# SYZYGY BUNDLES ON PICARD RANK ONE VARIETIES NEED NOT BE STABLE

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Abstract. We construct a smooth complex projective threefold $X$ of Picard rank 1 and a nontrivial globally generated line bundle $L$ on $X$ whose syzygy bundle $M_L$ is not semistable with respect to any polarization. This answers a question of Sarkar in the negative. The main result of this paper is obtained by generative AI, particularly ChatGPT 5.5 Pro and the Rethlas system.

## 1. Introduction

Let $X$ be a smooth projective variety over an algebraically closed field of characteristic zero and let $L$ be a globally generated line bundle on $X$ with $h^0(X, L) > 0$. The *syzygy bundle* $M_L$ is the kernel of the evaluation map $H^0(X, L) \otimes \mathcal{O}_X \to L$. Whether $M_L$ is stable with respect to a given polarization has been studied for a long time. For a curve of genus $g \geq 1$, the bundle $M_L$ is stable whenever $\deg L \geq 2g + 1$ [EL92], the bound was later improved in terms of the Clifford index, and examples of unstable $M_L$ on curves exist [Cam08]. In higher dimensions, Ein, Lazarsfeld, and Mustopa conjectured that $M_L$ is slope-stable whenever $L$ is sufficiently ample [ELM13], and this conjecture was proved by Rekuski [Rek24].

The question is sharpest when $X$ has Picard rank 1, since then the choice of the polarization is immaterial. Affirmative answers are known in several cases, including Enriques and bielliptic surfaces, abelian varieties, and other classes of varieties of Picard number one [MR22, CL21, JR25]. Prior to this paper, no example of a nontrivial globally generated line bundle $L$ on a smooth projective variety of Picard rank 1 and dimension $\geq 2$ with $M_L$ not stable seems to have been known. Sarkar raised the following question [Sar26, Question 1.1], which had appeared in a slightly different form as part of [FL22, Question 2] (see also [Liu26]).

**Question 1.1** ([Sar26, Question 1.1])**.** Let $X$ be a smooth projective variety of Picard rank 1 and dimension $\geq 2$, and let $L$ be a nontrivial globally generated line bundle on $X$. Is the syzygy bundle $M_L$ stable?

Sarkar proved that the answer is yes for a large class of varieties: if every ample line bundle on $X$ is globally generated, then $M_L$ is stable for every ample line bundle $L$ on $X$ [Sar26, Theorem A]; this applies in particular to complete intersections in projective space. The main result of this paper is that the answer to Question 1.1 in general is no.

**Theorem 1.2.** *There exist a smooth complex projective threefold $X$ with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ for an ample line bundle $H$, and a globally generated line bundle $L$ on $X$ with $L \not\cong \mathcal{O}_X$, such that the syzygy bundle*

$$M_L = \ker\big(H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L\big)$$

*is not slope-semistable, and in particular not slope-stable, with respect to any ample line bundle on $X$.*

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*One may take $X$ to be a general weighted complete intersection $X_{18,18,18,18} \subset \mathbb{P}(2,2,2,2,9,9,9,9)$, $H = \mathcal{O}_X(1)$, and $L = 11H$; the subbundle $M_{2H} \otimes_{\mathbb{C}} H^0(X, 9H) \cong M_{2H}^{\oplus 4}$ of $M_{11H}$ is then a destabilizing subbundle.*

We now explain how the example works. Write $W = \mathbb{P}(2,2,2,2,9,9,9,9)$, with coordinates $x_0, \dots, x_3$ of weight 2 and $z_0, \dots, z_3$ of weight 9. Since $\gcd(2,9) = 1$, the singular locus of $W$ is the disjoint union of the two coordinate strata $\{z_0 = \cdots = z_3 = 0\}$ and $\{x_0 = \cdots = x_3 = 0\}$, and a general complete intersection $X$ of four hypersurfaces of weighted degree 18 misses both strata, so $X$ is a smooth projective threefold carrying an ample line bundle $H = \mathcal{O}_X(1)$ with $H^3 = 1$. The affine cone over $X$ is a four-dimensional complete intersection with an isolated singularity, so Grothendieck's parafactoriality theorem forces $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Since the defining equations have degree 18, the sections of $mH$ for $m \leq 17$ are exactly the weighted degree-$m$ elements of the coordinate ring: $h^0(X, H) = 0$, the spaces $H^0(X, 2H)$ and $H^0(X, 9H)$ are spanned by the $x_i$ and the $z_j$ respectively, and the multiplication map

$$H^0(X, 2H) \otimes_{\mathbb{C}} H^0(X, 9H) \longrightarrow H^0(X, 11H)$$

is an isomorphism of 16-dimensional vector spaces. This multiplication isomorphism places four copies of $M_{2H}$ inside $M_{11H}$ as a subbundle $F$ with locally free quotient, and with respect to $H$ we have

$$\mu_H(F) = \frac{-8}{12} = -\frac{2}{3} > -\frac{11}{15} = \mu_H(M_{11H}),$$

so $M_{11H}$ is not semistable. Since $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, replacing $H$ by any other polarization rescales all slopes by the same positive factor, and the failure of semistability persists.

**Remark 1.3.** The example is consistent with [Sar26, Theorem A]: we have $h^0(X, H) = 0$, so the ample line bundle $H$ is not globally generated and the hypothesis of that theorem fails on $X$. It also shows that this hypothesis cannot be removed from the statement. The bundle $M_L$ in Theorem 1.2 is moreover not Gieseker-semistable with respect to any polarization; see Remark 6.4. Our example has dimension three, and we do not know whether the answer to Question 1.1 is negative for surfaces as well.

**Remark 1.4.** The main result of this paper was obtained using generative AI, particularly ChatGPT 5.5 Pro and the Rethlas system. The counterexample was proposed by ChatGPT 5.5 Pro, and every step of the proof presented below was then re-proved from scratch within the Rethlas system, whose automated verification layer was used as an adversarial check on each step. See [Ju+26] for a detailed introduction to the Rethlas system. Due to the limitation of generative AI, it is possible that we have missed some related references in the literature, and we welcome any comments from experts.

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## 2. Preliminaries

We work over the field of complex numbers $\mathbb{C}$. A *variety* is an integral scheme of finite type over $\mathbb{C}$. For general background on weighted projective spaces we refer to [Dol82, IF00], and for general background on slope stability to [HL10].

**Set-up 2.1.** Throughout this paper, $S = \mathbb{C}[x_0, x_1, x_2, x_3, z_0, z_1, z_2, z_3]$ denotes the polynomial ring graded by

$$\deg x_i = 2 \ (0 \leq i \leq 3), \qquad \deg z_j = 9 \ (0 \leq j \leq 3),$$

$S_d$ denotes the degree-$d$ graded piece, $S_+ = \bigoplus_{d>0} S_d$, and

$$W := \operatorname{Proj} S = \mathbb{P}(2, 2, 2, 2, 9, 9, 9, 9).$$

We write $\mathbb{P}^3_x \subset W$ for the closed stratum $\{z_0 = z_1 = z_2 = z_3 = 0\}$ and $\mathbb{P}^3_z \subset W$ for the closed stratum $\{x_0 = x_1 = x_2 = x_3 = 0\}$; each is a copy of $\mathbb{P}^3$, and they are disjoint. For $m \in \mathbb{Z}$, $\mathcal{O}_W(m)$ denotes the Serre twist, that is, the coherent sheaf associated with the graded $S$-module $S(m)$.

Given a quadruple $f = (f_1, f_2, f_3, f_4) \in (S_{18})^4$, we write

$$R := S/(f_1, f_2, f_3, f_4), \qquad X := \operatorname{Proj} R \subset W, \qquad C := \operatorname{Spec} R,$$

$\mathfrak{m} := R_+ \subset R$ for the irrelevant maximal ideal, and $U := C \setminus \{\mathfrak{m}\}$ for the punctured affine cone. For $m \in \mathbb{Z}$, $\mathcal{O}_X(m)$ denotes the coherent sheaf associated with the graded $R$-module $R(m)$. Following [IF00, Definition 6.3], we say that $X$ is *quasi-smooth* if $U$ is smooth.

**Definition 2.2.** Let $Y$ be a projective scheme over $\mathbb{C}$ and let $N$ be a globally generated line bundle on $Y$. The *syzygy bundle* of $N$ is

$$M_N := \ker\bigl(\mathrm{ev}_N \colon H^0(Y, N) \otimes_{\mathbb{C}} \mathcal{O}_Y \to N\bigr),$$

where $\mathrm{ev}_N$ is the evaluation map. Since $N$ is globally generated, $\mathrm{ev}_N$ is surjective.

**Definition 2.3.** Let $Y$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$, let $A$ be an ample line bundle on $Y$, and let $E$ be a nonzero torsion-free coherent sheaf on $Y$. The *slope* of $E$ with respect to $A$ is

$$\mu_A(E) := \frac{c_1(E) \cdot c_1(A)^{n-1}}{\operatorname{rank}(E)}.$$

We say that $E$ is *slope-semistable* (resp. *slope-stable*) with respect to $A$ if for every coherent subsheaf $F \subset E$ with $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$ we have $\mu_A(F) \le \mu_A(E)$ (resp. $\mu_A(F) < \mu_A(E)$). In this paper, "(semi)stable" means "slope-(semi)stable" unless stated otherwise.

## 3. The threefold

In this section, we construct the threefold of Theorem 1.2 and prove its basic properties: smoothness, irreducibility, the structure of the punctured affine cone, and the polarization.

We begin with two lemmas that will be used repeatedly. The first one lets us pass between a graded ring with an invertible degree-one element and its degree-zero part.

