FACTORIAL ASYMPTOTICS OF THE MATRYOSHKA NUMBERS

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ABSTRACT. We prove the factorial-growth asymptotics conjectured by Kotěšovec for the Matryoshka numbers, the sequence A177384 arising from the combinatorics of the cosmohedron. We also certify the value of the leading constant inside an interval of width below 10^{-9}. This answers a question on the cosmohedron of Ardila-Mantilla and her coauthors. The proof is elementary and is independent of the resurgence approach. The main result of this paper was obtained by the Rethlas system.

CONTENTS

1. INTRODUCTION

Background. The cosmohedron is a positive geometry recently introduced by Arkani-Hamed, Figueiredo, and Vazão to encode the wavefunction of the universe for a class of cosmological models [AFV25]. Its combinatorial structure was studied in detail in [AAFV26], where to the cosmohedron the authors attached several integer sequences counting maximal collections of nested intervals, called Matryoshkas. One such sequence is the central object of this paper.

Definition 1.1. The *Matryoshka numbers* are the integers $(a_n)_{n\geq 1}$ defined by $a_1 = 1$ and, for every integer $n \geq 2$,

$$a_n = \sum_{k=1}^{n-1} (k+1)\, a_k\, a_{n-k}. \tag{1.1}$$

These are, up to sign, the entries of the On-Line Encyclopedia of Integer Sequences A177384 [OEIS]. The first terms are

$$a_1, a_2, a_3, a_4, a_5, a_6 = 1,\ 2,\ 10,\ 72,\ 644,\ 6704.$$

The recurrence (1.1) is equivalent to a quadratic differential equation for the generating function. Writing $M(x) = \sum_{n\geq 1} a_n x^{n+1}$, the recurrence (1.1) is encoded by the relation $M = x^2 + M\,M'$, so (a_n) is a D-algebraic sequence. The factorial growth of such a sequence places it outside the D-finite range, and the boundary of decidability for the asymptotics of D-algebraic sequences is delicate; we refer to Melczer [Mel21] for the relevant background in analytic combinatorics,

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and to Flajolet and Sedgewick [FS09] and Bender [Ben74] for the general theory of asymptotic enumeration.

On the basis of numerical computation, Kotěšovec conjectured in the OEIS entry [OEIS] that a_n grows like $S\,n!\,n^4$ for a positive constant S near 0.0054283. This prediction was restated in [AAFV26] as a conjecture on the asymptotics of the maximal-Matryoshka count, where it is noted that the authors did not know how to establish it. An independent, non-elementary route to the constant, via resurgence and hyper-asymptotic analysis of the underlying Lambert-type series, was described by Broadhurst [Bro26]. In this paper, we prove the conjecture by elementary means and pin the constant S inside a certified interval of width below 10^{-9}.

Theorem 1.2 (Factorial asymptotics of the Matryoshka numbers)**.** *Let $(a_n)_{n\geq 1}$ be the Matryoshka numbers of Definition 1.1. Then there is a real number S with $0 < S < \infty$ such that*

$$\lim_{n\to\infty}\frac{a_n}{(n+4)!} = S, \qquad \lim_{n\to\infty}\frac{a_n}{n!\,n^4} = S, \qquad \lim_{n\to\infty}\frac{a_n}{S\,n!\,n^4} = 1. \tag{1.2}$$

Moreover

$$0.00542831750 \leq S \leq 0.00542831848, \tag{1.3}$$

an interval of width $\frac{49}{50000000000} < 10^{-9}$. In particular the benchmark value 0.0054283 lies in this interval.

Theorem 1.2 resolves the asymptotic conjecture for A177384 [OEIS, AAFV26]. Two comments on what is and is not asserted are in order. The existence of S in (1.2) is a genuine convergence theorem: it is proved by showing that $\log\bigl(a_n/(n+4)!\bigr)$ is a Cauchy sequence. The constant S has no known closed form, and the only meaningful sense in which a constant of this kind can be pinned is by a certified enclosure; the interval (1.3) is such an enclosure, obtained by transporting a certified finite computation to the limit through an effective tail estimate. The proof is independent of the resurgence computation of [Bro26]: every step below is a finite inequality or a rationally certified interval bound.

