i .

Proposition 5.4 generalizes [Ope26a, Lemma A.1] to arbitrary multiplicities and several boundary components.

Proof. If $r = 0$, then A is trivial, $T = S = X$, and all assertions hold. Assume that $r \geq 1$. Let $\nu : S \rightarrow T$ be the normalization and $X^\circ := X \setminus (D \cup G)$.

Step 1. In this step, we prove (1).

Let M be an ample line bundle on X . By [Har77, Theorem II.5.17 and Exercise II.7.5(d),(e)], there exists an integer $k > 0$ such that $M_i := M^{km_i}(-D_i)$ is very ample for every i . Let $\mathcal{H}_i := M^k$, and choose the $G_{i,j}$ successively. Suppose that the $G_{i',j'}$ already chosen are such that $D' := D + \sum G_{i',j'}$ is a reduced simple normal crossing divisor. Let Z be X or an irreducible component of an intersection of irreducible components of D' , which is smooth. By Bertini's theorem [Sta26, Tag 0FD6], a general member $G_{i,j}$ of $|M_i|$ meets every such Z in a smooth subscheme and contains none of them, as there are finitely many Z and the members containing Z form a proper linear subspace. Then $G_{i,j}$ is smooth, and nonempty as $n \geq 1$. Let $x \in G_{i,j}$ and $z_1, \dots, z_a$ local equations at x of the irreducible components of D' through x , which are part of a system of local coordinates. Let Z be the irreducible component of their intersection through x and ℓ a local equation of $G_{i,j}$. Since $G_{i,j} \cap Z$ is a smooth divisor on Z near x , the differential

of $\ell|_Z$ at x is nonzero, so $z_1, \dots, z_a, \ell$ are part of a system of local coordinates, and $D' + G_{i,j}$ is a reduced simple normal crossing divisor. This proves (1).

Step 2. In this step, we prove that T is integral and that S is a normal projective variety with a faithful action of A such that π identifies X with S/A .

The fiber product T is $\text{Spec}_X \mathcal{A}$, where $\mathcal{A} := \bigotimes_{i,j} \bigoplus_{a=0}^{m_i-1} \mathcal{H}_i^{-a}$ is a locally free $\mathcal{O}_X$ -algebra of rank $|A|$ on which A acts by $\mathcal{O}_X$ -algebra automorphisms. Let σ_i be a nonzero rational section of $\mathcal{H}_i$, $f_{i,j} := s_{D_i} s_{G_{i,j}} / \sigma_i^{m_i} \in \mathbb{C}(X)$, and $\mathcal{K}$ the field obtained from $\mathbb{C}(X)$ by adjoining roots $y_{i,j}$ of $t^{m_i} - f_{i,j}$ in an algebraic closure. We claim that $[\mathcal{K} : \mathbb{C}(X)] = |A|$. Order the pairs (i, j) , fix one, let $\mathcal{K}'$ be the field generated over $\mathbb{C}(X)$ by the $y_{i',j'}$ with (i', j') preceding (i, j) , and C an irreducible component of $G_{i,j}$. Replacing $f_{i',j'}$ by $f_{i',j'} \gamma^{m_{i'}}$ and $y_{i',j'}$ by $y_{i',j'} \gamma$ for $\gamma \in \mathbb{C}(X)^*$ changes neither $\mathcal{K}'$ nor $\mathcal{K}$, so we may assume that $\text{ord}_C(f_{i,j}) = 1$ and $\text{ord}_C(f_{i',j'}) = 0$ for $(i', j') \neq (i, j)$, as C has multiplicity one in $D_i + G_{i,j}$ and zero in $D_{i'} + G_{i',j'}$. Let w be a valuation of $\mathcal{K}'$ extending ord_C . Since $\mathcal{K}'$ is obtained by successively adjoining roots of polynomials $t^m - f$ with $w(f) = 0$, which remain separable over the residue field of w , the valuation w is unramified over ord_C . Hence $t^{m_i} - f_{i,j}$ is an Eisenstein polynomial over the valuation ring of w , so $[\mathcal{K}'(y_{i,j}) : \mathcal{K}'] = m_i$, and the claim follows by induction.

The generic fiber $\mathcal{A} \otimes_{\mathcal{O}_X} \mathbb{C}(X)$ is generated by the $y_{i,j}/\sigma_i$ subject to $(y_{i,j}/\sigma_i)^{m_i} = f_{i,j}$, has dimension $|A|$, and surjects onto $\mathcal{K}$, hence equals $\mathcal{K}$. Since $\mathcal{A}$ is locally free, T is integral with $\mathbb{C}(T) = \mathcal{K}$. The normalization ν is finite [Sta26, Tag 035S], so S is a normal projective variety and π is finite and surjective, and the action of A on T lifts to S [Har77, Exercise II.3.8]. Since A acts faithfully on $\mathcal{K} = \mathbb{C}(S)$ over $\mathbb{C}(X)$, as $y_{i,j} \neq 0$, and $|A| = [\mathcal{K} : \mathbb{C}(X)]$, the extension is Galois with group A , $\mathbb{C}(S)^A = \mathbb{C}(X)$, and A acts faithfully on S . By Lemma 5.1(1), π induces a morphism $\bar{\pi} : S/A \rightarrow X$. Let $W_0 \subset X$ be an affine open subset. Then $\pi^{-1}(W_0)$ is an A -invariant affine open subset whose coordinate ring R is integral over $\mathcal{O}_X(W_0)$, and $\bar{\pi}^{-1}(W_0) = p(\pi^{-1}(W_0))$ is affine with coordinate ring $R^A \subset \mathbb{C}(X)$ by Lemma 5.1(1). Since X is normal, $R^A = \mathcal{O}_X(W_0)$. Hence $\bar{\pi}$ is an isomorphism, and the fibers of π are the A -orbits.

Step 3. In this step, we prove (3) and that S is smooth.

Let x and W be as in (3). Since $D + G$ is a simple normal crossing divisor, $|K(x)| + |P(x)| \leq n$, so $|J_i(x)| \leq |P(x)| \leq N - 2$ for $i \in K(x)$, and we fix $j_0(i) \notin J_i(x)$ for every $i \in K(x)$. Shrinking W , we may assume that W is contained in an affine open subset $W_0 \subset X$ on which every $\mathcal{H}_i$ is trivial and $G_{i,j} \cap W_0 = \emptyset$ for $(i, j) \notin P(x)$. Trivializing the $\mathcal{H}_i$ on W_0 , $T|_{W_0}$ is the spectrum of $\mathcal{O}_X(W_0)[\xi_{i,j}]/(\xi_{i,j}^{m_i} - f_{i,j})$, where $f_{i,j} \in \mathcal{O}_X(W_0)$ is a local equation of $D_i + G_{i,j}$ and $\xi_{i,j}$ corresponds to $y_{i,j}$, and $c = (c_{i,j}) \in A$ acts by $\xi_{i,j} \rightarrow \zeta_i^{c_{i,j}} \xi_{i,j}$. On W we have $f_{i,j} = \eta_{i,j} z_i^{\delta_i} z_{i,j}^{\delta_{i,j}}$, where $\delta_i = 1$ if $i \in K(x)$ and $\delta_i = 0$ otherwise, $\delta_{i,j} = 1$ if $(i, j) \in P(x)$ and $\delta_{i,j} = 0$ otherwise, and $\eta_{i,j}$ is a nowhere vanishing holomorphic function on W , with a holomorphic m_i th root $\eta_{i,j}^{1/m_i}$ on the polydisc W . The rational functions $\xi_{i,j_0(i)}$ and $\xi_{i,j}/\xi_{i,j_0(i)}$ for $i \in K(x)$, and $\xi_{i,j}$ for $i \notin K(x)$, are integral over $\mathcal{O}_X(W_0)$, as their m_i th powers $f_{i,j_0(i)}$, $f_{i,j}/f_{i,j_0(i)}$, and $f_{i,j}$ are regular on W_0 , so they are regular on the normal variety $\pi^{-1}(W_0)$. Define holomorphic functions on $\pi^{-1}(W)$ by

$$u_i := \eta_{i,j_0(i)}^{-1/m_i} \xi_{i,j_0(i)}, \quad v_{i,j} := \frac{\eta_{i,j_0(i)}^{1/m_i}}{\eta_{i,j}^{1/m_i}} \cdot \frac{\xi_{i,j}}{\xi_{i,j_0(i)}} \quad (i \in K(x)),$$

$$v_{i,j} := \eta_{i,j}^{-1/m_i} \xi_{i,j} \quad (i \notin K(x))$$

for $(i, j) \in P(x)$, and define $\epsilon_{i,j}$ by the formula for $v_{i,j}$ for $(i, j) \notin P(x)$, where $j \neq j_0(i)$ when $i \in K(x)$. Then $u_i^{m_i} = z_i \circ \pi$, $v_{i,j}^{m_i} = z_{i,j} \circ \pi$, and $\epsilon_{i,j}^{m_i} = 1$. Thus every $\epsilon_{i,j}$ is locally constant with values in μ_{m_i} , and $\pi^{-1}(W)$ is the disjoint union of the open and closed subsets S_ϵ on which $(\epsilon_{i,j})_{i,j}$ takes a given value ϵ . The number of possible values of ϵ is the number in (3). Let $\Psi_\epsilon := (u_i, v_{i,j}, z_l \circ \pi) : S_\epsilon \rightarrow \widehat{W}$. It remains to show that each S_ϵ is nonempty and each Ψ_ϵ is a biholomorphism, since then (5.10) holds by construction.