**Lemma 3.1.** *Let $A$ be a commutative ring such that the Laurent polynomial ring $B := A[t, t^{-1}]$ is a finitely generated $\mathbb{C}$-algebra, let $\mathfrak{p} \subset A$ be a prime ideal, and let $\mathfrak{q} := \mathfrak{p}B + (t-1)$. Then $\mathfrak{q}$ is a prime ideal of $B$ with $\mathfrak{q} \cap A = \mathfrak{p}$, and:*

1. *$B_{\mathfrak{q}}/(t-1)B_{\mathfrak{q}} \cong A_{\mathfrak{p}}$ and $\dim B_{\mathfrak{q}} = \dim A_{\mathfrak{p}} + 1$;*
2. *if $B_{\mathfrak{q}}$ is a regular local ring, then $A_{\mathfrak{p}}$ is a regular local ring.*

*Proof.* We first note that $A$ is then itself a finitely generated $\mathbb{C}$-algebra: writing finitely many $\mathbb{C}$-algebra generators of $B$ as Laurent polynomials in $t$, the subalgebra $A' \subset A$ generated by all their coefficients satisfies $A'[t, t^{-1}] = B$, and comparing coefficients of $t^0$ gives $A' = A$. In particular $\operatorname{Spec} B \to \operatorname{Spec} A$ is a flat morphism of schemes of finite type over $\mathbb{C}$.

Evaluation at $t = 1$ defines a surjection $B \to A$ with kernel $(t-1)$, under which $\mathfrak{q}$ is the preimage of $\mathfrak{p}$; hence $B/\mathfrak{q} \cong A/\mathfrak{p}$ is a domain, $\mathfrak{q}$ is prime, and $\mathfrak{q} \cap A = \mathfrak{p}$. Localizing gives $B_{\mathfrak{q}}/(t-1)B_{\mathfrak{q}} \cong A_{\mathfrak{p}}$.

The inclusion $A \to B$ is flat, so the induced local homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is flat. By the dimension formula for flat morphisms [Har77, Proposition III.9.5], $\dim B_{\mathfrak{q}} - \dim A_{\mathfrak{p}}$ equals the dimension at $\mathfrak{q}$ of the fiber $\operatorname{Spec} \kappa(\mathfrak{p})[t, t^{-1}]$, which equals 1 since the point $\mathfrak{q}$ of the fiber is cut out by $t-1$, a nonzero nonunit in the principal ideal domain $\kappa(\mathfrak{p})[t, t^{-1}]$. This proves (1).

For (2), we first check that $t-1 \notin \mathfrak{q}^2 B_{\mathfrak{q}}$. The partial derivative $\partial_t \colon B \to B$ is $A$-linear and satisfies the Leibniz rule, so for $a, b \in \mathfrak{q}$ we have $\partial_t(ab) = a\,\partial_t b + b\,\partial_t a \in \mathfrak{q}$, and by the quotient rule the induced derivation on $B_{\mathfrak{q}}$ maps $\mathfrak{q}^2 B_{\mathfrak{q}}$ into $\mathfrak{q} B_{\mathfrak{q}}$; on the other hand $\partial_t(t-1) = 1 \notin \mathfrak{q} B_{\mathfrak{q}}$. Hence $t-1 \in \mathfrak{q} B_{\mathfrak{q}} \setminus \mathfrak{q}^2 B_{\mathfrak{q}}$. Writing $\mathfrak{n} := \mathfrak{q} B_{\mathfrak{q}}$ and $\bar{\mathfrak{n}}$ for the maximal ideal of $A_{\mathfrak{p}} = B_{\mathfrak{q}}/(t-1)$, we get

$$\dim_\kappa \bar{\mathfrak{n}}/\bar{\mathfrak{n}}^2 = \dim_\kappa \mathfrak{n}/\big(\mathfrak{n}^2 + (t-1)\big) = \dim_\kappa \mathfrak{n}/\mathfrak{n}^2 - 1 = \dim B_{\mathfrak{q}} - 1 = \dim A_{\mathfrak{p}},$$

where $\kappa$ is the common residue field, the second equality holds because the image of $t-1$ in $\mathfrak{n}/\mathfrak{n}^2$ is nonzero, and the third holds because $B_{\mathfrak{q}}$ is regular. Hence $A_{\mathfrak{p}}$ is regular. □

The second lemma collects graded miscellanea.

**Lemma 3.2.** *Let $R$ be a $\mathbb{Z}_{\geq 0}$-graded Noetherian ring.*

(1) *Every minimal prime over a homogeneous ideal of $R$ is homogeneous.*

*Assume moreover that $R$ is a domain with $R_0 = \mathbb{C}$, and let $\mathfrak{m} := R_+$ be the irrelevant maximal ideal.*

(2) *Every unit of $R$ lies in $\mathbb{C}^\times$.*
(3) *Let $M$ be a finitely generated graded $R$-module. If $M = \mathfrak{m}M$ then $M = 0$. Consequently $M$ can be generated by $\dim_{\mathbb{C}} M/\mathfrak{m}M$ homogeneous elements, and $\dim_{\mathbb{C}} M/\mathfrak{m}M$ is at most the minimal number of generators of the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$.*

*Proof.* (1) Let $J$ be a homogeneous ideal and $\mathfrak{p}$ a minimal prime over $J$. Let $\mathfrak{p}^* \subset \mathfrak{p}$ be the ideal generated by all homogeneous elements of $\mathfrak{p}$; it is homogeneous and contains $J$. It suffices to prove that $\mathfrak{p}^*$ is prime, since then $\mathfrak{p} = \mathfrak{p}^*$ by minimality. Suppose not; then there are $a, b \notin \mathfrak{p}^*$ with $ab \in \mathfrak{p}^*$, chosen so that the total number of homogeneous components of $a$ and $b$ is minimal. Let $a_{\mathrm{top}}$ and $b_{\mathrm{top}}$ be the top-degree components of $a$ and $b$. If $a_{\mathrm{top}} \in \mathfrak{p}^*$, then $(a - a_{\mathrm{top}})b = ab - a_{\mathrm{top}}b \in \mathfrak{p}^*$ while $a - a_{\mathrm{top}} \notin \mathfrak{p}^*$, contradicting minimality; so $a_{\mathrm{top}} \notin \mathfrak{p}^*$, and likewise $b_{\mathrm{top}} \notin \mathfrak{p}^*$. The top-degree component of $ab$ is $a_{\mathrm{top}}b_{\mathrm{top}}$, and the components of the element $ab$ of the homogeneous ideal $\mathfrak{p}^*$ lie in $\mathfrak{p}^*$; hence $a_{\mathrm{top}}b_{\mathrm{top}} \in \mathfrak{p}^* \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, one of $a_{\mathrm{top}}, b_{\mathrm{top}}$ lies in $\mathfrak{p}$; being homogeneous, it lies in $\mathfrak{p}^*$, a contradiction.

(2) If $uv = 1$, then in the domain $R$ the product of the top-degree components of $u$ and $v$ is nonzero of degree $\deg u_{\mathrm{top}} + \deg v_{\mathrm{top}}$, which forces $\deg u_{\mathrm{top}} = \deg v_{\mathrm{top}} = 0$, that is, $u \in R_0 = \mathbb{C}$.

(3) If $M \neq 0$, let $d$ be minimal with $M_d \neq 0$; then $(\mathfrak{m}M)_d$ is spanned by products of elements of positive degree with elements of $M$ of degree less than $d$, hence is zero, contradicting $M = \mathfrak{m}M$. For the second statement, let $\bar{m}_1, \ldots, \bar{m}_r$ be a $\mathbb{C}$-basis of $M/\mathfrak{m}M$ consisting of images of homogeneous elements $m_1, \ldots, m_r$ of $M$ (possible since $M/\mathfrak{m}M$ is a graded vector space), and let $M' \subset M$ be the graded submodule they generate. Then $M/M' = \mathfrak{m}(M/M')$, so $M = M'$ by the first statement, and $r = \dim_{\mathbb{C}} M/\mathfrak{m}M$. For the last statement, note that $M/\mathfrak{m}M$ is a module over $R/\mathfrak{m} = \mathbb{C}$ on which every element of $R \setminus \mathfrak{m}$ acts through its nonzero constant term, hence invertibly; therefore the localization map $M/\mathfrak{m}M \to M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}}$ is an isomorphism of $\mathbb{C}$-vector spaces, and any generating set of $M_{\mathfrak{m}}$ spans $M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}}$. □

We now describe the ambient space $W$. For $0 \leq i, j \leq 3$, let $D_+(x_i z_j) \subset W$ be the standard open subset where the weighted homogeneous element $x_i z_j$ of degree 11 is invertible, and set

$$t_{ij} := \frac{x_i^5}{z_j}, \tag{3.1}$$

a homogeneous element of degree $5 \cdot 2 - 9 = 1$ that is a unit on $D_+(x_i z_j)$.

**Lemma 3.3.** *With the notation of Set-up 2.1, the following statements hold.*

(1) *The open subsets $D_+(x_i z_j)$, $0 \leq i, j \leq 3$, cover $W \setminus (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z)$, and this locus is smooth. For every $m \in \mathbb{Z}$, the restriction of $\mathcal{O}_W(m)$ to $W \setminus (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z)$ is invertible, and the multiplication maps induce isomorphisms $\mathcal{O}_W(a) \otimes \mathcal{O}_W(b) \cong \mathcal{O}_W(a+b)$ on this locus for all $a, b \in \mathbb{Z}$.*

(2) $\operatorname{Sing} W = \mathbb{P}^3_x \sqcup \mathbb{P}^3_z$.
(3) *The Veronese subring* $S^{(18)} := \bigoplus_{n\geq 0} S_{18n}$ *is generated as a* $\mathbb{C}$*-algebra by* $S_{18}$*, the natural morphism* $W \to \operatorname{Proj} S^{(18)}$ *is an isomorphism, and* $W$ *embeds as a closed subscheme of a projective space* $\mathbb{P}^N$ *in such a way that* $\mathcal{O}_W(18)$ *is identified with the restriction of* $\mathcal{O}_{\mathbb{P}^N}(1)$*. In particular* $W$ *is projective and* $\mathcal{O}_W(18)$ *is a very ample line bundle on* $W$*.*

*Proof.* (1) A point of $W$ outside $\mathbb{P}^3_x \sqcup \mathbb{P}^3_z$ has some $x_i \neq 0$ and some $z_j \neq 0$, hence lies in $D_+(x_i z_j)$; this proves the covering claim. Fix $i, j$ and write $B := S[(x_i z_j)^{-1}]$, a $\mathbb{Z}$-graded ring, and $A := B_0$, so that $D_+(x_i z_j) = \operatorname{Spec} A$. The element $t := t_{ij} \in B_1$ of (3.1) is a unit. Every homogeneous $g \in B_d$ satisfies $g = t^d \cdot (g/t^d)$ with $g/t^d \in A$, and in a relation $\sum_d a_d t^d = 0$ with $a_d \in A$ all summands vanish because they are homogeneous of pairwise distinct degrees. Therefore

$$B = A[t, t^{-1}], \tag{3.2}$$

a Laurent polynomial ring over $A$. The ring $B$ is the localization of the polynomial ring $S$ at the single element $x_i z_j$; it is therefore a regular ring and a finitely generated $\mathbb{C}$-algebra, so $A_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p} \subset A$ by Lemma 3.1(2). Hence $D_+(x_i z_j)$ is smooth, and so is $W \setminus (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z)$.