Sketch of the proof. We outline the chain leading to Theorem 1.2. The starting point is a pair of crude bounds for the sequence and its ratios $r_n := a_n/a_{n-1}$: the factorial upper bound $a_n \leq 3^{n-1}n!$ (Proposition 2.2) and the lower ratio bound $r_n \geq n+2$ (Lemma 2.3). Dividing the recurrence (1.1) by a_{n-1} and extracting the five terms nearest the endpoints gives the exact ratio identity

$$r_n = (n+2) + \frac{2(n-1)}{r_{n-1}} + \frac{6}{r_{n-1}} + \frac{10(n-2)}{r_{n-2}r_{n-1}} + E_n, \qquad E_n \geq 0,$$

of Lemma 2.4, in which E_n is the convolution remainder. The size of E_n is controlled by a regime split: near the endpoints the majorants decay geometrically, while in the middle a binomial entropy lower bound (Lemma 3.2) combined with the inequality $H(x) - x\ln 3 \geq \frac{1}{8}$ on $[1/9, 1/2]$ (Lemma 3.1) beats the powers of 3. This yields $E_n \leq 1730/n^2$ (Proposition 3.3) and, after isolating the two leading remainder terms, the refinement $|E_n - 112/n^2| \leq 28000/n^3$ (Proposition 3.6).

Feeding these into the ratio identity yields a three-step bootstrap: $|r_n - (n+4)| \leq 14/n$, then $|r_n - (n+4) - 8/n| \leq 1900/n^2$, then $|r_n - (n+4) - 8/n - 32/n^2| \leq 32800/n^3$ (Lemmas 4.1, 4.2, 4.3). Writing $b_n = a_n/(n+4)!$, the first-order bound shows that the increments $\log(b_n/b_{n-1})$ are absolutely summable, so $\log b_n$ converges and $S = \lim b_n$ exists in $(0,\infty)$ (Proposition 5.1). The sharper bounds give effective logarithmic tail estimates, $|\log(S/b_N) - T_N| \leq 951/N^2$ and a third-order refinement (Propositions 5.2 and 5.3). A certified outward-rounded interval computation encloses b_{5000} (Proposition 6.1), and the assembly lemma (Lemma 6.2) transports this enclosure to S through the tail estimate, giving the interval (1.3). The passage from $(n+4)!$ to $n!\,n^4$ is the elementary fact that $(n+4)!/(n!\,n^4) = \prod_{j=1}^{4}(1+j/n) \to 1$.

The chain of the proof is thus: crude bounds, the exact ratio identity, the entropy remainder estimates, the three-step ratio bootstrap, existence of S by absolute summability, the effective tail bound, the certified enclosure of b_{5000}, and the assembly. To our knowledge this is the first elementary, self-contained proof of the factorial asymptotics of the Matryoshka numbers, and the first certified enclosure of the constant S.

Remark 1.3. The main result of this paper was obtained by the Rethlas system, an automated multi-agent proof-search and verification system; the proof, including the certified interval computation for b_{5000}, was re-verified by the Rethlas verifier. See [Ju+26] for an introduction to the Rethlas system. Due to the limitation of automated systems, it is possible that we have missed related references in the literature, and we welcome comments from experts.

Structure of the paper. We close the introduction by indicating how the rest of the paper is organized. Section 2 records the crude bounds, the exact ratio identity, and the lower ratio estimate. Section 3 proves the entropy inputs and the two remainder estimates for E_n. Section 4 runs the three-step ratio bootstrap. Section 5 proves the existence of S and the effective tail bounds. Section 6 establishes the certified enclosure and assembles the proof of Theorem 1.2.