Let $\phi_\epsilon : \widehat{W} \rightarrow T|_W$ be the holomorphic map over W defined by $\xi_{i,j_0(i)} = \eta_{i,j_0(i)}^{1/m_i} u_i$, by $\xi_{i,j} = \eta_{i,j}^{1/m_i} u_i v_{i,j}$ or $\eta_{i,j}^{1/m_i} \epsilon_{i,j} u_i$ for $i \in K(x)$ and $j \neq j_0(i)$ according to whether $(i,j) \in P(x)$ or not, and by $\xi_{i,j} = \eta_{i,j}^{1/m_i} v_{i,j}$ or $\eta_{i,j}^{1/m_i} \epsilon_{i,j}$ for $i \notin K(x)$ according to whether $(i,j) \in P(x)$ or not, so that $\phi_\epsilon \circ \Psi_\epsilon = \nu|_{S_\epsilon}$. Let $W^\circ := W \cap X^\circ$ and $\widehat{W}^\circ$ its preimage in $\widehat{W}$. Every $f_{i,j}$ is a unit on X° , so T is étale over X° , ν is an isomorphism over $T|_{X^\circ}$, π is étale over X° , and $T|_{W^\circ} \rightarrow W^\circ$ is a local biholomorphism. Since ϕ_ϵ commutes with the local biholomorphisms $\widehat{W}^\circ \rightarrow W^\circ$ and $T|_{W^\circ} \rightarrow W^\circ$, it is a local biholomorphism on $\widehat{W}^\circ$, and $\coprod_\epsilon \widehat{W}^\circ \rightarrow T|_{W^\circ}$ is bijective, as a point of $T|_{W^\circ}$ determines u , v , and ϵ . Hence Ψ_ϵ restricts to a biholomorphism $S_\epsilon \cap \pi^{-1}(W^\circ) \rightarrow \widehat{W}^\circ$ with inverse $\nu^{-1} \circ \phi_\epsilon$, and $S_\epsilon \neq \emptyset$.

Since ν is finite and $\phi_\epsilon \circ \Psi_\epsilon = \nu|_{S_\epsilon}$, the map Ψ_ϵ is proper with finite fibers, and surjective because its image is closed and contains $\widehat{W}^\circ$. Let $\Omega \subset \widehat{W}$ be a connected open subset and ψ a holomorphic function on $\Psi_\epsilon^{-1}(\Omega)$. Then $\psi \circ \Psi_\epsilon^{-1}$ is holomorphic on $\Omega \cap \widehat{W}^\circ$, the complement in Ω of the zero locus of $\prod_i u_i \prod_{i,j} v_{i,j}$, and locally bounded near $\Omega \setminus \widehat{W}^\circ$ as Ψ_ϵ is proper, so it extends to a holomorphic function $\tilde{\psi}$ on Ω [Dem12, Chapter II, Remark 4.2]. The continuous functions ψ and $\tilde{\psi} \circ \Psi_\epsilon$ agree on $\Psi_\epsilon^{-1}(\Omega \cap \widehat{W}^\circ)$, which is dense in $\Psi_\epsilon^{-1}(\Omega)$ as its complement lies in the proper closed subvariety $\pi^{-1}(D \cup G)$ of S , so $\psi = \tilde{\psi} \circ \Psi_\epsilon$. Suppose that a point $\hat{w} \in \widehat{W}$ has two preimages under Ψ_ϵ . For a small polydisc Ω around $\hat{w}$, the set $\Psi_\epsilon^{-1}(\Omega)$ is the disjoint union of neighborhoods of the finitely many points of $\Psi_\epsilon^{-1}(\hat{w})$. The function equal to 1 on one of them and to 0 on the others is holomorphic but not of the form $\tilde{\psi} \circ \Psi_\epsilon$, a contradiction. Thus Ψ_ϵ is a homeomorphism, and every holomorphic function on $\Psi_\epsilon^{-1}(\Omega)$ for a polydisc Ω is the pullback of a holomorphic function on Ω , so Ψ_ϵ is a biholomorphism. Consequently S is a complex manifold near $\pi^{-1}(x)$, hence smooth [Har77, Appendix B, Section 1]. If $x \in X^\circ$, then $K(x) = P(x) = \emptyset$ and $\widehat{W} = W$, so the $|A|$ charts S_ϵ map biholomorphically onto W . Since the fibers of π are the A -orbits by **Step 2** and each chart contains one point of every fiber, A permutes the charts simply transitively. This proves (3).

Step 4. In this step, we prove (4) and (5).

Let x , W , S_ϵ , u_i , $v_{i,j}$, and $\epsilon_{i,j}$ be as in **Step 3**. An element $c = (c_{i,j}) \in A$ acts on $\pi^{-1}(W)$ by $u_i \rightarrow \zeta_i^{c_{i,j_0(i)}} u_i$, $v_{i,j} \rightarrow \zeta_i^{c_{i,j} - c_{i,j_0(i)}} v_{i,j}$, and $\epsilon_{i,j} \rightarrow \zeta_i^{c_{i,j} - c_{i,j_0(i)}} \epsilon_{i,j}$ for $i \in K(x)$, by $v_{i,j} \rightarrow \zeta_i^{c_{i,j}} v_{i,j}$ and $\epsilon_{i,j} \rightarrow \zeta_i^{c_{i,j}} \epsilon_{i,j}$ for $i \notin K(x)$, and it fixes the $z_l \circ \pi$. Each S_ϵ contains exactly one point of $\pi^{-1}(x)$, the point $u = v = z = 0$. Hence c fixes a point of $\pi^{-1}(x)$ if and only if c preserves its S_ϵ , if and only if $c_{i,j} = c_{i,j_0(i)}$ for $i \in K(x)$ and $j \notin J_i(x)$ and $c_{i,j} = 0$ for $i \notin K(x)$ and $j \notin J_i(x)$. Such c equals $\sum_{i \in K(x)} a_i \tau_i + \sum_{(i,j) \in P(x)} b_{i,j} e_{i,j}$ with $a_i := c_{i,j_0(i)}$, $b_{i,j} := c_{i,j} - a_i$ for $i \in K(x)$, and $b_{i,j} := c_{i,j}$ for $i \notin K(x)$, uniquely since a_i is the coordinate of c at $(i, j_0(i))$. Thus I_x does not depend on the point, (5.11) holds, the action of c is (5.12), and $|J_i(x)| \leq N - 2$ by **Step 3**. This proves (4).

By (5.10), $\pi^* D_i$ and $\pi^* G_{i,j}$ are m_i times the reduced smooth divisors $(u_i = 0)$ and $(v_{i,j} = 0)$ in every S_ϵ . For $i \in K(x)$ there are at least two indices $j \notin J_i(x)$, so by (5.11), $e_{i,j} \in I_x$ if and only if $(i,j) \in P(x)$, and then $e_{i,j}$ acts on S_ϵ by $v_{i,j} \rightarrow \zeta_i v_{i,j}$ with fixed locus $(v_{i,j} = 0) = \pi^{-1}(G_{i,j}) \cap S_\epsilon$. Hence $S^{e_{i,j}} = (\pi^{-1}(G_{i,j}))_{\text{red}}$ is a smooth divisor and $\pi^* G_{i,j} = m_i S^{e_{i,j}}$. The image under the finite morphism π of an irreducible component of $S^{e_{i,j}}$ is an irreducible closed subset of $G_{i,j}$ of dimension $n - 1$, that is, an irreducible component of $G_{i,j}$. If $x \in G_{i,j}$ lies on no other irreducible component of $D + G$, then $K(x) = \emptyset$ and $P(x) = \{(i,j)\}$, so $I_x = \langle e_{i,j} \rangle$ by (5.11).

Let T' be an irreducible component of $S^{e_{i,j}}$ and $C := \pi(T')$. Since $\pi^* G_{i,j} = m_i S^{e_{i,j}}$ with $S^{e_{i,j}}$ reduced, $\text{ord}_{T'}(\pi^* G_{i,j}) = m_i$. The element $e_{i,j}$ fixes $T' \subset S^{e_{i,j}}$ pointwise. Conversely, let $a \in A$ fix T' pointwise. There exists $x' \in C$ lying on no other irreducible component of $D + G$, and $\pi|_{T'} : T' \rightarrow C$ is surjective, being finite and dominant, so a fixes a point of $T' \cap \pi^{-1}(x')$ and $a \in I_{x'} = \langle e_{i,j} \rangle$. Let W_0 be an affine open neighborhood of x' on which $\mathcal{H}_i$ is trivial and $W_0 \cap D_i = \emptyset$. Trivializing $\mathcal{H}_i$ on W_0 , $y_{i,j}$ is a regular function on $\pi^{-1}(W_0)$ with $y_{i,j}^{m_i} = f \circ \pi$,

where $f \in \mathcal{O}_X(W_0)$ is the function $s_{D_i} s_{G_{i,j}}$, whose divisor is $G_{i,j} \cap W_0$. Since $T' \cap \pi^{-1}(W_0) \neq \emptyset$, we have $y_{i,j} \in \mathcal{O}_{S,T'}$ and $m_i \operatorname{ord}_{T'}(y_{i,j}) = \operatorname{ord}_{T'}(\pi^* G_{i,j}) = m_i$. Thus $y_{i,j}$ generates $\mathfrak{m}_{T'}$, and $e_{i,j} y_{i,j} = \zeta_i y_{i,j}$ gives the action on $\mathfrak{m}_{T'}/\mathfrak{m}_{T'}^2$. This proves (5). $\square$

5.4. The quotient.

Proposition 5.5. *Let (X, B) , N , $\mathcal{H}_i$, $G_{i,j}$, G , s_{D_i} , $s_{G_{i,j}}$, $y_{i,j}$, S , π , A , and $e_{i,j}$ be as in Proposition 5.4. Let F be an abelian variety with an action of A by automorphisms of varieties such that every $a \in A$ which has two equal nonzero coordinates in one of the factors $(\mathbb{Z}/m_i \mathbb{Z})^N$ of A has no fixed point on F . Let A act diagonally on $S \times F$, let*

$$(5.13) \quad Q := (S \times F)/A, \quad h : Q \rightarrow X, \quad V := X \setminus G, \quad U := X \setminus (D \cup G),$$

where Q and $p : S \times F \rightarrow Q$ are as in Lemma 5.1 and h is induced by $\pi \circ \operatorname{pr}_S$. Then the following hold.