For the twists, the restriction of $\mathcal{O}_W(m)$ to $D_+(x_i z_j)$ is the sheaf associated with the $A$-module $B_m = t^m \cdot A$, which is free of rank one, and the multiplication maps $B_a \otimes_A B_b \to B_{a+b}$ send $t^a \otimes t^b$ to $t^{a+b}$, hence are isomorphisms of free rank-one modules.

(2) By (1), it suffices to show that every point of $\mathbb{P}^3_x \sqcup \mathbb{P}^3_z$ is singular. Let $p \in \mathbb{P}^3_x$; after renumbering, $p \in D_+(x_0)$, and $D_+(x_0) = \operatorname{Spec} A'$ with $A' = (S[x_0^{-1}])_0$. A monomial $x^a z^b x_0^{-r}$ of degree zero satisfies $2|a| + 9|b| = 2r$, so $|b|$ is even, and the monomial is a product of the elements

$$u_k := \frac{x_k}{x_0}\ (1 \leq k \leq 3), \qquad w_{jl} := \frac{z_j z_l}{x_0^9}\ (0 \leq j \leq l \leq 3).$$

Hence the $\mathbb{C}$-algebra homomorphism

$$\varphi\colon \mathbb{C}[U_1, U_2, U_3, W_{jl}\ (0 \leq j \leq l \leq 3)] \longrightarrow A', \qquad U_k \mapsto u_k, \quad W_{jl} \mapsto w_{jl},$$

from a polynomial ring in $3 + 10 = 13$ variables is surjective. Grade the source by the number of $W$-factors and $A'$ by half the total $z$-degree; then $\varphi$ preserves the gradings. A weight-0 element of $\ker \varphi$ is a polynomial relation among $u_1, u_2, u_3$, which are algebraically independent; a weight-1 element of $\ker \varphi$ has the form $\sum_{j\leq l} p_{jl}(U) W_{jl}$ with $\sum_{j \leq l} p_{jl}(u)\, z_j z_l / x_0^9 = 0$, which forces all $p_{jl} = 0$ since the monomials $z_j z_l$ are linearly independent over the field $\mathbb{C}(u_1, u_2, u_3)$. Hence $\ker \varphi$ lives in weights $\geq 2$, so $\ker \varphi \subset (W_{jl} : j \leq l)^2$.

Now let $\mathfrak{p}$ be any point of $\mathbb{P}^3_x \cap D_+(x_0)$, that is, a prime ideal of $A'$ containing all the $w_{jl}$, and let $\mathfrak{n} \subset \mathbb{C}[U, W]$ be its preimage, a prime ideal containing all the $W_{jl}$; thus $\ker \varphi \subset (W_{jl} : j \leq l)^2 \subset \mathfrak{n}^2$. Consequently the maximal ideal of the local ring $A'_{\mathfrak{p}} = \mathbb{C}[U, W]_{\mathfrak{n}} / \ker \varphi$ has the same embedding dimension as that of the regular local ring $\mathbb{C}[U, W]_{\mathfrak{n}}$, namely $\operatorname{ht} \mathfrak{n}$. On the other hand, $A'$ is a finitely generated domain whose fraction field is the function field of $W$, so $\dim A' = \dim W = 7$ by [Har77, Theorem I.1.8A], and the dimension formula for finitely generated domains over a field [Har77, Theorem I.1.8A] gives

$$\dim A'_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{n} / \ker \varphi) = \operatorname{ht} \mathfrak{n} - \operatorname{ht}(\ker \varphi) = \operatorname{ht} \mathfrak{n} - (13 - 7) = \operatorname{ht} \mathfrak{n} - 6.$$

So the embedding dimension of $\mathcal{O}_{W,\mathfrak{p}} = A'_{\mathfrak{p}}$ exceeds its Krull dimension by 6, and every point of $\mathbb{P}^3_x \cap D_+(x_0)$ — closed or not — is singular. Renumbering the coordinates covers all of $\mathbb{P}^3_x$.

The stratum $\mathbb{P}^3_z$ is handled the same way: for the chart $D_+(z_0)$, the chart ring $A'' = (S[z_0^{-1}])_0$ is generated by $v_l := z_l / z_0$ $(1 \leq l \leq 3)$ and the $\binom{12}{3} = 220$ elements $x^a / z_0^2$ with $|a| = 9$ (a degree-zero monomial $x^a z^b z_0^{-r}$ satisfies $2|a| + 9|b| = 9r$, so $9 \mid 2|a|$ and hence $9 \mid |a|$). Grading the corresponding polynomial ring in $3 + 220 = 223$ variables by the number of $x$-block variables and repeating the argument — the monomials $x^a$ with $|a| = 9$ are linearly independent over $\mathbb{C}(v_1, v_2, v_3)$, and any product of two generators involving the $x$-variables has $x$-degree at least

18 — shows that the kernel of the corresponding presentation lies in the square of the ideal generated by the $x$-block variables. As before, at every point of $\mathbb{P}^3_z \cap D_+(z_0)$ the embedding dimension of the local ring of $W$ exceeds its dimension by $223 - 7 = 216$, so every point of $\mathbb{P}^3_z$ is singular. This proves (2).

(3) Let $x^a z^b$ be a monomial of degree $18n$, so that $2|a| + 9|b| = 18n$. Reducing modulo 2 shows that $|b|$ is even, and reducing modulo 9 shows that 9 divides $|a|$, since 2 is invertible modulo 9. Write $|a| = 9\alpha$ and $|b| = 2\beta$; then $18\alpha + 18\beta = 18n$, so $\alpha + \beta = n$. Splitting the $x$-part into $\alpha$ monomials of $x$-degree 9 and the $z$-part into $\beta$ monomials of $z$-degree 2 expresses $x^a z^b$ as a product of $n$ monomials of degree 18. Hence $S^{(18)}$ is generated by $S_{18}$.

The inclusion $S^{(18)} \subset S$ induces a morphism $\varphi_{18}\colon W \to \operatorname{Proj} S^{(18)}$ defined on all of $W$, because the degree-18 monomials $x_i^9$ and $z_j^2$ have no common zero on $W$. For $g \in S_{18}$ we have $\varphi_{18}^{-1}(D_+(g)) = D_+(g)$, and the corresponding ring map $(S^{(18)}[g^{-1}])_0 \to (S[g^{-1}])_0$ is an isomorphism: it is injective, and every degree-zero fraction $h/g^k$ with $h \in S_{18k} = (S^{(18)})_k$ already lies in the source. Hence $\varphi_{18}$ is an isomorphism. Choosing a basis of the finite-dimensional space $S_{18}$ gives a surjective degree-preserving homomorphism from a standard graded polynomial ring $\mathbb{C}[y_0, \ldots, y_N]$ onto $S^{(18)}$ (where $\deg y_r = 1$), hence a closed immersion $\operatorname{Proj} S^{(18)} \hookrightarrow \mathbb{P}^N$ by [Har77, Exercise II.3.12], under which $\mathcal{O}_{\mathbb{P}^N}(1)$ pulls back to $\mathcal{O}_{\operatorname{Proj} S^{(18)}}(1)$ by [Har77, Proposition II.5.12]. Finally, on each chart $D_+(g)$ with $g \in S_{18}$, the identification $\varphi_{18}$ matches $(S^{(18)}[g^{-1}])_1$ with $(S[g^{-1}])_{18}$, so $\mathcal{O}_{\operatorname{Proj} S^{(18)}}(1)$ pulls back to $\mathcal{O}_W(18)$. In particular $W$ is projective and $\mathcal{O}_W(18)$ is very ample. $\square$

We next produce the relevant dense open set of complete intersections.

**Lemma 3.4.** *There is a dense Zariski-open subset $\Omega \subset (S_{18})^4$ such that for every $f = (f_1, f_2, f_3, f_4) \in \Omega$, in the notation of Set-up 2.1:*

(1) *$X \cap (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z) = \emptyset$;*

(2) *$U$ is nonempty, smooth, and pure of dimension 4; in particular, $X$ is quasi-smooth.*

*Proof.* Write $\mathbb{A} := (S_{18})^4$, an affine space over $\mathbb{C}$.

(1) Restriction to $\mathbb{P}^3_x$ sets all $z_j$ to zero, and weighted degree-18 polynomials in the $x$-variables are exactly the ordinary degree-9 forms, so restriction gives a surjection $S_{18} \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(9))$. Quadruples of degree-9 forms on $\mathbb{P}^3$ with empty common zero locus form a Zariski-open subset, which is nonempty because of the quadruple $(x_0^9, x_1^9, x_2^9, x_3^9)$; its preimage $\mathbb{A}_x \subset \mathbb{A}$ is a dense open subset such that $X \cap \mathbb{P}^3_x = \emptyset$ for all $f \in \mathbb{A}_x$. Symmetrically, restriction to $\mathbb{P}^3_z$ gives ordinary quadrics in the $z$-variables, the quadruple $(z_0^2, z_1^2, z_2^2, z_3^2)$ has empty common zero locus, and we obtain a dense open $\mathbb{A}_z \subset \mathbb{A}$ such that $X \cap \mathbb{P}^3_z = \emptyset$ for all $f \in \mathbb{A}_z$.