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2. Preliminaries and crude bounds

This section fixes the basic facts about the Matryoshka numbers: positivity, the factorial upper bound, the lower ratio bound, and the exact ratio identity used throughout.

Throughout the paper, for every integer $m \geq 2$ we write $r_m = a_m/a_{m-1}$ for the ratio of consecutive Matryoshka numbers, and log denotes the natural logarithm.

Lemma 2.1. *Every Matryoshka number a_n is a positive integer. Moreover $a_2 = 2$, $a_3 = 10$, $a_4 = 72$, $a_5 = 644$, and $a_6 = 6704$, and the ratios satisfy $r_2 = 2$ and $r_3 = 5$.*

Proof. We prove by strong induction on n that a_n is a positive integer. For $n = 1$ this holds because $a_1 = 1$. Let $n \geq 2$ and assume $a_1, \ldots, a_{n-1}$ are positive integers. For every k with $1 \leq k \leq n-1$ the indices k and $n-k$ lie in $\{1, \ldots, n-1\}$, so a_k and a_{n-k} are positive integers by the induction hypothesis, and $k+1$ is a positive integer. Hence each summand $(k+1)a_k a_{n-k}$ in (1.1) is a positive integer, and the finite sum a_n is a positive integer.

The initial values follow by direct substitution into (1.1):

$$\begin{aligned}
a_2 &= (1+1)a_1a_1 = 2,\\
a_3 &= 2a_1a_2 + 3a_2a_1 = 2\cdot 1\cdot 2 + 3\cdot 2\cdot 1 = 10,\\
a_4 &= 2a_1a_3 + 3a_2a_2 + 4a_3a_1 = 20 + 12 + 40 = 72,\\
a_5 &= 2a_1a_4 + 3a_2a_3 + 4a_3a_2 + 5a_4a_1 = 144 + 60 + 80 + 360 = 644,\\
a_6 &= 2a_1a_5 + 3a_2a_4 + 4a_3a_3 + 5a_4a_2 + 6a_5a_1 = 1288 + 432 + 400 + 720 + 3864 = 6704.
\end{aligned}$$

Finally $r_2 = a_2/a_1 = 2$ and $r_3 = a_3/a_2 = 5$. □

Proposition 2.2. *For every integer $n \geq 1$ one has $a_n \leq 3^{n-1} n!$.*

Proof. For an integer $n \geq 2$ set $T_n = \sum_{k=1}^{n-1}(k+1)/\binom{n}{k}$. We first show $T_n \leq 3$ for all $n \geq 2$. Directly, $T_2 = 2/\binom{2}{1} = 1$ and $T_3 = 2/\binom{3}{1} + 3/\binom{3}{2} = \frac{2}{3} + 1 = \frac{5}{3}$. Now let $n \geq 4$. The $k = 1$ term of T_n is $2/n$ and the $k = n-1$ term is $n/\binom{n}{n-1} = 1$. For $2 \leq k \leq n-2$ the binomial coefficients are symmetric and increase up to the middle of the row, so $\binom{n}{k} \geq \binom{n}{2} = \frac{n(n-1)}{2}$. Hence

$$\sum_{k=2}^{n-2} \frac{k+1}{\binom{n}{k}} \leq \frac{2}{n(n-1)} \sum_{k=2}^{n-2}(k+1) = \frac{2}{n(n-1)} \cdot \frac{(n-3)(n+2)}{2} = \frac{(n-3)(n+2)}{n(n-1)} < 1,$$

because $(n-3)(n+2) = n^2 - n - 6 < n(n-1)$ for $n \geq 4$. Therefore $T_n < 2/n + 1 + 1 \leq 3$ for $n \geq 4$, and $T_n \leq 3$ in all cases.