- (1) Q is a normal projective variety and h is surjective.
- (2) Let $q \in Q$, $x := h(q)$, and, for $i \in K(x)$, z_i a holomorphic function on an analytic neighborhood of x generating the ideal of D_i in the ring $\mathcal{O}_{X,x}^{\text{an}}$ of germs of holomorphic functions at x . Then there exist a connected analytic neighborhood Q_q of q and holomorphic functions r_i on Q_q for $i \in K(x)$ such that

$$(5.14) \quad h^* z_i = r_i^{m_i} \quad \text{on } Q_q \quad (i \in K(x)).$$

- (3) $h^{-1}(V)$ is smooth, and for every $q \in h^{-1}(V)$ some local lift of h to the root chart at $h(q)$ is a submersion at q .
- (4) $h|_{h^{-1}(U)} : h^{-1}(U) \rightarrow U$ is a holomorphically locally trivial fiber bundle with fiber F .
- (5) Let Y be a smooth projective variety, $\rho : Y \rightarrow Q$ a projective birational morphism which is an isomorphism over $h^{-1}(V)$, and $g := h \circ \rho$. Then for every $y \in g^{-1}(V)$ some local lift of g to the root chart at $g(y)$ is a submersion at y , and $g|_{g^{-1}(U)} : g^{-1}(U) \rightarrow U$ is a holomorphically locally trivial fiber bundle with fiber F . Let E be a prime divisor on Y , $y \in E$ a point at which E is smooth, $x := g(y)$, z_i for $i \in K(x)$ as in (2), and Q_q and r_i given by (2) for $q := \rho(y)$. Then

$$(5.15) \quad \operatorname{ord}_E(g^* z_i) = m_i \operatorname{ord}_E(\rho^* r_i) \quad (i \in K(x)),$$

where $\operatorname{ord}_E(\psi)$ denotes, for a holomorphic function ψ near y and a generator t of the ideal of E in $\mathcal{O}_{Y,y}^{\text{an}}$, the largest integer k with $\psi \in t^k \mathcal{O}_{Y,y}^{\text{an}}$. Hence $g^* D_i = m_i C_i$ for an effective divisor C_i on Y for every i . Moreover, such Y and ρ exist.

- (6) Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ be elliptic curves, $t_i \in \mathcal{E}_i$ a point of order m_i , and

$$(5.16) \quad F := \prod_{i=1}^r F_{m_i, N} = \prod_{i=1}^r \prod_{1 \leq j < k \leq N} \mathcal{E}_i^{m_i}$$

with the action of $A = \prod_{i=1}^r A_{m_i, N}$ of Lemma 5.3(4), where $F_{m_i, N}$ is defined by (5.2) with $\mathcal{E} = \mathcal{E}_i$ and $t = t_i$. Then F satisfies the hypothesis above, $\dim F = \sum_{i=1}^r m_i \binom{N}{2}$, and $F^{e_{i,j}}$ is nonempty, smooth, and connected for every i and j .

Propositions 5.5 and 6.3 replace the blowup construction of [Ope26a, Proposition A.3].

Proof. If $r = 0$, then A is trivial, $Q = X \times F$, h is the projection, and all assertions hold, with F a point in (6). Assume that $r \geq 1$. We use the notation $K(x)$, $J_i(x)$, $P(x)$, and I_x of Proposition 5.4 and identify every chart of S over W with $\widehat{W}$ by Proposition 5.4(3).

Step 1. In this step, we prove (1) and (2).

By Lemma 5.1(1), Q is a normal projective variety, and h is surjective as π is. Let q , x , and z_i be as in (2), $(s_0, f_0) \in p^{-1}(q)$, and $H \subset A$ its stabilizer, so that $H \subset I_x$ by Proposition 5.4(4). Let $k = \sum_{i \in K(x)} a_i \tau_i + \sum_{(i,j) \in P(x)} b_{i,j} e_{i,j} \in H$ be as in (5.11). Suppose that $a_i \neq 0$ for some $i \in K(x)$. Since $|J_i(x)| \leq N - 2$, there exist $j \neq j'$ not in $J_i(x)$, and the coordinates of k at

(i, j) and (i, j') are both a_i . By the hypothesis on F , k has no fixed point on F , contradicting $kf_0 = f_0$. Hence $a_i = 0$ for every $i \in K(x)$ and $k \in H$.

Fix $i \in K(x)$ and $j \notin J_i(x)$. Let $W_{0,i}$ be an affine open neighborhood of x on which $\mathcal{H}_i$ is trivial and $W_{0,i} \cap G_{i,j} = \emptyset$. Trivializing $\mathcal{H}_i$ on $W_{0,i}$, $y_{i,j}$ is a regular function on $\pi^{-1}(W_{0,i})$ with $y_{i,j}^{m_i} = f_i \circ \pi$, where $f_i \in \mathcal{O}_X(W_{0,i})$ is the function $s_{D_i} s_{G_{i,j}}$, whose divisor is $D_i \cap W_{0,i}$. Thus f_i generates the ideal of D_i in $\mathcal{O}_{X,x}$, so $df_i(x) \neq 0$ as D_i is smooth, and f_i generates the ideal of D_i in $\mathcal{O}_{X,x}^{\text{an}}$, as D_i and $(f_i = 0)$ are smooth hypersurfaces through x with $D_i \subset (f_i = 0)$. Every $k \in H$ has coordinate $a_i = 0$ at (i, j) , as $(i, j) \notin P(x)$, so $\psi_i := y_{i,j} \circ \text{pr}_S$ is an H -invariant regular function on $\pi^{-1}(W_{0,i}) \times F$.

Let $W_1 \subset \bigcap_{i \in K(x)} W_{0,i}$ be an adapted coordinate neighborhood of x , shrunk so that $z_i = \omega_i f_i$ on W_1 for every $i \in K(x)$, where ω_i is a nowhere vanishing holomorphic function on W_1 , with a holomorphic m_i th root ω_i^{1/m_i} on the polydisc W_1 . Since the points $a(s_0, f_0)$ for $a \in A/H$ are distinct, there exists an analytic neighborhood $\mathcal{N}_0$ of (s_0, f_0) in $\pi^{-1}(W_1) \times F$ with $a\mathcal{N}_0 \cap \mathcal{N}_0 = \emptyset$ for every $a \in A \setminus H$. The connected component $\mathcal{N}$ of $\bigcap_{k \in H} k\mathcal{N}_0$ containing (s_0, f_0) is a connected H -invariant analytic neighborhood of (s_0, f_0) with $a\mathcal{N} \cap \mathcal{N} = \emptyset$ for every $a \in A \setminus H$. By Lemma 5.1(2), $Q_q := p(\mathcal{N})$ is a connected analytic neighborhood of q contained in $h^{-1}(W_1)$, and $\psi_i|_{\mathcal{N}} = \phi_i \circ p|_{\mathcal{N}}$ for a holomorphic function ϕ_i on Q_q . Let $r_i := (\omega_i^{1/m_i} \circ h)\phi_i$. Then

$$r_i^{m_i} \circ p = (\omega_i \circ \pi \circ \text{pr}_S) \psi_i^{m_i} = (\omega_i f_i) \circ \pi \circ \text{pr}_S = z_i \circ h \circ p \quad \text{on } \mathcal{N},$$

and (5.14) follows as $p|_{\mathcal{N}} : \mathcal{N} \rightarrow Q_q$ is surjective. This proves (2).

Step 2. In this step, we prove (3) and (4).

Let $x \in V$. Then $P(x) = \emptyset$, so every nonzero element of I_x has some $a_i \neq 0$ by (5.11) and no fixed point on F by **Step 1**, and the stabilizer of every point of $S \times F$ over x is trivial. Reordering the indices, let $K(x) = \{1, \dots, s\}$, let W be an adapted coordinate neighborhood of x with $W \cap G = \emptyset$ to which Proposition 5.4(3) applies, and $\varpi_W : \widetilde{W} \rightarrow W$ the root chart (2.1) over W . Since $P(x) = \emptyset$, the polydisc $\widetilde{W}$ has coordinates $u_1, \dots, u_s, z_{s+1}, \dots, z_n$, so $\widetilde{W} = \widehat{W}$, and by (5.10) the identification of every chart S' of S over W with $\widetilde{W}$ identifies $\pi|_{S'}$ with ϖ_W . Let $q \in h^{-1}(x)$, $(s_0, f_0) \in p^{-1}(q)$, S' the chart containing s_0 , and $\mathcal{N}$ a connected analytic neighborhood of (s_0, f_0) in $S' \times F$ with $a\mathcal{N} \cap \mathcal{N} = \emptyset$ for every $a \in A \setminus \{1\}$. By Lemma 5.1(2), $Q_q := p(\mathcal{N})$ is a connected analytic neighborhood of q and $p|_{\mathcal{N}} : \mathcal{N} \rightarrow Q_q$ is a biholomorphism, so Q is smooth at q [Har77, Appendix B, Section 1]. The map $\tilde{h} := \text{pr}_S \circ (p|_{\mathcal{N}})^{-1} : Q_q \rightarrow S' = \widetilde{W}$ satisfies $\varpi_W \circ \tilde{h} = h|_{Q_q}$, so it is a local lift of h to the root chart at x , and it is a submersion at q as pr_S is. This proves (3).

Let $x \in U$, W as above, and S_1 a chart of S over W . By Proposition 5.4(3), $\pi|_{S_1} : S_1 \rightarrow W$ is biholomorphic and $\pi^{-1}(W) = \coprod_{a \in A} aS_1$. The open subset $\mathcal{N} := S_1 \times F$ of $S \times F$ satisfies $a\mathcal{N} \cap \mathcal{N} = \emptyset$ for $a \neq 1$ and $a\mathcal{N} = p^{-1}(h^{-1}(W))$, and its points have trivial stabilizers. By Lemma 5.1(2), $p|_{\mathcal{N}} : S_1 \times F \rightarrow h^{-1}(W)$ is a biholomorphism with $h \circ p|_{\mathcal{N}} = \pi \circ \text{pr}_{S_1}$, so $h^{-1}(W)$ is biholomorphic to $W \times F$ over W . This proves (4).

Step 3. In this step, we prove (5) and (6).

Let $y \in g^{-1}(V)$, and let $\tilde{h}$ be a local lift of h to the root chart at $g(y)$ which is a submersion at $\rho(y)$, given by (3). Since ρ is an isomorphism over $h^{-1}(V)$, the map $\tilde{h} \circ \rho$ is a local lift of g which is a submersion at y , and ρ identifies $g^{-1}(U)$ with $h^{-1}(U)$ over U , so the bundle assertion follows from (4). Let $E, y, x, z_i, q, Q_q, r_i$, and t be as in (5). Since $\mathcal{O}_{Y,y}^{\text{an}}(t)$ is the ring of germs of holomorphic functions on E at y , a domain, ord_E is additive on products. By (5.14), $g^*z_i = (\rho^*r_i)^{m_i}$ on $\rho^{-1}(Q_q)$, which gives (5.15). If $g(E) \not\subset D_i$, then E is not a component of g^*D_i . If $g(E) \subset D_i$, then for $y \in E$ at which E is smooth and $x := g(y) \in D_i$, the divisor g^*D_i is the divisor of g^*z_i near y , so the coefficient of E in g^*D_i is $\text{ord}_E(g^*z_i) \in m_i\mathbb{Z}$ by (5.15). Hence $C_i := \frac{1}{m_i}g^*D_i$ is an effective divisor. Finally, $h^{-1}(V)$ lies in the smooth locus of Q by (3), so Theorem 2.10 provides Y and ρ . This proves (5).