(2) Let $V := \mathbb{A}^8 \setminus \{0\}$, a smooth variety of dimension 8, and consider the universal zero locus

$$Z := \{(v, f) \in V \times \mathbb{A} \mid f_1(v) = f_2(v) = f_3(v) = f_4(v) = 0\}$$

with its two projections. The morphism $Z \to V$ is the kernel of the morphism of trivial vector bundles $(S_{18})^4 \otimes \mathcal{O}_V \to \mathcal{O}_V^4$ given by evaluation, which is surjective at every point of $V$: at each $v \in V$ some coordinate $x_i$ or $z_j$ is nonzero, and then $x_i^9$ or $z_j^2$ is an element of $S_{18}$ not vanishing at $v$. Hence $Z \to V$ is a vector bundle of rank $4 \dim S_{18} - 4$, and $Z$ is a smooth variety of dimension $\dim \mathbb{A} + 4$.

The projection $\mathrm{pr}\colon Z \to \mathbb{A}$ is surjective: for every $f$, the closed subset $\{f_1 = \cdots = f_4 = 0\} \subset \mathbb{A}^8$ contains the origin, and each of its irreducible components has codimension at most 4 by Krull's height theorem [Mat89, Theorem 13.5], hence has dimension at least 4 and meets $V$. By generic smoothness [Har77, Corollary III.10.7] applied to pr, there is a dense open subset $\mathbb{A}' \subset \mathbb{A}$ such that $\mathrm{pr}^{-1}(\mathbb{A}') \to \mathbb{A}'$ is a smooth morphism. Fix $f \in \mathbb{A}'$ and a point $v$ of the fiber $Z_f = \mathrm{pr}^{-1}(f)$. Smooth morphisms are flat with regular fibers [Har77, Theorem III.10.2], so $\mathcal{O}_{Z_f, v}$ is regular, and by the dimension formula for flat morphisms [Har77, Proposition III.9.5],

$$\dim \mathcal{O}_{Z_f, v} = \dim \mathcal{O}_{Z,(v,f)} - \dim \mathcal{O}_{\mathbb{A}, f}.$$

At closed points $v$ and $f$ this equals $\dim Z - \dim \mathbb{A} = 4$, since $Z$ and $\mathbb{A}$ are irreducible. The fiber $Z_f$ is the punctured affine cone $U = \{f_1 = \cdots = f_4 = 0\} \cap V$, so for every $f \in \mathbb{A}'$ the scheme $U$ is smooth, pure of dimension 4, and nonempty by the surjectivity of pr.

Setting $\Omega := \mathbb{A}_x \cap \mathbb{A}_z \cap \mathbb{A}'$ finishes the proof. □

The following lemma isolates the commutative algebra of the cone at its vertex.

**Lemma 3.5.** *Let $f \in (S_{18})^4$ and assume that $U$ is nonempty and pure of dimension 4. Then:*

(1) $\operatorname{ht}(f_1, f_2, f_3, f_4) = 4$, *and $f_1, f_2, f_3, f_4$ is a regular sequence in $S$;*
(2) $R_{\mathfrak{m}}$ *is a complete intersection local ring of dimension 4, and* $\operatorname{depth}_{\mathfrak{m}} R = 4$*;*
(3) *the Hilbert series of $R$ is*

$$\sum_{d\geq 0}(\dim_{\mathbb{C}} R_d)\, t^d = \frac{(1-t^{18})^4}{(1-t^2)^4(1-t^9)^4} = \frac{(1-t+t^2-\cdots+t^8)^4}{(1-t)^4}, \tag{3.3}$$

*and consequently* $\dim_{\mathbb{C}} R_d = \frac{d^3}{6} + O(d^2)$ *as* $d \to \infty$.

*Proof.* (1) Let $I := (f_1, f_2, f_3, f_4)$ and let $\mathfrak{p}$ be a minimal prime over $I$. Since $I$ is homogeneous, $\mathfrak{p}$ is homogeneous by Lemma 3.2(1), hence $\mathfrak{p} \subset S_+$; and $\operatorname{ht}\mathfrak{p} \leq 4 < 8 = \operatorname{ht} S_+$ by Krull's height theorem [Mat89, Theorem 13.5], so $\mathfrak{p} \subsetneq S_+$ and the closed subset $V(\mathfrak{p}) \subset \operatorname{Spec} R$ has positive dimension. The irreducible components of $U$ are the sets $V(\mathfrak{p}_i) \setminus \{\mathfrak{m}\}$, where $\mathfrak{p}_i$ runs over the minimal primes of $I$; in particular $V(\mathfrak{p}) \setminus \{\mathfrak{m}\}$ is one of them. Since $U$ is pure of dimension 4 and removing the closed point $\{\mathfrak{m}\}$ from the positive-dimensional irreducible set $V(\mathfrak{p})$ does not change its dimension, $\dim S/\mathfrak{p} = 4$. By the dimension formula for finitely generated domains over a field [Har77, Theorem I.1.8A] applied to the polynomial ring $S$, we get $\operatorname{ht}\mathfrak{p} = 8 - 4 = 4$. Hence $\operatorname{ht} I = 4$.

The local ring $S_{S_+}$ is a regular local ring of dimension 8, hence Cohen–Macaulay [Mat89, Theorem 17.8]. The ideal $IS_{S_+}$ is generated by the four elements $f_1, f_2, f_3, f_4$ and has height 4, so these elements form a regular sequence in $S_{S_+}$ by the characterization of regular sequences in Cohen–Macaulay local rings via heights [Mat89, Theorem 17.4].

It remains to pass from $S_{S_+}$ to $S$. We claim that for a homogeneous ideal $J \subset S$ and a homogeneous $g \in S$ with $g \in JS_{S_+}$, we have $g \in J$. Indeed, $ug \in J$ for some $u \notin S_+$; writing $u = c + u'$ with $c \in \mathbb{C}^\times$ and $u' \in S_+$, the homogeneous component of degree $\deg g$ of $ug \in J$ is $cg$ plus terms coming from $u'g$ of different degrees; since $J$ is homogeneous, $cg \in J$, so $g \in J$. Now suppose $f_{k+1}g \in (f_1, \ldots, f_k)$ for some $0 \leq k \leq 3$ and some $g \in S \setminus (f_1, \ldots, f_k)$; comparing homogeneous components (the ideal $(f_1, \ldots, f_k)$ is homogeneous), we may take $g$ homogeneous. By the claim, $g \notin (f_1, \ldots, f_k)S_{S_+}$, while $f_{k+1}g \in (f_1, \ldots, f_k) \subset (f_1, \ldots, f_k)S_{S_+}$; so $f_{k+1}$ is a zerodivisor modulo $(f_1, \ldots, f_k)S_{S_+}$, a contradiction. Hence $f_1, f_2, f_3, f_4$ is a regular sequence in $S$.

(2) By (1), $R_{\mathfrak{m}} = S_{S_+}/IS_{S_+}$ is the quotient of a regular local ring of dimension 8 by a regular sequence of length 4, hence a complete intersection local ring of dimension 4; it is Cohen–Macaulay [Mat89, Theorem 17.4], so $\operatorname{depth}_{\mathfrak{m}} R = \operatorname{depth} R_{\mathfrak{m}} = 4$.

(3) The Hilbert series of $S$ is $(1-t^2)^{-4}(1-t^9)^{-4}$ by counting monomials. If $g \in S$ is homogeneous of degree $e$ and is a nonzerodivisor on a graded quotient $S/J$ with $J$ homogeneous, the exact sequence

$$0 \longrightarrow (S/J)(-e) \xrightarrow{g} S/J \longrightarrow S/(J+(g)) \longrightarrow 0$$

multiplies the Hilbert series by $(1-t^e)$. Applying this four times along the regular sequence of (1) gives the first equality in (3.3). For the second equality, note that

$$\frac{(1-t^{18})(1-t)}{(1-t^2)(1-t^9)} = \frac{(1+t^9)(1-t)}{1-t^2} = \frac{1+t^9}{1+t} = 1-t+t^2-\cdots+t^8.$$

Writing $g(t) := (1-t+t^2-\cdots+t^8)^4 = \sum_j g_j t^j$, a polynomial with $g(1)=1$, and using $(1-t)^{-4} = \sum_{d\geq 0}\binom{d+3}{3}t^d$, we obtain

$$\dim_{\mathbb{C}} R_d = \sum_j g_j \binom{d-j+3}{3} = g(1)\,\frac{d^3}{6} + O(d^2) = \frac{d^3}{6} + O(d^2). \qquad \square$$

The next lemma describes the punctured cone as a Zariski-locally trivial $\mathbb{G}_m$-torsor over $X$ and identifies all section spaces. Let $\mathbb{G}_m$ act on $C$ through the grading of $R$, so that $\lambda \in \mathbb{G}_m$ acts on a homogeneous function $g$ of degree $d$ by $\sigma_\lambda^* g = \lambda^d g$; the action preserves $U$, and we write $\pi\colon U \to X$ for the canonical morphism. For $0 \leq i,j \leq 3$, let

$$X_{ij} := X \cap D_+(x_i z_j) = \operatorname{Spec}\big(R[(x_i z_j)^{-1}]\big)_0, \qquad U_{ij} := \{x_i z_j \neq 0\} \subset U,$$

so that $U_{ij} = \operatorname{Spec} R[(x_i z_j)^{-1}]$ and $\pi(U_{ij}) = X_{ij}$.

