

UPPER BOUND OF MINIMAL LOG DISCREPANCIES

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ABSTRACT. We show that the minimal log discrepancy of any variety of fixed dimension has an upper bound.

We work over the field of complex numbers $\mathbb{C}$. The goal of this paper is to prove the conjecture on the upper bound of mlds ([Kaw11, Problem 1.2], [Nak16, Conjecture 1.4]), a weaker version of [Sho88, Problem 5a]), provided that [OAI26, Proposition 2.1] is correct. In particular the finiteness of mlds holds for pairs with finite coefficients and bounded Gorenstein indices [Nak16, Corollary 1.3].

Lemma 0.1. *Let d be a positive integer. Let X be a d -dimensional Gorenstein variety and $x \in X$ a closed point. Then $\text{lct}(X; \mathfrak{m}_x) \leq d$.*

Proof. Since $\dim X = d$ there exists $\mathfrak{q} = (f_1, \dots, f_d) \subset \mathfrak{m}_x$ such that $\mathfrak{m}_x^{k+1} = \mathfrak{q}\mathfrak{m}_x^k$ for any $k \gg 0$. By [LLS08, Corollary 1.4(i)] applied to $\Delta = 0, \mathfrak{c} = \mathfrak{m}_x, \mathfrak{b} = \mathcal{O}_{X,x}, \lambda = 1$, and $m = r = d$, we have

$$\mathcal{J}(X; \mathfrak{m}_x^d) \subset \mathfrak{q} \cdot \mathcal{J}(X; \mathfrak{m}_x^{d-1}) \subset \mathfrak{q} \subset \mathfrak{m} \subsetneq \mathcal{O}_X.$$

Thus $(X, \mathfrak{m}_x^d)$ is not klt. Thus lemma follows. $\square$

Kawakita suggested that Lemma 0.1 may also follow from [Kol+92, Theorem 18.22].

Theorem 0.2. *Let d be a positive integer. Then there exists a positive integer $N = N(d)$ depending only on d satisfying the following: for any d -dimensional pair (X, B) and any point $x \in X$, we have $\text{mld}(X \ni x, B) \leq N(d)$.*

Proof. Let $f : Y \rightarrow X$ be a $\mathbb{Q}$ -factorial dlt modification of (X, B) and let $B_Y := f_*^{-1}(B \wedge \text{Supp } B) + \text{Exc}(f)$. Possibly replacing (X, B) with (Y, B_Y) , we may assume that (X, B) is $\mathbb{Q}$ -factorial dlt. Possibly replacing B with 0, we may assume that (X, B) is $\mathbb{Q}$ -factorial klt. Possibly replacing (X, B) with a $\mathbb{Q}$ -factorial terminalization, we may assume that X is terminal. Possibly replacing B with 0, we may assume that $B = 0$ and X is $\mathbb{Q}$ -factorial terminal. Possibly cutting X by hyperplane sections and induction on dimension, we only need to show that for any closed point $x \in X$, $\text{mld}(X \ni x) \leq N(d)$ for some $N(d)$. In the following we fix such a closed point x . Let $h : W \rightarrow X$ be a resolution of X such that

$$\mathfrak{m}_x \mathcal{O}_W = \mathcal{O}_W(-F), \quad F = \sum b_i F_i$$

where $F_i \geq 0$ and $\text{Supp } F = h^{-1}(x)$. Let $a_i := a(F_i, X) > 1$ for each i and let $t := \min_i \frac{a_i - \frac{1}{2}}{b_i}$. Possibly switching indices, we may assume that $t = \frac{a_1 - \frac{1}{2}}{b_1}$. Let $E := F_1$, then we have $a_i - tb_i \geq \frac{1}{2}$ for any i and $a(E, X) = \frac{1}{2} + t \text{mult}_E \mathfrak{m}_x$. By Lemma 0.1, we have

$$t < \min_i \frac{a_i}{b_i} = \text{lct}(X, \mathfrak{m}_x) \leq d,$$

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hence $0 < t < d$. Possibly shrinking X to an affine neighborhood of x and picking general elements $H_1, \dots, H_{d+1} \in |\mathfrak{m}_x|$, we let $\Delta := \frac{t}{d+1} \sum_{i=1}^{d+1} H_i$ and consider (X, Δ) . Then we have $h^* H_i = F + h_*^{-1} H_i$ for any i . Since H_i are general, $(W, \text{Exc}(h) + \sum_{i=1}^{d+1} h_*^{-1} H_i)$ is log smooth. For any h -exceptional prime divisor D , we have $a(D, X, \Delta) = a(D, X) > 1$ if $\text{center}_X D \neq x$ and

$$a(F_i, X, \Delta) = a(F_i, X) - t \text{mult}_{F_i} F = a_i - tb_i = \frac{1}{2} + (a_i - \frac{1}{2}) - tb_i \geq \frac{1}{2}.$$

for any i . In particular,

$$a(E, X, \Delta) = a_1 - \frac{a_1 - \frac{1}{2}}{b_1} b_1 = \frac{1}{2}.$$

By [BCHM10, Corollary 1.4.3], there exists an extraction $g : Z \rightarrow X$ of E . Let $E_Z := \text{center}_Z E$. Since X is $\mathbb{Q}$ -factorial, Z is $\mathbb{Q}$ -factorial and $-E_Z$ is ample/ X . Thus $-K_Z$ is ample/ X . We have that (X, Δ) is $\frac{1}{d+1}$ -lc, so $(Z, \frac{1}{2}E + g_*^{-1}\Delta)$ is $\frac{1}{d+1}$ -lc, and so Z is $\frac{1}{d+1}$ -lc. By [OAI26, Proposition 2.1], there exists a real number $c = c(d, \frac{1}{d+1})$ depending only on d , such that $\text{let}(Z, \mathfrak{m}_x \mathcal{O}_Z) \geq c(d, \frac{1}{d+1})$.

This implies that $c(d, \frac{1}{d+1}) \text{mult}_E \mathfrak{m}_x \leq 1$, hence $\text{mult}_E \mathfrak{m}_x \leq \frac{1}{c(d, \frac{1}{d+1})}$. We have

$$\text{mld}(X \ni x) \leq a(E, X) = a(E, X, \Delta) + \text{mult}_E \Delta = \frac{1}{2} + \frac{t}{d+1} \cdot (d+1) \text{mult}_E \mathfrak{m}_x < \frac{1}{2} + \frac{d}{c(d, \frac{1}{d+1})}.$$

The theorem follows. $\square$

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