Since $\mathcal{E}_i$ is a one-dimensional complex torus, it has points of order m_i . By Lemma 5.3(4), every $a \in A$ with two equal nonzero coordinates in one factor $A_{m_i, N}$ has no fixed point on F , and $F^{e_{i,j}}$ is nonempty, smooth, and connected, and by Lemma 5.3(1), $\dim F = \sum_{i=1}^r m_i \binom{N}{2}$. This proves (6). $\square$

6. DIVISORIAL VALUATIONS

In this section, we prove Proposition 6.3.

Definition 6.1. Let W be a variety of dimension d . For an integral closed subvariety $V \subset W$, let η_V be its generic point and $\mathcal{O}_{W,V} := \mathcal{O}_{W,\eta_V}$. A *divisorial valuation* of $\mathbb{C}(W)$ is a discrete valuation v of $\mathbb{C}(W)$ which is trivial on $\mathbb{C}$, has value group $\mathbb{Z}$, and whose residue field $k(v)$ satisfies $\text{trdeg}_{\mathbb{C}} k(v) = d - 1$. Let $\mathcal{O}_v \subset \mathbb{C}(W)$ be its valuation ring, $\mathfrak{m}_v$ its maximal ideal, and $k(c)$ the residue field of a point c of a scheme. If W is proper, the *center* of v on W is the unique point $c_W(v) \in W$ such that $\mathcal{O}_v$ dominates $\mathcal{O}_{W,c_W(v)}$ [Har77, Exercise II.4.5].

If W is normal and E is a prime divisor on W , then ord_E has residue field $\mathbb{C}(E)$, so ord_E is a divisorial valuation of $\mathbb{C}(W)$, with center η_E if W is proper.

Lemma 6.2. *Let W be a normal projective variety of dimension d , $O \subset W$ a nonempty open subset, and $v_1, \dots, v_s$ distinct divisorial valuations of $\mathbb{C}(W)$ whose centers on W lie in $W \setminus O$.*

- (1) *There exist a normal projective variety W' and a projective birational morphism $\rho : W' \rightarrow W$ which is an isomorphism over O , such that for each i there exists a prime divisor E_i on W' with $v_i = \text{ord}_{E_i}$.*
- (2) *If O is smooth, then W' in (1) can be chosen to be smooth.*

Proof. We first prove (1) for $s = 1$. Let $v := v_1$ and $c := c_W(v)$. If $c = \eta_E$ for a prime divisor E , then $\mathcal{O}_v = \mathcal{O}_{W,E}$ [Har77, Theorem I.6.1A], so $v = \text{ord}_E$ and $W' := W$ works. Assume that $\dim \overline{\{c\}} \leq d - 2$. Choose $f_1, \dots, f_{d-1} \in \mathcal{O}_v$ with residues in $k(v)$ algebraically independent over $\mathbb{C}$, an affine open neighborhood $W_0 = \text{Spec } R_0$ of c , and $a_1, \dots, a_{d-1}, b \in R_0$ with $f_j = a_j/b$, so $R_0 \subset \mathcal{O}_{W,c} \subset \mathcal{O}_v$ and $v(a_j) = v(b)$. Let $I_0 := (b, a_1, \dots, a_{d-1}) \subset R_0$, $j : W_0 \hookrightarrow W$ the inclusion, $I := \ker(\mathcal{O}_W \rightarrow j_*(\mathcal{O}_{W_0}/I_0))$, a coherent ideal sheaf with $I|_{W_0} = I_0$ [Sta26, Tag 01R8], and J the ideal sheaf of the reduced closed subscheme $W \setminus O$. As $c \in W \setminus O \neq W$, $v(J) := \min\{v(x) \mid x \in J_c\}$ is finite and positive. Choose N with $Nv(J) > v(b)$ and let

$$(6.1) \quad I' := I + J^N.$$

Then $I'|_O = \mathcal{O}_O$, and $v(x) \geq v(b)$ for $x \in I'_c$.

Let $\tilde{b} : \tilde{W} \rightarrow W$ be the blowup of I' . As $I' \supset J^N \neq 0$, $\tilde{W}$ is a projective variety and $\tilde{b}$ is projective and birational [Har77, Proposition II.7.16], and an isomorphism over O [Sta26, Tag 02OS]. The subring $\tilde{R} \subset \mathbb{C}(W)$ generated by R_0 and the x/b with $x \in I'(W_0)$ is the affine blowup algebra $R_0[I'(W_0)/b]$, so $\text{Spec } \tilde{R}$ is an open subset of $\tilde{W}$ [Sta26, Tag 0804]. As $\tilde{R} \subset \mathcal{O}_v$, the point $\tilde{c} \in \text{Spec } \tilde{R}$ corresponding to $\mathfrak{m}_v \cap \tilde{R}$ is the center of v on $\tilde{W}$. The $f_j \in \tilde{R}$ have algebraically independent residues in $k(\tilde{c}) \subset k(v)$, so $\dim \overline{\{\tilde{c}\}} = \text{trdeg}_{\mathbb{C}} k(\tilde{c}) \geq d - 1$ [Har77, Exercise II.3.20(b)], and $\tilde{c}$ is not the generic point as $\mathcal{O}_v \neq \mathbb{C}(W)$. Thus $\tilde{E} := \overline{\{\tilde{c}\}}$ is a prime divisor. Let $\nu : W' \rightarrow \tilde{W}$ be the normalization, so ν is finite and birational, W' is a normal projective variety [Sta26, Tags 0BXR and 0GK4], and ν is an isomorphism over $\tilde{b}^{-1}(O) \cong O$ as O is normal. For $c' := c_{W'}(v)$, the inclusion $\mathcal{O}_{\tilde{W},\nu(c')} \subset \mathcal{O}_{W',c'}$ gives $\nu(c') = \tilde{c}$, so ν maps $E' := \overline{\{c'\}}$ finitely onto $\tilde{E}$ and E' is a prime divisor [Har77, Exercise II.3.20(b)]. As $\mathcal{O}_v$ dominates $\mathcal{O}_{W',E'}$, $v = \text{ord}_{E'}$ [Har77, Theorem I.6.1A]. This proves (1) for $s = 1$ with $\rho := \tilde{b} \circ \nu$.

In general, we argue by induction on $s \geq 1$. Let $\rho_{s-1} : W_{s-1} \rightarrow W$ and $E_1, \dots, E_{s-1}$ be given by induction, $c_s := c_{W_{s-1}}(v_s)$, and $O_s := W_{s-1} \setminus \overline{\{c_s\}}$. As $\rho_{s-1}(c_s) = c_W(v_s) \notin O$, $\rho_{s-1}^{-1}(O) \subset O_s$. If $\eta_{E_i} \notin O_s$, then $E_i \subset \overline{\{c_s\}} \neq W_{s-1}$, so $c_s = \eta_{E_i}$ and $v_s = \text{ord}_{E_i} = v_i$ [Har77, Theorem I.6.1A], a contradiction. The case $s = 1$ for W_{s-1}, O_s, v_s gives a projective birational morphism $W_s \rightarrow W_{s-1}$

from a normal projective variety, an isomorphism over O_s , and a prime divisor E_s on W_s with $v_s = \text{ord}_{E_s}$. As $O_s \supset \rho_{s-1}^{-1}(O)$ contains every η_{E_i} , the strict transforms E'_i of E_i on W_s satisfy $v_i = \text{ord}_{E'_i}$. This proves (1) with $W' := W_s$.

For (2), let O be smooth. By Theorem 2.10, there exists a projective birational morphism $\sigma : W'' \rightarrow W'$ from a smooth projective variety which is an isomorphism over the smooth locus of W' . This locus contains $\rho^{-1}(O) \cong O$ and every η_{E_i} , as W' is normal [Har77, Section II.6]. Hence $\rho \circ \sigma$ is an isomorphism over O , and $v_i = \text{ord}_{E''_i}$ for the strict transform E''_i of E_i on W'' . $\square$

Proposition 6.3. *Let X and S be smooth projective varieties, A a finite group acting faithfully on S , and $\pi : S \rightarrow X$ an A -invariant finite surjective morphism which induces an isomorphism $S/A \cong X$. Let F be a smooth projective variety with an action of A , let A act diagonally on $V_0 := S \times F$, and let*

$$(6.2) \quad Q := V_0/A, \quad p : V_0 \rightarrow Q, \quad g_0 : Q \rightarrow X$$

be the quotient, the quotient morphism, and the morphism induced by $\pi \circ \text{pr}_S$. Let $P_1, \dots, P_b$ be distinct prime divisors on X . For each λ , let T_λ be a prime divisor on S with $\pi(T_\lambda) = P_\lambda$, $m_\lambda := \text{ord}_{T_\lambda}(\pi^*P_\lambda)$, $H_\lambda := \{a \in A \mid a|_{T_\lambda} = \text{id}_{T_\lambda}\}$, $\mathfrak{m}_{T_\lambda}$ the maximal ideal of $\mathcal{O}_{S, T_\lambda}$, and z_λ a uniformizer of $\mathcal{O}_{X, P_\lambda}$. Assume that $m_\lambda \geq 2$, that H_λ is cyclic of order m_λ and acts faithfully on the one-dimensional $\mathbb{C}(T_\lambda)$ -vector space $\mathfrak{m}_{T_\lambda}/\mathfrak{m}_{T_\lambda}^2$, and that $F^{H_\lambda} \neq \emptyset$. Let Z_λ be an irreducible component of F^{H_λ} , Q_{reg} the smooth locus of Q , and

$$(6.3) \quad O := Q_{\text{reg}} \setminus g_0^{-1}\left(\bigcup_{\lambda=1}^b P_\lambda\right).$$

Then the following hold.

- (1) *For each λ , there exists a divisorial valuation ν_λ of $\mathbb{C}(Q)$ whose center on Q is the generic point of $p(T_\lambda \times Z_\lambda)$, such that*

$$(6.4) \quad \nu_\lambda(g_0^* z_\lambda) = 1$$

and $\nu_\lambda|_{\mathbb{C}(X)} = \text{ord}_{P_\lambda}$.