**Lemma 3.6.** *Let $f \in (S_{18})^4$ and assume that $X \cap (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z) = \emptyset$. Then:*

1. *the open subsets $X_{ij}$ cover $X$, the open subsets $U_{ij}$ cover $U$, and for each $i,j$ the degree-one unit $t_{ij} \in R[(x_i z_j)^{-1}]$ induces an isomorphism $U_{ij} \cong X_{ij} \times \mathbb{G}_m$ over $X_{ij}$, under which $\pi$ becomes the first projection and the $\mathbb{G}_m$-action becomes the translation action on the second factor;*
2. *for every $m \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(m)$ is invertible, trivialized on $X_{ij}$ by $t_{ij}^m$, and the multiplication maps induce isomorphisms $\mathcal{O}_X(a) \otimes \mathcal{O}_X(b) \cong \mathcal{O}_X(a+b)$ for all $a,b \in \mathbb{Z}$;*
3. *if moreover $\operatorname{depth}_{\mathfrak{m}} R \geq 2$, then $\Gamma(U, \mathcal{O}_U) = R$, and the canonical maps give isomorphisms*
$$R_m \;\cong\; H^0(X, \mathcal{O}_X(m)) \qquad \text{for all } m \in \mathbb{Z}.$$
*In particular $H^0(X, \mathcal{O}_X) = R_0 = \mathbb{C}$, and $H^0(X, \mathcal{O}_X(m)) = 0$ for $m < 0$.*

*Proof.* (1) We first prove the covering claims. The ideal generated by the sixteen elements $x_i z_j$ contains the product of the homogeneous ideals $\mathfrak{a} := (x_0, \ldots, x_3)R$ and $\mathfrak{b} := (z_0, \ldots, z_3)R$. Let $\mathfrak{q}$ be a point of $U$, that is, a prime of $R$ with $\mathfrak{q} \neq \mathfrak{m}$, and suppose that all $x_i z_j$ vanish at $\mathfrak{q}$; then $\mathfrak{q} \supset \mathfrak{a}$ or $\mathfrak{q} \supset \mathfrak{b}$, say $\mathfrak{q} \supset \mathfrak{a}$. Choose a minimal prime $\mathfrak{p}$ over $\mathfrak{a}$ with $\mathfrak{p} \subset \mathfrak{q}$; it is homogeneous by Lemma 3.2(1). If $R_+ \subset \mathfrak{p}$, then $\mathfrak{m} = R_+ \subset \mathfrak{q}$, so $\mathfrak{q} = \mathfrak{m}$ by the maximality of $\mathfrak{m}$, a contradiction. Hence $R_+ \not\subset \mathfrak{p}$, so $\mathfrak{p}$ is a point of $V_+(\mathfrak{a}) = X \cap \mathbb{P}^3_z$, which is empty, again a contradiction. The case $\mathfrak{q} \supset \mathfrak{b}$ is symmetric, using $X \cap \mathbb{P}^3_x = \emptyset$. Hence the $U_{ij}$ cover $U$. If instead $\mathfrak{q}$ is a point of $X$, that is, a homogeneous prime with $R_+ \not\subset \mathfrak{q}$, the same argument applies, the first case now being impossible outright; so the $X_{ij}$ cover $X$.

For the product decomposition, fix $i,j$ and write $B' := R[(x_i z_j)^{-1}]$ and $A := B'_0$. The graded ring $B'$ contains the degree-one unit $t_{ij}$, so exactly as in (3.2) we get $B' = A[t_{ij}, t_{ij}^{-1}]$, that is, $U_{ij} \cong X_{ij} \times \mathbb{G}_m$. Under this isomorphism $\pi$ is the first projection, and the grading corresponds to the $t_{ij}$-adic grading, so the $\mathbb{G}_m$-action is the translation on the second factor.

(2) On $X_{ij}$, the sheaf $\mathcal{O}_X(m)$ is associated with the $A$-module $B'_m = t_{ij}^m \cdot A$, which is free of rank one; the multiplication maps $B'_a \otimes_A B'_b \to B'_{a+b}$ send $t_{ij}^a \otimes t_{ij}^b$ to $t_{ij}^{a+b}$, hence are isomorphisms of free rank-one modules.

(3) Since $R$ is Noetherian and $\operatorname{depth}_{\mathfrak{m}} R \geq 2$, restriction gives an isomorphism $\Gamma(C, \mathcal{O}_C) \to \Gamma(U, \mathcal{O}_U)$ by [Gro05, Exposé III, Corollaire 3.5], that is, $\Gamma(U, \mathcal{O}_U) = R$. On the other hand, $U$ is covered by the affine opens $U_{ij}$, whose pairwise intersections are again principal opens of the same shape, and

$$\Gamma(U_{ij}, \mathcal{O}_U) = R[(x_i z_j)^{-1}] = \bigoplus_{m \in \mathbb{Z}} \big(R[(x_i z_j)^{-1}]\big)_m = \bigoplus_{m \in \mathbb{Z}} \Gamma(X_{ij}, \mathcal{O}_X(m)),$$

compatibly with restrictions. Taking the kernel of the Čech difference map $\prod \Gamma(U_{ij}, \mathcal{O}_U) \to \prod \Gamma(U_{ij} \cap U_{kl}, \mathcal{O}_U)$, which is a map of graded vector spaces, and comparing degree-$m$ pieces, we obtain

$$R = \Gamma(U, \mathcal{O}_U) = \bigoplus_{m\in\mathbb{Z}} H^0(X, \mathcal{O}_X(m)),$$

where the degree-$m$ piece of $R$ maps isomorphically onto $H^0(X, \mathcal{O}_X(m))$ by the canonical map. The final statements follow since $R_0 = \mathbb{C}$, since $R$ is generated in nonnegative degrees, and since $R_m = 0$ for $m < 0$. □

**Proposition 3.7.** *Let $\Omega$ be as in Lemma 3.4 and let $f \in \Omega$. Then $X$ is a smooth irreducible projective threefold contained in $W \setminus \operatorname{Sing} W$, the sheaf $H := \mathcal{O}_X(1)$ is an ample line bundle with $\mathcal{O}_X(m) \cong H^{\otimes m}$ for all $m \in \mathbb{Z}$, and $H^3 = 1$. Moreover $h^0(X, mH) = \dim_{\mathbb{C}} R_m$ for all $m \in \mathbb{Z}$.*

*Proof.* By Lemma 3.4, $X \cap (\mathbb{P}^3_x \sqcup \mathbb{P}^3_z) = \emptyset$ and $U$ is nonempty, smooth, and pure of dimension 4; by Lemma 3.3(2), $X \subset W \setminus \operatorname{Sing} W$. The scheme $X$ is projective, being a closed subscheme of the projective scheme $W$ (Lemma 3.3(3)).

*Smoothness and dimension.* By Lemma 3.6(1), $X$ is covered by the charts $X_{ij} = \operatorname{Spec} A$ with $U_{ij} = \operatorname{Spec} A[t_{ij}, t_{ij}^{-1}]$ open in the smooth scheme $U$. Here $A[t_{ij}, t_{ij}^{-1}] = R[(x_i z_j)^{-1}]$ is a finitely generated $\mathbb{C}$-algebra, so for a prime $\mathfrak{p}$ of $A$, Lemma 3.1 applied with $B = A[t_{ij}, t_{ij}^{-1}]$ produces a prime $\mathfrak{q}$ of $B$ with $B_{\mathfrak{q}}$ regular of dimension $\dim A_{\mathfrak{p}} + 1$, and shows that $A_{\mathfrak{p}}$ is regular. Hence $X$ is smooth; moreover its local rings at closed points have dimension $4 - 1 = 3$, since those of $U$ have dimension 4. So $X$ is smooth of pure dimension 3, and it is nonempty because $U$ is.

*Irreducibility.* By Lemma 3.5(2), $\operatorname{depth}_{\mathfrak{m}} R = 4 \geq 2$, so $H^0(X, \mathcal{O}_X) = \mathbb{C}$ by Lemma 3.6(3). In particular $\Gamma(X, \mathcal{O}_X)$ has no nontrivial idempotents, so $X$ is connected. A connected smooth scheme of finite type over $\mathbb{C}$ is irreducible, because its irreducible components are disjoint (each point lies on a unique component, the local rings being domains) and closed, and a connected scheme is not a disjoint union of two nonempty closed subsets. Hence $X$ is irreducible.

*The polarization.* By Lemma 3.6(2), each $\mathcal{O}_X(m)$ is invertible and $\mathcal{O}_X(a) \otimes \mathcal{O}_X(b) \cong \mathcal{O}_X(a+b)$; iterating with $a = b = 1$ gives $\mathcal{O}_X(m) \cong H^{\otimes m}$ for all $m \in \mathbb{Z}$. By Lemma 3.3(3), there is a closed immersion $W \hookrightarrow \mathbb{P}^N$ with $\mathcal{O}_W(18) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_W$. The composition $X \hookrightarrow W \hookrightarrow \mathbb{P}^N$ is a closed immersion, and $\mathcal{O}_W(18)|_X \cong \mathcal{O}_X(18)$: both sheaves are, on each chart $X_{ij}$, canonically the degree-18 graded piece of the chart's homogeneous coordinate ring, compatibly with the trivializations of Lemma 3.3(1) and Lemma 3.6(2). Hence $H^{\otimes 18} \cong \mathcal{O}_{\mathbb{P}^N}(1)|_X$ is very ample, and $H$ is ample by [Har77, Theorem II.7.6].

*Sections and the self-intersection number.* By Lemma 3.6(3), $h^0(X, mH) = \dim_{\mathbb{C}} R_m$ for all $m \in \mathbb{Z}$. Since $H$ is ample, [Har77, Proposition III.5.3] applied to $\mathcal{F} = \mathcal{O}_X$ gives $H^i(X, mH) = 0$ for all $i > 0$ and all $m \gg 0$, so $\chi(X, mH) = h^0(X, mH) = \dim_{\mathbb{C}} R_m$ for $m \gg 0$. By the Hirzebruch–Riemann–Roch theorem [Har77, Appendix A, Theorem 4.1],

$$\chi(X, mH) = \deg\big(\operatorname{ch}(H^{\otimes m}) \cdot \operatorname{td}(\mathcal{T}_X)\big)_3 = \frac{H^3}{6} m^3 + O(m^2),$$

where $(\ )_3$ denotes the degree-3 component and $\mathcal{T}_X$ the tangent bundle. Comparing with $\dim_{\mathbb{C}} R_m = \frac{m^3}{6} + O(m^2)$ from Lemma 3.5(3) yields $H^3 = 1$. □

## 4. The Picard group

In this section, we prove that $\operatorname{Pic}(X) = \mathbb{Z} \cdot H$ for the threefolds constructed in Section 3. The route goes through the affine cone: the vertex is an isolated complete intersection singularity of dimension 4, so the local ring at the vertex is factorial by Grothendieck's results in SGA 2; a graded argument then makes the whole cone factorial, so the punctured cone has trivial Picard group, and descent along the torsor $\pi \colon U \to X$ identifies $\operatorname{Pic}(X)$ with the character group of $\mathbb{G}_m$.

**Lemma 4.1.** *Let $R$ be a $\mathbb{Z}_{\geq 0}$-graded Noetherian domain with $R_0 = \mathbb{C}$ and irrelevant maximal ideal $\mathfrak{m} = R_+$. If $R_{\mathfrak{m}}$ is factorial, then $R$ is factorial.*

*Proof. Step 1: every homogeneous height-one prime ideal $P$ of $R$ is generated by one homogeneous element.* Since $P$ is a proper homogeneous ideal, $P \subset \mathfrak{m}$, and $PR_{\mathfrak{m}}$ is a height-one prime of the factorial Noetherian domain $R_{\mathfrak{m}}$. A height-one prime of a factorial Noetherian domain is principal: pick a nonzero $a$ in it, factor $a$ into prime elements, observe that some prime factor $q$ lies in the given prime, and note that $(q)$ and the given prime are then height-one primes with $(q)$ contained in it, hence equal. So $PR_{\mathfrak{m}}$ is generated by one element, and therefore $\dim_{\mathbb{C}} P/\mathfrak{m}P \le 1$ by the last statement of Lemma 3.2(3); by the second statement of Lemma 3.2(3), $P$ is generated by one homogeneous element $p$, that is, $P = (p)$, and $p \neq 0$.

*Step 2: every nonzero homogeneous element of $R$ is a unit times a product of homogeneous prime elements.* Let $h$ be a nonzero homogeneous element; we induct on $\deg h$. If $\deg h = 0$ then $h$ is a unit. If $\deg h > 0$, then $h$ is not a unit by Lemma 3.2(2), so there is a minimal prime $P$ over $(h)$, contained in any given prime containing $h$. It satisfies $\operatorname{ht} P \le 1$ by Krull's height theorem [Mat89, Theorem 13.5], hence $\operatorname{ht} P = 1$ since $R$ is a domain and $h \neq 0$, and it is homogeneous by Lemma 3.2(1). By Step 1, $P = (p)$ with $p$ a homogeneous prime element, necessarily of positive degree (a degree-zero generator would be a unit). Then $h = ph'$ with $h'$ homogeneous of degree $\deg h - \deg p < \deg h$, and the induction hypothesis applies to $h'$.

*Step 3: the homogeneous localization of $R$ is a principal ideal domain.* Let $T$ be the multiplicative set of nonzero homogeneous elements of $R$ and let $R_T$ be the localization, a $\mathbb{Z}$-graded ring in which every nonzero homogeneous element is invertible. The degree-zero part $K := (R_T)_0$ is a field. If $R = R_0$ there is nothing to prove in the rest of the lemma, so assume $R \neq R_0$; then the degrees of nonzero homogeneous elements of $R_T$ form a nonzero subgroup $e\mathbb{Z} \subset \mathbb{Z}$. Pick a homogeneous unit $s \in R_T$ of degree $e$; every homogeneous element of $R_T$ of degree $ne$ is $K$-proportional to $s^n$, so $R_T = K[s, s^{-1}]$, a localization of the polynomial ring $K[s]$. Since $K[s]$ is a principal ideal domain, so is its localization $R_T$.

*Step 4: every height-one prime ideal $Q$ of $R$ is principal.* If $Q$ contains a nonzero homogeneous element $h$, then $Q$ contains a minimal prime $P$ of $(h)$, which as in Step 2 is a homogeneous height-one prime; then $Q = P$ because $\operatorname{ht} Q = 1$, and $Q$ is principal by Step 1. So assume $Q$ contains no nonzero homogeneous element, that is, $Q \cap T = \emptyset$. By the correspondence of primes under localization, $QR_T$ is a prime ideal of $R_T$ with $QR_T \cap R = Q$; it is nonzero, hence contains a prime element $\alpha$ of the principal ideal domain $R_T$ with $QR_T = \alpha R_T$, and clearing denominators we may take $\alpha \in Q$. By the ascending chain condition, we may choose $q \in Q$ with $qR_T = QR_T$ such that $(q)$ is maximal among the ideals $(q')$ with $q' \in Q$ and $q'R_T = QR_T$. We claim $Q = (q)$. Let $x \in Q$. Since $QR_T = qR_T$, there are $u \in R$ and $s \in T$ with $sx = qu$. By Step 2, $s = c\,p_1 \cdots p_k$ with $c$ a unit and each $p_r$ a homogeneous prime element; we induct on $k$. If $k = 0$, then $s$ is a unit and $x = q\,(s^{-1}u) \in (q)$. If $k \ge 1$, the prime $p_1$ divides $qu$. If $p_1$ divided $q$, say $q = p_1 q'$, then $p_1 \notin Q$ (because $Q \cap T = \emptyset$) and $q \in Q$ would force $q' \in Q$, with $q'R_T = qR_T = QR_T$ (as $p_1$ is a unit in $R_T$) and $(q) \subsetneq (q')$ (as $p_1$ is not a unit in $R$, and $R$ is a domain), contradicting the maximality of $(q)$. Hence $p_1 \mid u$, and writing $u = p_1 u'$ and canceling $p_1$ in $sx = qu$ (permissible in a domain) gives $(c\,p_2 \cdots p_k)x = qu'$, which has one fewer prime factor on the left. By induction, $x \in (q)$. Therefore $Q = (q)$.

*Step 5: conclusion.* Since $R$ is a Noetherian domain, every nonzero nonunit is a finite product of irreducible elements, by the ascending chain condition on principal ideals. Let $q$ be irreducible. A minimal prime $P$ over $(q)$ has height one as in Step 2, and $P = (p)$ is principal by Steps 1 and 4; then $q = pc$ for some $c$, and irreducibility of $q$ makes $c$ a unit, so $(q) = (p)$ is prime. Thus every irreducible element of $R$ is prime, and factorizations into irreducibles are then unique up to reordering and units: if $p_1 \cdots p_r = q_1 \cdots q_s$ with all factors irreducible, then the prime $p_1$ divides some $q_j$, so $q_j = p_1 \cdot (\text{unit})$, and we conclude by induction after canceling. Hence $R$ is factorial. $\square$

**Proposition 4.2.** *Let $\Omega$ be as in Lemma 3.4 and let $f \in \Omega$. Then, with $H = \mathcal{O}_X(1)$ as in Proposition 3.7, the map*

$$\mathbb{Z} \longrightarrow \operatorname{Pic}(X), \qquad n \longmapsto H^{\otimes n} = \mathcal{O}_X(n),$$

*is an isomorphism. In particular $X$ has Picard rank 1.*

*Proof. Step 1: $R$ is a normal domain.* The scheme $U$ is covered by the opens $U_{ij} \cong X_{ij} \times \mathbb{G}_m$ (Lemma 3.6(1)); each nonempty $X_{ij}$ is an open subset of the irreducible scheme $X$ (Proposition 3.7), hence irreducible and dense in $X$, so the nonempty $U_{ij}$ are irreducible with pairwise nonempty intersections, and $U$ is irreducible. Being also smooth, $U$ is integral. By Lemma 3.5(2) and Lemma 3.6(3), $R = \Gamma(U, \mathcal{O}_U)$, so $R$ is a subring of the function field of $U$; hence $R$ is a domain, $\mathbb{Z}_{\geq 0}$-graded with $R_0 = \mathbb{C}$.

For normality we verify Serre's criterion [Mat89, Theorem 23.8]. Every prime $\mathfrak{p} \neq \mathfrak{m}$ of $R$ is a point of $U$, so $R_\mathfrak{p}$ is a regular local ring, hence Cohen–Macaulay [Mat89, Theorem 17.8]; and $R_\mathfrak{m}$ is Cohen–Macaulay of dimension 4 by Lemma 3.5(2). Hence $R$ satisfies $(S_2)$; and $R$ satisfies $(R_1)$ because every height-one prime differs from $\mathfrak{m}$ (which has height 4) and so lies on the smooth locus $U$. By Serre's criterion, $R$ is normal.

*Step 2: $R_\mathfrak{m}$ is factorial.* By Lemma 3.5(2), $R_\mathfrak{m}$ is a complete intersection local ring of dimension 4. Every localization of $R_\mathfrak{m}$ at a nonmaximal prime is a localization of $R$ at a prime $\mathfrak{p} \neq \mathfrak{m}$, hence a regular local ring as in Step 1, hence factorial by the Auslander–Buchsbaum theorem [Gro05, Exposé XI, Théorème 3.13(i)]. In particular $R_\mathfrak{m}$ is factorial in codimension $\leq 3$, so by Grothendieck's theorem on Samuel's conjecture [Gro05, Exposé XI, Corollaire 3.14], which rests on the parafactoriality theorem [Gro05, Exposé XI, Théorème 3.13(ii)], the local ring $R_\mathfrak{m}$ is factorial.

*Step 3: $\operatorname{Pic}(U) = 0$ and $\Gamma(U, \mathcal{O}_U)^\times = \mathbb{C}^\times$.* By Step 2 and Lemma 4.1, $R$ is factorial, so $\operatorname{Cl}(C) = 0$ by [Har77, Proposition II.6.2]. The complement of $U$ in the normal integral scheme $C$ is the single closed point $\{\mathfrak{m}\}$, of codimension $4 \geq 2$, so restriction induces an isomorphism $\operatorname{Cl}(C) \cong \operatorname{Cl}(U)$ by [Har77, Proposition II.6.5]. Since $U$ is smooth, its local rings are regular, hence factorial by [Gro05, Exposé XI, Théorème 3.13(i)], so $U$ is locally factorial and $\operatorname{Pic}(U) \cong \operatorname{Cl}(U) = 0$ by [Har77, Corollary II.6.16]. Moreover $\Gamma(U, \mathcal{O}_U)^\times = R^\times = \mathbb{C}^\times$ by Lemma 3.2(2).