- (2) *There exist a smooth projective variety Y and a projective birational morphism $\rho : Y \rightarrow Q$ which is an isomorphism over O satisfying the following. Let $g := g_0 \circ \rho$. Then for each λ , there exist a prime divisor E_λ on Y with $\nu_\lambda = \text{ord}_{E_\lambda}$ and $g(E_\lambda) = P_\lambda$, and nonempty open subsets $E_\lambda^\circ \subset E_\lambda$ and $P_\lambda^\circ \subset P_\lambda$ with $g(E_\lambda^\circ) \supset P_\lambda^\circ$, such that g is smooth at every point of E_λ° .*
- (3) *Let $D_1, \dots, D_r$ be smooth prime divisors on X , none of which equals any P_λ , and $\ell_1, \dots, \ell_r \geq 2$ integers. An analytic local equation of D_α at $x \in D_\alpha$ is a holomorphic function near x which generates the ideal of D_α in the ring $\mathcal{O}_{X, x}^{\text{an}}$ of germs of holomorphic functions at x . Assume that for every α and every $x \in D_\alpha$, there exists an analytic local equation z_α of D_α at x such that for every $y_0 \in g_0^{-1}(x)$, the function $g_0^* z_\alpha$ is the ℓ_α th power of a holomorphic function near y_0 . Then for every α , every $x \in D_\alpha$, every analytic local equation z'_α of D_α at x , and every $y \in g^{-1}(x)$, the function $g^* z'_\alpha$ is the ℓ_α th power of a holomorphic function near y . In particular, $\text{ord}_E(g^* D_\alpha) \in \ell_\alpha \mathbb{Z}$ for every α and every prime divisor E on Y .*
- (4) *Let $U_0 \subset X$ be an open subset such that π is étale over U_0 and $U_0 \cap P_\lambda = \emptyset$ for every λ . Then $g|_{g^{-1}(U_0)} : g^{-1}(U_0) \rightarrow U_0$ is a holomorphically locally trivial fiber bundle with fiber F , and g is smooth at every point of $g^{-1}(U_0)$.*

Proposition 6.3 replaces the blowup of the fixed loci in [Ope26a, Proposition A.3].

Proof. By Lemma 5.1(1), Q is a normal projective variety, p is finite, $\mathbb{C}(Q) = L^A$ where $L := \mathbb{C}(V_0)$, and g_0 exists. We fix λ , write P, T, m, H, Z, z for $P_\lambda, T_\lambda, m_\lambda, H_\lambda, Z_\lambda, z_\lambda$, and let $d := \dim Q$, $C := T \times Z$, and $R := \mathcal{O}_{V_0, C}$. **Steps 1–3** construct $\nu := \nu_\lambda$ by restricting to $\mathbb{C}(Q)$ a monomial valuation of L with center η_C , and **Steps 4 and 5** prove (2)–(4).

Step 1. In this step, we construct a regular system of parameters $u, a_1, \dots, a_a$ of R consisting of H -eigenvectors.

Let η generate H . As H fixes T pointwise, it acts on $\mathcal{O}_{S,T}$ and trivially on $\mathbb{C}(T)$, so η acts on $\mathfrak{m}_T/\mathfrak{m}_T^2$ by some $\zeta \in \mathbb{C}(T)$ with $\zeta^m = 1$, which lies in μ_m and is primitive by faithfulness. For a uniformizer t of $\mathcal{O}_{S,T}$, $u := \frac{1}{m} \sum_{j=0}^{m-1} \zeta^{-j} \eta^j(t) \equiv t \pmod{\mathfrak{m}_T^2}$ is a uniformizer with $\eta(u) = \zeta u$. As $\text{ord}_T(\pi^* z) = m$ and $\pi^* z$ is H -invariant,

$$(6.5) \quad \pi^* z = u^m \epsilon$$

for some H -invariant unit ϵ of $\mathcal{O}_{S,T}$.

By Lemma 5.2, Z is smooth and $T_f Z = (T_f F)^H$ for closed points $f \in Z$. Let $a := \text{codim}_F Z$, $\mathfrak{m}_Z$ the maximal ideal of $\mathcal{O}_{F,Z}$, and $N^\vee$ the conormal sheaf of Z in F , on which H acts $\mathcal{O}_Z$ -linearly. By averaging, $(N^\vee)^H$ is a direct summand of $N^\vee$ with fiber $(N^\vee \otimes k(f))^H = ((T_f F/T_f Z)^H)^\vee = 0$ at f , so $(N^\vee)^H = 0$ and 1 is not an eigenvalue of η on $N^\vee \otimes \mathbb{C}(Z) = \mathfrak{m}_Z/\mathfrak{m}_Z^2$. Choose a basis of this space of eigenvectors of η with eigenvalues $\zeta^{w_1}, \dots, \zeta^{w_a}$, $1 \leq w_j \leq m-1$, lift them to $x_1, \dots, x_a \in \mathfrak{m}_Z$, and let $a_j := \frac{1}{m} \sum_{i=0}^{m-1} \zeta^{-w_j i} \eta^i(x_j)$. Then $\eta(a_j) = \zeta^{w_j} a_j$ and $a_j \equiv x_j \pmod{\mathfrak{m}_Z^2}$, so $a_1, \dots, a_a$ generate $\mathfrak{m}_Z$ by Nakayama's lemma. As $V_0 = S \times F$ and $C = T \times Z$, the ring R is regular of dimension $a+1$ with residue field $k_0 := \mathbb{C}(C)$ and $\mathfrak{m}_R = (u, a_1, \dots, a_a)$, and H acts on R and trivially on k_0 .

Step 2. In this step, we construct a divisorial valuation v of L with center η_C , $v(u) = 1$, and $v(a_j) = w_j$.

Let $\widehat{R}$ be the completion of R and $\widehat{\mathfrak{m}}$ its maximal ideal, so $L = \text{Frac}(R) \subset \text{Frac}(\widehat{R})$. By [Sta26, Tag 0C0S] and its proof, $\widehat{R}$ contains a subfield k' mapping isomorphically onto k_0 , and the continuous k' -algebra homomorphism

$$(6.6) \quad k'[[T_0, T_1, \dots, T_a]] \rightarrow \widehat{R}, \quad T_0 \rightarrow u, \quad T_j \rightarrow a_j,$$

is an isomorphism, as $u, a_1, \dots, a_a$ map to a basis of $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$. We identify the two sides of (6.6), and for integers $b_0, \dots, b_a \geq 0$ call

$$(6.7) \quad b_0 + \sum_{j=1}^a w_j b_j$$

the weight of the monomial $u^{b_0} a_1^{b_1} \dots a_a^{b_a}$. As the weights $1, w_1, \dots, w_a$ are positive, only finitely many monomials have a given weight. For $0 \neq f \in \widehat{R}$, let $v(f)$ be the least weight of a monomial of f and $\text{in}(f) \in k'[u, a_1, \dots, a_a]$ the sum of the terms of f of weight $v(f)$. Then $\text{in}(fg) = \text{in}(f) \text{in}(g)$, so $v(fg) = v(f) + v(g)$, and $v(f+g) \geq \min\{v(f), v(g)\}$. Hence v extends to a valuation of $\text{Frac}(\widehat{R})$, and we also write v for its restriction to L . As units of $\widehat{R}$ have value 0, elements of $\widehat{\mathfrak{m}}$ have positive value, and $v(u) = 1$, the valuation v is trivial on $\mathbb{C}$ with value group $\mathbb{Z}$, $\mathcal{O}_v$ dominates R , and the center of v on V_0 is η_C .

Via residues, we regard $k_0 \cong k'$ as a subfield of $k(v)$ and let $t_j \in k(v)$ be the residue of $a_j/u^{w_j} \in L$. We claim that

$$(6.8) \quad k(v) = k_0(t_1, \dots, t_a)$$

with $t_1, \dots, t_a$ algebraically independent over k_0 . Every element of $k(v)$ is the residue of some f/g with $f, g \in \widehat{R}$ and $v(f) = v(g) =: e$, hence of $\text{in}(f)/\text{in}(g)$, as $v(f/g - \text{in}(f)/\text{in}(g)) > 0$. Each monomial of weight e equals $u^e \prod_j (a_j/u^{w_j})^{b_j}$, so $\text{in}(f)/\text{in}(g)$ is a quotient of polynomials in the a_j/u^{w_j} over k' with denominator of value 0. Thus $k(v) = k_0(t_1, \dots, t_a)$. For a nonzero polynomial ψ in a variables over k_0 and $N \gg 0$, $u^N \psi(a_1/u^{w_1}, \dots, a_a/u^{w_a})$ is a nonzero polynomial in $u, a_1, \dots, a_a$ whose monomials all have weight N , so $v(\psi(a_1/u^{w_1}, \dots, a_a/u^{w_a})) = 0$ and $\psi(t_1, \dots, t_a) \neq 0$. As k_0 and the t_j are residues of elements of L , the residue field of $v|_L$ is also (6.8), and

$$\text{trdeg}_{\mathbb{C}} k(v) = \text{trdeg}_{\mathbb{C}} k_0 + a = \dim C + a = d - 1.$$

Step 3. In this step, we restrict v to $\mathbb{C}(Q)$ and prove (1).

For $e \geq 0$, let $I_e \subset \widehat{R}$ be the ideal generated by the monomials of weight at least e , so $v(f) = \max\{e \mid f \in I_e\}$ for $0 \neq f \in \widehat{R}$. The action of H on R extends to $\widehat{R}$, and η multiplies each monomial by ζ raised to its weight (6.7), so $\eta(I_e) = I_e$. As $\widehat{m}I_e \subset I_{e+1}$ and H acts trivially on k_0 , η acts on I_e/I_{e+1} , which is spanned over k_0 by the monomials of weight e , by ζ^e . Hence $m \mid v(f)$ for $0 \neq f \in \widehat{R}^H$, as the class of f in $I_{v(f)}/I_{v(f)+1}$ is nonzero and H -invariant. For $0 \neq f \in L^H$, write $f = x/y$ with $x, y \in R$. Replacing y by $\prod_{j=0}^{m-1} \eta^j(y)$ and x by fy , we may assume that $x, y \in R^H$, so $v(f) = v(x) - v(y) \in m\mathbb{Z}$. Thus $v_H := \frac{1}{m}v|_{L^H}$ is a valuation of L^H with values in $\mathbb{Z}$, and $v_H(\pi^*z) = \frac{1}{m}(m + v(\epsilon)) = 1$ by (6.5). As $\mathbb{C}(Q) = L^A \subset L^H$, we may define

$$(6.9) \quad \nu := v_H|_{\mathbb{C}(Q)}.$$

As $p^*g_0^*z = \text{pr}_S^*\pi^*z$, (6.4) holds, and ν has value group $\mathbb{Z}$.