*Step 4: every line bundle on $X$ is isomorphic to some $\mathcal{O}_X(n)$.* Write $\sigma \colon \mathbb{G}_m \times U \to U$ for the action and $\operatorname{pr} \colon \mathbb{G}_m \times U \to U$ for the projection. First note that

$$\Gamma(\mathbb{G}_m \times U, \mathcal{O}) = R[\lambda, \lambda^{-1}], \qquad \Gamma(\mathbb{G}_m \times U, \mathcal{O}^\times) = \mathbb{C}^\times \cdot \lambda^{\mathbb{Z}}. \tag{4.1}$$

Indeed, computing $\Gamma(\mathbb{G}_m \times U, \mathcal{O})$ by the Čech kernel over the affine cover $\{\mathbb{G}_m \times U_{ij}\}$ and using that $(-) \otimes_\mathbb{C} \mathbb{C}[\lambda, \lambda^{-1}]$ is exact gives the first equality from $\Gamma(U, \mathcal{O}_U) = R$; for the second, in a Laurent polynomial ring over a domain the units are the units of the coefficient ring times powers of the variable, and $R^\times = \mathbb{C}^\times$.

Let $N$ be a line bundle on $X$. Since $\pi \circ \sigma = \pi \circ \operatorname{pr}$, there is a canonical isomorphism $\Theta \colon \operatorname{pr}^*(\pi^* N) \to \sigma^*(\pi^* N)$, and the associativity of the action gives the cocycle relation for $\Theta$ over $\mathbb{G}_m \times \mathbb{G}_m \times U$. By Step 3, $\pi^* N$ is trivial; fix an isomorphism $\phi \colon \mathcal{O}_U \to \pi^* N$. Then

$$\psi := \left(\sigma^* \phi\right)^{-1} \circ \Theta \circ \operatorname{pr}^* \phi$$

is an automorphism of $\mathcal{O}_{\mathbb{G}_m \times U}$, that is, a unit $\psi \in \mathbb{C}^\times \cdot \lambda^{\mathbb{Z}}$ by (4.1); write $\psi = c\,\lambda^n$. The cocycle relation for $\Theta$ translates into $\psi(\lambda\mu) = \psi(\lambda)\psi(\mu)$ for the two $\mathbb{G}_m$-factors, that is, $c(\lambda\mu)^n = c\lambda^n \cdot c\mu^n$, which forces $c = 1$. Therefore $\phi$ intertwines the $\mathbb{G}_m$-equivariant structure on $\mathcal{O}_U$ twisted by the character $\lambda \mapsto \lambda^n$ (in which $\lambda$ acts on a function $g$ by $\lambda^n \sigma_\lambda^* g$) with the canonical equivariant structure of $\pi^* N$.

Now push forward to $X$. Over each chart, $\Gamma(U_{ij}, \mathcal{O}_U) = \bigoplus_{d \in \mathbb{Z}} t_{ij}^d \, \Gamma(X_{ij}, \mathcal{O}_X)$ by Lemma 3.6(1), and $\sigma_\lambda^*$ acts on the summand indexed by $d$ through $\lambda^d$; these identifications glue, by Lemma 3.6(2), to a decomposition of $\pi_* \mathcal{O}_U$ into eigensheaves in which the weight-$d$ eigensheaf of the canonical action is $\mathcal{O}_X(d)$. Consequently, for the structure twisted by $\lambda \mapsto \lambda^n$, the weight-0 eigensheaf of $\pi_*\mathcal{O}_U$ is $\mathcal{O}_X(-n)$. On the other hand, by the projection formula, $\pi_*(\pi^*N) = N \otimes \pi_*\mathcal{O}_U$ with the canonical equivariant structure acting through the second factor, so the weight-0 eigensheaf of $\pi_*(\pi^*N)$ is $N \otimes \mathcal{O}_X = N$. Applying $\pi_*$ to the equivariant isomorphism $\phi$ and taking weight-0 parts gives $N \cong \mathcal{O}_X(-n)$.

*Step 5: the twists are pairwise nonisomorphic.* Suppose $\mathcal{O}_X(n) \cong \mathcal{O}_X(n')$ with $k := n-n' > 0$. By Lemma 3.6(2), $\mathcal{O}_X(k) \cong \mathcal{O}_X$ and hence $\mathcal{O}_X(rk) \cong \mathcal{O}_X$ for all $r > 0$, so $\dim_{\mathbb{C}} R_{rk} = h^0(X, \mathcal{O}_X(rk)) = 1$ for all $r > 0$ by Proposition 3.7. This contradicts $\dim_{\mathbb{C}} R_d = \frac{d^3}{6} + O(d^2) \to \infty$ from Lemma 3.5(3). Hence $n \mapsto \mathcal{O}_X(n)$ is injective, and it is surjective onto $\operatorname{Pic}(X)$ by Step 4. $\square$

## 5. Section spaces and global generation

In this section, we compute the low-degree section spaces of the twists and deduce the global generation statements. Throughout, $f \in \Omega$ with $\Omega$ as in Lemma 3.4, and $H = \mathcal{O}_X(1)$, $mH := H^{\otimes m} \cong \mathcal{O}_X(m)$ as in Proposition 3.7.

**Proposition 5.1.** *For $0 \le m \le 17$, the composition $S_m \to R_m \to H^0(X, mH)$ is an isomorphism. In particular:*

1. *$h^0(X, H) = 0$;*
2. *$H^0(X, 2H)$ has basis the images of $x_0, x_1, x_2, x_3$, so $h^0(X, 2H) = 4$;*
3. *$H^0(X, 9H)$ has basis the images of $z_0, z_1, z_2, z_3$, so $h^0(X, 9H) = 4$;*
4. *$H^0(X, 11H)$ has basis the images of the sixteen monomials $x_i z_j$, so $h^0(X, 11H) = 16$;*
5. *the multiplication map*
$$H^0(X, 2H) \otimes_{\mathbb{C}} H^0(X, 9H) \longrightarrow H^0(X, 11H)$$
*is an isomorphism of 16-dimensional vector spaces.*

*Proof.* Each element of the ideal $I = (f_1, \ldots, f_4)$ is a sum $\sum_a q_a f_a$ with $q_a \in S$, and every homogeneous component of degree $m \le 17$ of such a sum vanishes, because it would require a component of some $q_a$ of degree $m - 18 < 0$. Hence $I_m = 0$, and $S_m \to R_m$ is an isomorphism for $0 \le m \le 17$; composing with the isomorphism $R_m \cong H^0(X, mH)$ of Proposition 3.7 proves the first claim.

A monomial $x^a z^b$ has degree $2|a| + 9|b|$. The equation $2u + 9v = m$ with $u, v \in \mathbb{Z}_{\ge 0}$ has no solution for $m = 1$; only $(u, v) = (1, 0)$ for $m = 2$; only $(0, 1)$ for $m = 9$; and only $(1, 1)$ for $m = 11$. Since distinct monomials are linearly independent, this gives (1)–(4). For (5): under the identifications of Lemma 3.6(2), multiplication of sections is induced on each chart by the ring structure of $R[(x_i z_j)^{-1}]$, so the multiplication map of (5) is identified with the multiplication $S_2 \otimes_{\mathbb{C}} S_9 \to S_{11}$, which sends the basis $\{x_i \otimes z_j\}$ bijectively onto the basis $\{x_i z_j\}$ and is therefore an isomorphism. $\square$

**Proposition 5.2.** *The line bundles $2H$, $9H$, and $11H$ are globally generated, while $H$ is ample and not globally generated. The line bundle $L := 11H$ is ample, globally generated, and not isomorphic to $\mathcal{O}_X$.*

*Proof.* Under the trivializations of Lemma 3.6(2), a global section $s \in S_m \subset H^0(X, mH)$ vanishes at a point $p \in X_{ij}$ if and only if the regular function $s/t_{ij}^m$ vanishes at $p$; consequently, the common zero locus on $X$ of a set of weighted homogeneous elements of degree $m$, viewed as sections of $mH$, is the intersection of $X$ with their common zero locus in $W$. A line bundle is globally generated if and only if its global sections have no common zero, since at a point where some section is nonzero the evaluation map to the one-dimensional fiber is surjective.

The common zero locus of $x_0, \ldots, x_3$ on $X$ is $X \cap \mathbb{P}^3_z = \emptyset$, so $2H$ is globally generated. The common zero locus of $z_0, \ldots, z_3$ on $X$ is $X \cap \mathbb{P}^3_x = \emptyset$, so $9H$ is globally generated. At every point $p \in X$ some $x_i$ and some $z_j$ are nonzero, so the section $x_i z_j$ of $11H$ does not vanish at $p$; hence $11H$ is globally generated.

The line bundle $H$ is ample by Proposition 3.7, and $h^0(X,H)=0$ by Proposition 5.1(1); since $X\neq\emptyset$, the zero map $H^0(X,H)\otimes\mathcal{O}_X\to H$ is not surjective, so $H$ is not globally generated. Finally, $L=11H$ is a positive power of the ample line bundle $H$, hence ample; and $L\ncong\mathcal{O}_X$ because $h^0(X,L)=16\neq 1=h^0(X,\mathcal{O}_X)$. $\square$

## 6. The destabilizing subbundle and the proof of the main theorem

In this section, we prove Theorem 1.2. We first record three general facts about syzygy bundles.

**Lemma 6.1.** *Let $Y$ be a scheme, let $E,F,G$ be vector bundles on $Y$, and let $\alpha\colon E\to F$ and $\beta\colon F\to G$ be surjective morphisms of $\mathcal{O}_Y$-modules. Then $\ker\alpha$, $\ker\beta$, and $\ker(\beta\circ\alpha)$ are vector bundles, and the sequence*

$$0\longrightarrow\ker\alpha\longrightarrow\ker(\beta\circ\alpha)\xrightarrow{\alpha}\ker\beta\longrightarrow 0,$$

*with the first map the inclusion and the second the restriction of $\alpha$, is exact.*

*Proof.* A surjection of vector bundles is locally split: locally, lift a basis of the target through the surjection. Hence the kernels of $\alpha$, of $\beta$, and of $\beta\circ\alpha$ (the last being surjective as a composition of surjections) are locally direct summands of vector bundles, hence vector bundles.