For a finite extension L'/K and a valuation v' of L' , $[k(v') : k(v'|_K)] \leq [L' : K]$: a nontrivial relation over K among $x_1, \dots, x_k \in \mathcal{O}_{v'}$ with residues linearly independent over $k(v'|_K)$, divided by a coefficient of minimal value, reduces to a nontrivial relation among the residues. As L is finite over L^A and $k(v_H) = k(v|_{L^H})$, $k(v)$ is finite over $k(\nu)$. Hence $\text{trdeg}_{\mathbb{C}} k(\nu) = d - 1$, and ν is a divisorial valuation of $\mathbb{C}(Q)$.

As $p(\eta_C)$ is the generic point of $p(C)$, the ring R dominates $\mathcal{O}_{Q,p(\eta_C)}$, and so does $\mathcal{O}_\nu = \mathcal{O}_v \cap \mathbb{C}(Q)$ by (6.9). Thus the center of ν on Q is $p(\eta_C)$. Finally, $\pi(\eta_T) = \eta_P$, so $\mathcal{O}_{X,P} \subset \mathcal{O}_{S,T} \subset R$ and units of $\mathcal{O}_{X,P}$ are units of R . As every element of $\mathbb{C}(X)^\times$ is $z^j \epsilon'$ with $j \in \mathbb{Z}$ and ϵ' a unit of $\mathcal{O}_{X,P}$, and $\nu(z) = 1$, we obtain

$$(6.10) \quad \nu|_{\mathbb{C}(X)} = \text{ord}_P.$$

This proves (1). As the P_λ are distinct, so are $\nu_1, \dots, \nu_b$ by (6.10).

Step 4. In this step, we construct Y , ρ , and $E_1, \dots, E_b$ by Lemma 6.2 and prove (2).

The open subset O is smooth, and nonempty as g_0 is dominant. By (1), the center of ν_λ on Q lies in $g_0^{-1}(P_\lambda) \subset Q \setminus O$. Lemma 6.2 gives a smooth projective variety Y , a projective birational morphism $\rho : Y \rightarrow Q$ which is an isomorphism over O , and prime divisors $E_1, \dots, E_b$ on Y with $\nu_\lambda = \text{ord}_{E_\lambda}$. Let $g := g_0 \circ \rho$, fix λ , and write P, E, z, ν for $P_\lambda, E_\lambda, z_\lambda, \nu_\lambda$. As η_E is the center of ν on Y , $\mathcal{O}_\nu \cap \mathbb{C}(X) = \mathcal{O}_{X,P}$ dominates $\mathcal{O}_{X,g(\eta_E)} \subset \mathcal{O}_{Y,E}$ by (6.10), so $g(\eta_E) = \eta_P$ and $g(E) = P$.

Let $P' \subset P$ be the open subset of points x at which P is smooth and z generates the ideal of P in $\mathcal{O}_{X,x}$, $X_1 \subset X$ the open subset on which z is regular, so that $\eta_P \in P' \subset X_1$, and E_{sm} the smooth locus of E . As $g(\eta_E) = \eta_P$, $g|_{E_{\text{sm}}} : E_{\text{sm}} \rightarrow P$ is dominant, so $E_{\text{sm}} \cap g^{-1}(P'') \rightarrow P''$ is smooth for some nonempty open subset $P'' \subset P'$ [Har77, Corollary III.10.7], with surjective differential $T_y E \rightarrow T_{g(y)} P$ at closed points y [Har77, Proposition III.10.4]. On $g^{-1}(X_1) \ni \eta_E$, the regular function g^*z has divisor $E + R'$ with $R' \geq 0$ and $E \not\subset \text{Supp } R'$, as $\text{ord}_E(g^*z) = \nu(z) = 1$. Let

$$E^\circ := (E_{\text{sm}} \cap g^{-1}(P'')) \setminus \text{Supp } R',$$

a nonempty open subset of E containing η_E .

Let $y \in E^\circ$ be a closed point and $x := g(y)$. As Y is smooth, E has a local equation $t \in \mathcal{O}_{Y,y}$ [Har77, Proposition II.6.11 and Remark II.6.11.1A], and $dt_y \neq 0$ as E is smooth at y . The rational function $a := g^*z/t$ has order 0 along every prime divisor through y , so a and a^{-1} are regular at y [Har77, Proposition II.6.3A]. Thus

$$(6.11) \quad g^*z = ta$$

near y with $a(y) \neq 0$, so $d(g^*z)_y = a(y)dt_y \neq 0$ as $t(y) = 0$. Since $z|_{P'} = 0$, dz_x vanishes on $T_x P$, while $dz_x \circ dg_y = d(g^*z)_y \neq 0$. Hence the image of dg_y contains the hyperplane $T_x P = dg_y(T_y E)$ and a vector outside it, so g is smooth at y . Finally, $g(E^\circ)$ is constructible [Har77, Exercise II.3.19] and contains η_P , so it contains a nonempty open subset P° of P [Har77, Exercise II.3.18(b)]. This proves (2).

Step 5. In this step, we descend the factorizations $g_0^* z_\alpha = \phi_0^{\ell_\alpha}$ and the bundle structure over U_0 to Y , which proves (3) and (4).

For (3), let $\alpha, x \in D_\alpha, z_\alpha$ be as in (3), z'_α an analytic local equation of D_α at x , and $y \in g^{-1}(x)$. Then $z'_\alpha = \epsilon z_\alpha$ for a holomorphic function ϵ near x with $\epsilon(x) \neq 0$, which has a holomorphic ℓ_α th root ϵ_1 on a polydisc around x where it has no zero. By assumption, $g_0^* z_\alpha = \phi_0^{\ell_\alpha}$ for a holomorphic function ϕ_0 near $\rho(y)$. Hence $g^* z'_\alpha = \phi^{\ell_\alpha}$ near y with $\phi := (\epsilon_1 \circ g)(\phi_0 \circ \rho)$, which proves the first assertion of (3).

For the second assertion, let E be a prime divisor on Y . If $g(E) \not\subset D_\alpha$, then $\text{ord}_E(g^* D_\alpha) = 0$. Otherwise, let $y \in E$ be a closed point at which E is smooth, $x := g(y)$, and $z \in \mathcal{O}_{X,x}$ a local equation of D_α [Har77, Proposition II.6.11 and Remark II.6.11.1A]. As D_α is smooth at x , z is not in the square of the maximal ideal of $\mathcal{O}_{X,x}$, so $dz_x \neq 0$ and z is an analytic local equation of D_α at x . By the first assertion, $g^* z = \phi^{\ell_\alpha}$ for a holomorphic function ϕ near y . Let $t \in \mathcal{O}_{Y,y}$ be a local equation of E , which is prime in the ring $\mathcal{O}_{Y,y}^{\text{an}}$ of germs of holomorphic functions at y , a unique factorization domain [Dem12, Chapter II, Theorem (2.10)], as E is smooth at y . Write $\phi = t^k \phi'$ with $t \nmid \phi' \in \mathcal{O}_{Y,y}^{\text{an}}$, so $g^* z = t^{\ell_\alpha k} \phi'^{\ell_\alpha}$ with $t \nmid \phi'^{\ell_\alpha}$. As $\mathcal{O}_{Y,y}^{\text{an}}$ is faithfully flat over $\mathcal{O}_{Y,y}$, the largest power of t dividing $g^* z$ is the same in both rings, so $\text{ord}_E(g^* D_\alpha) = \text{ord}_E(g^* z) = \ell_\alpha k$.

For (4), let $x \in U_0$ be a closed point. As A acts faithfully on S and $\mathbb{C}(S)^A = \mathbb{C}(X)$ by Lemma 5.1(1), $[\mathbb{C}(S) : \mathbb{C}(X)] = |A|$. As π is finite, flat, and unramified over U_0 [Har77, Exercise III.10.3], $\pi^{-1}(x)$ is reduced of length $|A|$ and an A -orbit by Lemma 5.1(1), so $\pi^{-1}(x) = As_1$ for any $s_1 \in \pi^{-1}(x)$, with $as_1 \neq s_1$ for $1 \neq a \in A$. As π is étale at s_1 and $\dim S = \dim X$, $d\pi_{s_1}$ is bijective [Har77, Proposition III.10.4], and as S is compact, there are analytic neighborhoods W_1 of s_1 and $W \subset U_0$ of x with $\pi|_{W_1} : W_1 \rightarrow W$ biholomorphic and $\pi^{-1}(W) = \coprod_{a \in A} aW_1$. Let $M := W_1 \times F$. Then $aM \cap M = \emptyset$ for $1 \neq a \in A$, and $AM = p^{-1}(g_0^{-1}(W))$ as $g_0 \circ p = \pi \circ \text{pr}_S$. By Lemma 5.1(2), $p(M) = g_0^{-1}(W)$ is open in Q and $p|_M : M \rightarrow g_0^{-1}(W)$ is a biholomorphism with $g_0 \circ p|_M = \pi \circ \text{pr}_{W_1}$, so $g_0^{-1}(W) \cong W \times F$ over W . In particular, $g_0^{-1}(U_0)$ is smooth and disjoint from $g_0^{-1}(\bigcup_\lambda P_\lambda)$, so ρ is an isomorphism over $g_0^{-1}(U_0) \subset O$. Thus $g|_{g^{-1}(U_0)}$ is a holomorphically locally trivial fiber bundle with fiber F , and g is smooth on $g^{-1}(U_0)$ as $W \times F \rightarrow W$ is a submersion. This finishes the proof. $\square$

7. PROOF OF PROPOSITION 1.2

Proof of Proposition 1.2. If $r = 0$, then $Y := X, g := \text{id}_X, U := X$, and a point F satisfy (1)–(4) and (1.2). Assume that $r \geq 1$. Let $N := n + 1$, and let $G_{i,j}, G, S, \pi : S \rightarrow X, A, e_{i,j}$, and ζ_i be as in Proposition 5.4.

Step 1. In this step, we construct F, Y , and g by Propositions 5.5 and 6.3.

Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ be elliptic curves, $t_i \in \mathcal{E}_i$ a point of order m_i , and F the abelian variety (5.16) with the action of A of Lemma 5.3(4). By Proposition 5.5(6), F satisfies the hypothesis of Proposition 5.5, $\dim F = \sum_{i=1}^r m_i \binom{N}{2}$, and $F^{e_{i,j}} \neq \emptyset$ for every i and j . Let $Q, p : S \times F \rightarrow Q, h : Q \rightarrow X, V$, and U be as in (5.13).

By Proposition 5.4(1), $D + G$ is a reduced simple normal crossing divisor. Hence the irreducible components $P_1, \dots, P_b$ of G are distinct prime divisors on X , none of them equals any D_i , and each P_λ is an irreducible component of a unique $G_{i,j}$. Fix λ and this (i, j) . By Proposition 5.4(5), $S^{e_{i,j}} = (\pi^{-1}(G_{i,j}))_{\text{red}}$ is a smooth divisor on S whose irreducible components dominate irreducible components of $G_{i,j}$. Since π is finite, it is closed [Har77, Exercise II.3.5(b)], so each such component maps onto an irreducible component of $G_{i,j}$, and these images cover $G_{i,j} = \pi(S^{e_{i,j}})$ as π is surjective. Hence $P_\lambda = \pi(T_\lambda)$ for some irreducible component T_λ of $S^{e_{i,j}}$, which is a prime divisor on S . Let $m_\lambda := \text{ord}_{T_\lambda}(\pi^* P_\lambda)$, $H_\lambda := \{a \in A \mid a|_{T_\lambda} = \text{id}_{T_\lambda}\}$, and $\mathfrak{m}_{T_\lambda}$ the maximal ideal of $\mathcal{O}_{S,T_\lambda}$. Since $G_{i,j}$ is reduced and P_λ lies in no other irreducible component of $G_{i,j}$, $m_\lambda = \text{ord}_{T_\lambda}(\pi^* G_{i,j})$. By Proposition 5.4(5), $m_\lambda = m_i \geq 2$, $H_\lambda = \langle e_{i,j} \rangle$ is cyclic of order m_i , and $e_{i,j}$ acts on $\mathfrak{m}_{T_\lambda} / \mathfrak{m}_{T_\lambda}^2$ by the primitive m_i th root of unity ζ_i , so H_λ acts faithfully on $\mathfrak{m}_{T_\lambda} / \mathfrak{m}_{T_\lambda}^2$. Let Z_λ be an irreducible component of $F^{H_\lambda} = F^{e_{i,j}} \neq \emptyset$.

By Proposition 5.4(2), the hypotheses of Proposition 6.3 hold for $X, S, A, \pi, F, P_1, \dots, P_b, T_\lambda, m_\lambda, H_\lambda$, and Z_λ , with $V_0 = S \times F$ and $g_0 = h$ in (6.2), and the open subset (6.3) is $O = Q_{\text{reg}} \setminus h^{-1}(G)$. By Proposition 6.3(2), there exist a smooth projective variety Y , a projective birational morphism $\rho : Y \rightarrow Q$ which is an isomorphism over O , and for each λ a prime divisor E_λ on Y and nonempty open subsets $E_\lambda^\circ \subset E_\lambda$ and $P_\lambda^\circ \subset P_\lambda$ such that $g := h \circ \rho$ is smooth at every point of E_λ° and $g(E_\lambda^\circ) \supset P_\lambda^\circ$. Since ρ is proper [Har77, Theorem II.4.9] and birational, it is surjective, so g is surjective by Proposition 5.5(1). Since $h^{-1}(V) = Q \setminus h^{-1}(G)$ is smooth by Proposition 5.5(3), $h^{-1}(V) \subset O$, and ρ is an isomorphism over $h^{-1}(V)$.

Step 2. In this step, we prove (1)–(4) and (1.2).

By **Step 1**, Proposition 5.5(5) applies to Y, ρ , and $g = h \circ \rho$. Hence $g^*D_i = m_i C_i$ for an effective divisor C_i on Y for every i , which is (1), and $g|_{g^{-1}(U)} : g^{-1}(U) \rightarrow U$ is a holomorphically locally trivial fiber bundle with fiber the abelian variety F . As $D \cup G \neq X$, the open subset $U \subset X \setminus D$ is nonempty. This proves (2), and (1.2) holds as $N = n + 1$.

Let P be a prime divisor on X with $P \neq D_i$ for every i . If $P = P_\lambda$ for some λ , let $P^\circ := P_\lambda^\circ$. By **Step 1**, every $x \in P^\circ$ equals $g(y)$ for some $y \in E_\lambda^\circ$, and g is smooth at y . Otherwise, P is not an irreducible component of $D + G$, so $P \not\subset D \cup G$, and $P^\circ := P \cap U$ is a nonempty open subset of P . By Proposition 5.4(3), π is étale over U , and $U \cap P_\lambda = \emptyset$ for every λ , so g is smooth at every point of $g^{-1}(U)$ by Proposition 6.3(4). Since g is surjective, $g^{-1}(x) \neq \emptyset$ for every $x \in P^\circ$. This proves (3).

Fix i , and let $D_i^\circ := D_i \setminus (\bigcup_{k \neq i} D_k \cup G)$. Since D_i lies in no other irreducible component of $D + G$, D_i° is a nonempty open subset of $D_i \setminus \bigcup_{k \neq i} D_k$. Let $x \in D_i^\circ$ and $y \in g^{-1}(x)$. Since $x \in V$, some local lift of g to the root chart at x is a submersion at y by Proposition 5.5(5). This proves (4), and the proposition follows. $\square$

8. FUNDAMENTAL GROUPS

In this section, we prove Proposition 8.3 and Theorem 1.1.

Definition 8.1. Let M be a smooth quasi-projective variety, Σ a reduced effective divisor on M , Σ' an irreducible component of Σ , and $y \in M \setminus \Sigma$. Let $\Delta \subset M$ be a closed holomorphic disc meeting Σ transversally in a single smooth point of Σ on Σ' , $\partial\Delta$ its positively oriented boundary circle, and β a path in $M \setminus \Sigma$ from y to a point of $\partial\Delta$. The loop $\beta \cdot \partial\Delta \cdot \beta^{-1}$ is a *positively oriented meridian* about Σ' at y . For $M = X$ and $\Sigma = D$ as in Definition 2.1, this is the notion of Definition 2.2.

Lemma 8.2. *Let M be a smooth quasi-projective variety.*

- (1) *Let $Z \subsetneq M$ be a closed subset in the Zariski topology and $y \in M \setminus Z$. Then $M \setminus Z$ is path connected and $\pi_1(M \setminus Z, y) \rightarrow \pi_1(M, y)$ is surjective.*
- (2) *Let Σ be a reduced effective divisor on M with irreducible components $\Sigma_1, \dots, \Sigma_k$, $y \in M \setminus \Sigma$, and μ_j a positively oriented meridian about Σ_j at y for each j . Then the kernel of $\pi_1(M \setminus \Sigma, y) \rightarrow \pi_1(M, y)$ is the normal subgroup generated by $\mu_1, \dots, \mu_k$.*
- (3) *In the situation of (2), any two positively oriented meridians about Σ_j at y are conjugate in $\pi_1(M \setminus \Sigma, y)$.*

Proof. The variety M is a connected complex manifold [Har77, Appendix B, Section 1]. Stratify Z into finitely many locally closed smooth subvarieties of real codimension at least 2. By transversality, every path in M with endpoints in $M \setminus Z$ is homotopic relative to its endpoints to a path transverse to all strata, hence disjoint from Z . This proves (1).

For (3), let Δ be as in Definition 8.1 for $\Sigma' = \Sigma_j$ and W a coordinate polydisc about $\Delta \cap \Sigma$ with $\Sigma \cap W = (z_1 = 0)$. Shrinking Δ , we may assume $\Delta \subset W$. Since z_1 has a single simple zero on Δ and $\pi_1(W \setminus \Sigma) \cong \mathbb{Z}$ via the winding number of z_1 , $\partial\Delta$ is freely homotopic in $W \setminus \Sigma$ to the boundary circle of $\{(z_1, 0, \dots, 0) \mid |z_1| \leq \epsilon\}$ for $0 < \epsilon \ll 1$. The smooth points of Σ on Σ_j form a nonempty open subset of the smooth locus of Σ_j , path connected by (1). Covering a path in it

between the centers of two such discs by finitely many polydiscs W , their boundary circles are freely homotopic in $M \setminus \Sigma$. This proves (3).

For (2), let N be the normal subgroup generated by $\mu_1, \dots, \mu_k$. Each μ_j bounds a disc in M , so N lies in the kernel. Conversely, let γ be a loop in $M \setminus \Sigma$ at y , null-homotopic in M , which we may take smooth, and $\phi : \mathbb{D} \rightarrow M$ a null-homotopy on the closed unit disc with $\phi(\rho e^{2\pi\sqrt{-1}\tau}) = \gamma(\tau)$ on the annulus $\mathbb{A} := \{1/2 \leq \rho \leq 1\}$, so that $\phi(\mathbb{A}) \cap \Sigma = \emptyset$. Since the singular locus Z' of Σ has real codimension at least 4, by smooth approximation and transversality relative to $\mathbb{A}$ we may assume that ϕ is smooth, $\phi(\mathbb{D}) \cap Z' = \emptyset$, and ϕ is transverse to $\Sigma \setminus Z'$, so that $\phi^{-1}(\Sigma) = \{p_1, \dots, p_l\}$ consists of interior points. Removing small discs $\mathbb{D}_s$ about the p_s , we write γ in $\pi_1(M \setminus \Sigma, y)$ as a product of conjugates of the loops $\phi(\partial\mathbb{D}_s)$ transported to y along arcs. In a coordinate polydisc W about $\phi(p_s)$ with $\Sigma \cap W = (z_1 = 0)$, $d(z_1 \circ \phi)_{p_s}$ is invertible, so $\phi(\partial\mathbb{D}_s)$ has winding number ± 1 for $\mathbb{D}_s$ small, hence is freely homotopic in $W \setminus \Sigma$ to the boundary circle of a coordinate disc $\{(z_1, a_2, \dots, a_d) \mid |z_1| \leq \epsilon\}$ or its inverse, so the transported loop is conjugate to a positively oriented meridian about the component of Σ through $\phi(p_s)$ or its inverse, hence to $\mu_j^{\pm 1}$ by (3). Thus $\gamma \in N$. $\square$