The inclusion $\ker\alpha\subset\ker(\beta\circ\alpha)$ is clear, and $\alpha$ maps $\ker(\beta\circ\alpha)$ into $\ker\beta$. A local section of $\ker(\beta\circ\alpha)$ killed by $\alpha$ is a local section of $\ker\alpha$, which gives exactness in the middle, and injectivity on the left is clear. For surjectivity on the right, let $t$ be a section of $\ker\beta$ over an open subset; since $\alpha$ is a surjective morphism of sheaves, after passing to a suitable open cover there are sections $e$ of $E$ with $\alpha(e)=t$, and then $\beta(\alpha(e))=\beta(t)=0$, so $e$ is a section of $\ker(\beta\circ\alpha)$ mapping to $t$. $\square$

**Lemma 6.2.** *Let $Y$ be a smooth projective variety over $\mathbb{C}$ and let $N$ be a globally generated line bundle on $Y$. Then $M_N$ is a vector bundle,*

$$\operatorname{rank}M_N=h^0(Y,N)-1,\qquad\det M_N\cong N^{-1},\qquad c_1(M_N)=-c_1(N).$$

*Proof.* The evaluation map $\mathrm{ev}_N\colon H^0(Y,N)\otimes\mathcal{O}_Y\to N$ is a surjection of vector bundles, so $M_N=\ker\mathrm{ev}_N$ is a vector bundle as in Lemma 6.1, of rank $h^0(Y,N)-1$ by rank additivity in the exact sequence

$$0\longrightarrow M_N\longrightarrow H^0(Y,N)\otimes\mathcal{O}_Y\longrightarrow N\longrightarrow 0.$$

Taking determinants in this sequence gives $\mathcal{O}_Y\cong\det M_N\otimes N$, so $\det M_N\cong N^{-1}$ and $c_1(M_N)=-c_1(N)$. $\square$

**Proposition 6.3.** *Let $Y$ be a smooth projective variety over $\mathbb{C}$, and let $A$ and $B$ be globally generated line bundles on $Y$ such that the multiplication map*

$$\mu\colon H^0(Y,A)\otimes_{\mathbb{C}}H^0(Y,B)\longrightarrow H^0(Y,A\otimes B)$$

*is an isomorphism. Then there is a short exact sequence of vector bundles*

$$0\longrightarrow M_A\otimes_{\mathbb{C}}H^0(Y,B)\longrightarrow M_{A\otimes B}\longrightarrow A\otimes M_B\longrightarrow 0.$$

*Proof.* Put $V:=H^0(Y,A)$ and $T:=H^0(Y,B)$. Consider the morphisms of vector bundles

$$\alpha:=\mathrm{ev}_A\otimes\mathrm{id}\colon V\otimes_{\mathbb{C}}T\otimes_{\mathbb{C}}\mathcal{O}_Y\longrightarrow A\otimes_{\mathbb{C}}T,\qquad\beta:=\mathrm{id}\otimes\mathrm{ev}_B\colon A\otimes_{\mathbb{C}}T\longrightarrow A\otimes B,$$

where $A\otimes_{\mathbb{C}}T$ denotes $A\otimes_{\mathcal{O}_Y}(T\otimes_{\mathbb{C}}\mathcal{O}_Y)$. The map $\alpha$ is surjective because $\mathrm{ev}_A$ is surjective and $-\otimes_{\mathbb{C}}T$ is exact, and $\beta$ is surjective because $\mathrm{ev}_B$ is surjective and tensoring with the line bundle $A$ is exact. On a pure tensor $s\otimes t$ of global sections, the composition $\beta\circ\alpha$ produces the section $st=\mu(s\otimes t)$ of $A\otimes B$; hence, under the isomorphism $\mu\colon V\otimes T\to H^0(Y,A\otimes B)$, the composition $\beta\circ\alpha$ is identified with $\mathrm{ev}_{A\otimes B}$, and $\ker(\beta\circ\alpha)\cong M_{A\otimes B}$. Moreover $\ker\alpha=M_A\otimes_{\mathbb{C}}T$, by tensoring the defining sequence of $M_A$ with $T$, and $\ker\beta=A\otimes M_B$, by tensoring the defining sequence of $M_B$ with $A$. Lemma 6.1 now gives the asserted exact sequence. $\square$

*Proof of Theorem 1.2.* Let $\Omega \subset (S_{18})^4$ be the dense open subset of Lemma 3.4; it is nonempty because $(S_{18})^4$ is an irreducible affine space over $\mathbb{C}$. Choose $f \in \Omega$, and let $X$ and $H = \mathcal{O}_X(1)$ be as in Proposition 3.7, so that $X$ is a smooth irreducible projective threefold, $H$ is ample with $H^3 = 1$, and $\operatorname{Pic}(X) = \mathbb{Z} \cdot H$ by Proposition 4.2. Let $L := 11H$; by Proposition 5.2, $L$ is a nontrivial, ample, globally generated line bundle.

By Proposition 5.1, $h^0(X, 2H) = h^0(X, 9H) = 4$ and $h^0(X, 11H) = 16$, and the multiplication map $H^0(X, 2H) \otimes H^0(X, 9H) \to H^0(X, 11H)$ is an isomorphism; by Proposition 5.2, $2H$ and $9H$ are globally generated. Proposition 6.3 applied with $A = 2H$ and $B = 9H$ yields a short exact sequence of vector bundles

$$0 \longrightarrow M_{2H} \otimes_{\mathbb{C}} H^0(X, 9H) \longrightarrow M_{11H} \longrightarrow 2H \otimes M_{9H} \longrightarrow 0. \tag{6.1}$$

Set $F := M_{2H} \otimes_{\mathbb{C}} H^0(X, 9H) \cong M_{2H}^{\oplus 4}$. By Lemma 6.2,

$$\operatorname{rank} M_{2H} = 3, \quad c_1(M_{2H}) = -2H, \qquad \operatorname{rank} M_{11H} = 15, \quad c_1(M_{11H}) = -11H,$$

so $\operatorname{rank} F = 12$ and $c_1(F) = -8H$. By (6.1), $F$ is a subbundle of $M_{11H} = M_L$ with $0 < \operatorname{rank} F < \operatorname{rank} M_L$. Using $H^3 = 1$,

$$\mu_H(F) = \frac{(-8H) \cdot H^2}{12} = -\frac{2}{3}, \qquad \mu_H(M_L) = \frac{(-11H) \cdot H^2}{15} = -\frac{11}{15},$$

and $-\frac{2}{3} = -\frac{10}{15} > -\frac{11}{15}$. Hence $F$ violates the semistability inequality, and $M_L$ is not semistable with respect to $H$; in particular it is not stable with respect to $H$.

Finally, let $A$ be any ample line bundle on $X$. Since $\operatorname{Pic}(X) = \mathbb{Z} \cdot H$, we have $A \cong aH$ for some $a \in \mathbb{Z}$, and we claim that $a > 0$. An ample line bundle on a projective scheme has a very ample positive power [Har77, Theorem II.7.6], and a very ample line bundle on the positive-dimensional projective variety $X$ has at least two linearly independent global sections: the coordinates of an embedding realizing a very ample line bundle pull back to global sections of it that generate it, so if $h^0 \le 1$ these pullbacks would be pairwise proportional, the image of the embedding would be a single point, and $X$ would be a point. If $a < 0$, then for $k > 0$ the power $(aH)^{\otimes k} = akH$ satisfies $h^0(X, akH) = \dim_{\mathbb{C}} R_{ak} = 0$ by Proposition 3.7, a contradiction; if $a = 0$, then every positive power of $A$ is $\mathcal{O}_X$, with only constant sections, again a contradiction. So $a > 0$, and for every nonzero torsion-free coherent sheaf $E$ on the threefold $X$,

$$\mu_A(E) = \frac{c_1(E) \cdot (aH)^2}{\operatorname{rank} E} = a^2 \, \mu_H(E).$$

Multiplying the strict inequality $\mu_H(F) > \mu_H(M_L)$ by $a^2 > 0$ gives $\mu_A(F) > \mu_A(M_L)$. Hence $M_L$ is not semistable, and in particular not stable, with respect to any ample line bundle on $X$. This completes the proof. $\square$

**Remark 6.4.** The bundle $M_L$ of Theorem 1.2 is not Gieseker-semistable with respect to any ample line bundle $A$ on $X$ either. Recall that a torsion-free coherent sheaf $E$ of positive rank is *Gieseker-semistable* with respect to $A$ if $p_F(n) \le p_E(n)$ for every nonzero proper subsheaf $F \subset E$ and all $n \gg 0$, where $p_E(n) := \chi(E \otimes A^{\otimes n}) / \operatorname{rank} E$ is the reduced Hilbert polynomial. By the Hirzebruch–Riemann–Roch theorem [Har77, Appendix A, Theorem 4.1], on the smooth projective threefold $X$,

$$p_E(n) = \frac{c_1(A)^3}{6} n^3 + \frac{1}{2}\Big(\mu_A(E) + c_1(A)^2 \cdot \operatorname{td}_1(\mathcal{T}_X)\Big) n^2 + O(n),$$

where $\operatorname{td}_1(\mathcal{T}_X) = -\frac{1}{2} K_X$: indeed, the $n^3$- and $n^2$-coefficients of $\chi(E \otimes A^{\otimes n}) = \deg(\operatorname{ch}(E) \cdot e^{n\, c_1(A)} \cdot \operatorname{td}(\mathcal{T}_X))_3$ are $\operatorname{rank}(E)\, c_1(A)^3/6$ and $c_1(A)^2 \cdot (c_1(E) + \operatorname{rank}(E) \operatorname{td}_1(\mathcal{T}_X))/2$. Thus the $n^3$-coefficient of $p_E$ is independent of $E$, and the $n^2$-coefficient is a constant independent of $E$ plus $\frac{1}{2}\mu_A(E)$. Consequently, if $E$ is Gieseker-semistable with respect to $A$, then comparing $n^2$-coefficients in $p_F(n) \le p_E(n)$ shows that $\mu_A(F) \le \mu_A(E)$ for every subsheaf $F \subset E$ with $0 < \operatorname{rank} F < \operatorname{rank} E$; that is, Gieseker semistability implies slope-semistability with respect to the same polarization.

Since $M_L$ is not slope-semistable with respect to any ample line bundle on $X$ by Theorem 1.2, it is not Gieseker-semistable with respect to any ample line bundle on $X$.

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