Proposition 8.3. *Let (X, B) be a projective orbifold pair as in Definition 2.1 and $g : Y \rightarrow X$ a surjective morphism from a smooth projective variety Y . Assume that there exists a nonempty open subset $U \subset X \setminus D$ such that g is smooth at every point of $g^{-1}(U)$ with connected fibers over U , and that for each i there exists an effective Weil divisor C_i on Y with $g^*D_i = m_i C_i$. Let $y_0 \in g^{-1}(U)$, $x_0 := g(y_0)$, Σ the reduced divisor $\text{Supp } g^*D$ on Y , $Y^0 := Y \setminus \Sigma$, $g^0 := g|_{Y^0} : Y^0 \rightarrow X \setminus D$, and $j : Y^0 \hookrightarrow Y$ the inclusion. Let γ_i be a positively oriented meridian about D_i at x_0 for each i , $\Gamma := \pi_1^{\text{orb}}(X, B; x_0)$ as in Definition 2.2, and $q : \pi_1(X \setminus D, x_0) \rightarrow \Gamma$ the quotient map. Then:*

(1) *There exists a unique surjective homomorphism $\Phi : \pi_1(Y, y_0) \rightarrow \Gamma$ such that*

$$(8.1) \quad \Phi \circ j_* = q \circ g_*^0.$$

(2) *Let E be an irreducible component of Σ , $e_{E,i} := \text{ord}_E(g^*D_i)$, $I_E := \{i \mid e_{E,i} > 0\}$, and μ_E a positively oriented meridian about E at y_0 . Then there exist pairwise commuting positively oriented meridians γ'_i about D_i at x_0 , $i \in I_E$, such that*

$$(8.2) \quad g_*^0(\mu_E) = \prod_{i \in I_E} (\gamma'_i)^{e_{E,i}}.$$

(3) *If $H \subset \pi_1(Y, y_0)$ is an abelian subgroup of finite index d , then $\Phi(H)$ is an abelian subgroup of Γ of index at most d .*

The proof follows [Ope26a, Proposition 3.1].

Proof. Since g is surjective, $g^*D = \sum_{i=1}^r m_i C_i$ is an effective divisor on Y and $\text{Supp } \Sigma = g^{-1}(D)$.

Step 1. In this step, we prove (2).

By Lemma 8.2(3), it suffices to prove (8.2) for one μ_E , since conjugating it by $g_*^0(\theta)$, $\theta \in \pi_1(Y^0, y_0)$, replaces the γ'_i by commuting positively oriented meridians about the D_i .

Since $E \not\subset g^{-1}(D_i)$ for $i \notin I_E$, there exists a smooth point $y \in E$ of Σ with $x := g(y) \in D_i$ if and only if $i \in I_E$. Reordering the D_i , we may assume $I_E = \{1, \dots, s\}$. Let W be an adapted coordinate neighborhood of x with coordinates $z_1, \dots, z_n$ (Definition 2.3) and $V \subset g^{-1}(W)$ a coordinate polydisc about y with coordinates $t, t_2, \dots, t_{\dim Y}$ and $\Sigma \cap V = (t = 0)$. For $i \in I_E$, the divisor of $z_i \circ g$ on V is $e_{E,i}(E \cap V)$, so after shrinking V we have

$$(8.3) \quad z_i \circ g = t^{e_{E,i}} u_i \quad (i \in I_E)$$

with u_i nowhere vanishing on V .

Let $\Delta := \{(t, 0, \dots, 0) \mid |t| \leq \epsilon\} \subset V$ for $0 < \epsilon \ll 1$, $\delta(\tau) := (\epsilon e^{2\pi\sqrt{-1}\tau}, 0, \dots, 0)$ its boundary circle, β_Y a path in Y^0 from y_0 to $y' := \delta(0)$, and $\mu_E := \beta_Y \cdot \delta \cdot \beta_Y^{-1}$, a positively oriented meridian about E . Let $w := g(y') = (w_1, \dots, w_n)$ and $\beta := g \circ \beta_Y$. As $W \setminus D$ is a product of punctured discs and discs, $\pi_1(W \setminus D, w)$ is free abelian on the classes λ_i , $i \in I_E$, of the

loops $\tau \rightarrow (w_1, \dots, e^{2\pi\sqrt{-1}\tau} w_i, \dots, w_n)$, a loop c having class $\prod_{i \in I_E} \lambda_i^{n_i(c)}$, $n_i(c)$ the winding number of $z_i \circ c$. By (8.3), $z_i \circ g \circ \delta(\tau) = e^{e_{E,i}} e^{2\pi\sqrt{-1}e_{E,i}\tau} u_i(\delta(\tau))$ with u_i nowhere zero on Δ , so $n_i(g \circ \delta) = e_{E,i}$ and in $\pi_1(X \setminus D, x_0)$

$$g_*^0(\mu_E) = [\beta \cdot (g \circ \delta) \cdot \beta^{-1}] = \prod_{i \in I_E} (\gamma'_i)^{e_{E,i}}, \quad \gamma'_i := [\beta \cdot \lambda_i \cdot \beta^{-1}].$$

The γ'_i commute as the λ_i do, and are positively oriented meridians about D_i at x_0 , as λ_i is the oriented boundary of the closed holomorphic disc $\{(w_1, \dots, z_i, \dots, w_n) \mid |z_i| \leq |w_i|\} \subset W$, which meets D transversally in one point of $D_i \setminus \bigcup_{k \neq i} D_k$. This proves (2).

Step 2. In this step, we prove (1) and (3).

By Lemma 8.2(1) and (2) for $M = Y$ and $Z = \Sigma$, j_* is surjective with kernel N the normal subgroup generated by positively oriented meridians $\mu_{E_1}, \dots, \mu_{E_k}$ at y_0 about the irreducible components $E_1, \dots, E_k$ of Σ . For $E := E_{k'}$ and γ'_i as in (2), $e_{E,i} = m_i \operatorname{ord}_E(C_i)$ as $g^* D_i = m_i C_i$, and $(\gamma'_i)^{m_i}$ is conjugate to $\gamma_i^{m_i}$ by Lemma 8.2(3) for $M = X$ and $\Sigma = D$, so $q \circ g_*^0(\mu_E) = 1$ by (8.2). Thus $q \circ g_*^0$ factors through $\pi_1(Y, y_0) = \pi_1(Y^0, y_0)/N$, giving Φ with (8.1), unique because j_* is surjective.

As q is surjective, by (8.1) it suffices that g_*^0 is surjective, and by Lemma 8.2(1) for $M = X \setminus D$, $Z = (X \setminus D) \setminus U$ and $g^{-1}(U) \subset Y^0$, that $g_U := g|_{g^{-1}(U)}$, a proper submersion with connected fibers, induces a surjection $\pi_1(g^{-1}(U), y_0) \rightarrow \pi_1(U, x_0)$. Let $\mathcal{H}$ be the orthogonal complement of $\ker(dg_U)$ for a Riemannian metric on $g^{-1}(U)$, so that $dg_U : \mathcal{H}_y \rightarrow T_{g(y)}U$ is an isomorphism. Let $c : [0, 1] \rightarrow U$ be a smooth loop at x_0 . Let ξ be a smooth time-dependent vector field on U with $\xi(\tau, c(\tau)) = c'(\tau)$, obtained by extending $c'(\tau)$ in finitely many charts covering $c([0, 1])$ and gluing by a partition of unity. Let $\tilde{c}$ be the maximal integral curve with $\tilde{c}(0) = y_0$ of $\tilde{\xi}(\tau, y) := (dg_U|_{\mathcal{H}_y})^{-1}\xi(\tau, g(y))$. Then $g \circ \tilde{c}$ and c are integral curves of ξ from x_0 , so $g \circ \tilde{c} = c$, and $\tilde{c}$ is defined on $[0, 1]$ as it stays in the compact set $g_U^{-1}(c([0, 1]))$. For a path σ in the connected manifold $g^{-1}(x_0)$ from $\tilde{c}(1)$ to y_0 , $g \circ (\tilde{c} \cdot \sigma)$ is homotopic to c . As every loop in U is homotopic to a smooth one, g_{U*} is surjective. This proves (1).

The group $\Phi(H)$ is abelian, and as Φ is surjective, $kH \rightarrow \Phi(k)\Phi(H)$ maps $\pi_1(Y, y_0)/H$ onto $\Gamma/\Phi(H)$, so $[\Gamma : \Phi(H)] \leq d$. This proves (3). $\square$

Proof of Theorem 1.1. If $\dim X = 0$, then $r = 0$ and $\pi_1^{\text{orb}}(X, B; x_0)$ is trivial. Assume that $\dim X \geq 1$. Since (X, B) is special, $\mathcal{R}$ is special by Proposition 3.2(4). If $r = 0$, then $\mathcal{R} = X$ and $\pi_1^{\text{orb}}(X, B; x_0) = \pi_1(X, x_0)$ is virtually abelian by Theorem 2.11, as X is a connected compact Kähler manifold.

Assume that $r \geq 1$, and let $g : Y \rightarrow X$, C_i , U , and F be as in Proposition 1.2. Then g satisfies hypotheses (1), (3), and (4) of Proposition 4.3 by Proposition 1.2(1), (3), and (4), and hypothesis (2) by Proposition 1.2(2), as g is smooth at every point of $g^{-1}(U)$ and the fibers of g over U are biholomorphic to the abelian variety F , connected with Ω_F^1 trivialized by translations of a basis of T_0^*F . Thus Y is special by Proposition 4.3, and for $y_0 \in g^{-1}(U)$, Theorem 2.11 gives an abelian subgroup $H \subset \pi_1(Y, y_0)$ of finite index d . By Proposition 1.2(1) and (2), Proposition 8.3(3) applies with $x'_0 := g(y_0)$, so $\Phi(H) \subset \pi_1^{\text{orb}}(X, B; x'_0)$ is an abelian subgroup of index at most d .

Finally, let σ be a path in $X \setminus D$ from x'_0 to x_0 (Lemma 8.2(1)). Conjugation by σ is an isomorphism $\pi_1(X \setminus D, x'_0) \rightarrow \pi_1(X \setminus D, x_0)$ mapping positively oriented meridians about D_i at x'_0 to ones at x_0 , conjugate to γ_i by Lemma 8.2(3), hence the normal subgroup defining $\pi_1^{\text{orb}}(X, B; x'_0)$ onto $\langle \gamma_i^{m_i} \mid 1 \leq i \leq r \rangle$. Thus $\pi_1^{\text{orb}}(X, B; x_0) \cong \pi_1^{\text{orb}}(X, B; x'_0)$ is virtually abelian. $\square$

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