We now prove the claim by strong induction on n. For $n = 1$, $a_1 = 1 = 3^0\, 1!$. Let $n \geq 2$ and assume $a_m \leq 3^{m-1} m!$ for $1 \leq m < n$. Then, by (1.1) and the induction hypothesis,

$$a_n \leq \sum_{k=1}^{n-1}(k+1)3^{k-1}k!\, 3^{n-k-1}(n-k)! = 3^{n-2}n! \sum_{k=1}^{n-1} \frac{k+1}{\binom{n}{k}} = 3^{n-2}n!\, T_n \leq 3^{n-1}n!.$$

This completes the induction. ☐

Lemma 2.3. *For every integer $n \geq 3$ one has $a_n \geq (n+2)a_{n-1}$, and hence $r_n \geq n+2$.*

Proof. Fix $n \geq 3$. In (1.1) the indices $k = 1$ and $k = n-1$ are present and distinct. The $k = 1$ summand is $2a_1 a_{n-1} = 2a_{n-1}$ and the $k = n-1$ summand is $n\, a_{n-1} a_1 = n\, a_{n-1}$, using $a_1 = 1$ from Lemma 2.1; their total is $(n+2)a_{n-1}$. Every remaining summand is nonnegative by positivity. Hence $a_n \geq (n+2)a_{n-1}$, and dividing by $a_{n-1} > 0$ gives $r_n \geq n+2$. ☐

We now record the exact ratio identity obtained by isolating the five terms of (1.1) nearest the endpoints.

Lemma 2.4. *For every integer $n \geq 6$ define*

$$E_n = \sum_{k=3}^{n-4} \frac{(k+1)a_k a_{n-k}}{a_{n-1}}, \tag{2.1}$$

with the convention that the sum is 0 when its index set is empty. Then $E_n \geq 0$ and

$$r_n = (n+2) + \frac{2(n-1)}{r_{n-1}} + \frac{6}{r_{n-1}} + \frac{10(n-2)}{r_{n-2}r_{n-1}} + E_n. \tag{2.2}$$

Proof. By Lemma 2.1 all a_m are positive, so the ratios are defined and positive. Fix $n \geq 6$. Dividing (1.1) by a_{n-1} gives $r_n = \sum_{k=1}^{n-1}(k+1)a_k a_{n-k}/a_{n-1}$. The five indices $1, 2, n-3, n-2, n-1$ are distinct because $n \geq 6$, and the remaining indices are exactly $3 \leq k \leq n-4$. The endpoint terms $k = 1$ and $k = n-1$ contribute $2a_1 a_{n-1}/a_{n-1} = 2$ and $n\, a_{n-1}a_1/a_{n-1} = n$, with total $n+2$. Using $a_2 = 2$, $a_3 = 10$ from Lemma 2.1 and $a_{n-2}/a_{n-1} = 1/r_{n-1}$, $a_{n-3}/a_{n-1} = 1/(r_{n-2}r_{n-1})$, the terms $k = n-2$, $k = 2$, and $k = n-3$ contribute respectively

$$\frac{(n-1)\cdot 2 \cdot a_{n-2}}{a_{n-1}} = \frac{2(n-1)}{r_{n-1}}, \quad \frac{3 \cdot 2 \cdot a_{n-2}}{a_{n-1}} = \frac{6}{r_{n-1}}, \quad \frac{(n-2)\cdot 10 \cdot a_{n-3}}{a_{n-1}} = \frac{10(n-2)}{r_{n-2}r_{n-1}}.$$

The remaining terms are exactly E_n. Each summand in E_n is nonnegative by positivity, so $E_n \geq 0$. Adding the contributions gives (2.2). ☐

3. The remainder estimate

The purpose of this section is to bound the convolution remainder E_n. The bound is obtained by majorizing each term, splitting the sum into geometric blocks near the endpoints and an entropy-dominated middle block, and then isolating the two leading terms for the refined estimate.

We first record the two analytic inputs.

Lemma 3.1. *Define* $H(x) = -x\log x - (1-x)\log(1-x)$ *for* $0 < x < 1$*. For every real* x *with* $\frac{1}{9} \le x \le \frac{1}{2}$ *one has* $H(x) - x\log 3 \ge \frac{1}{8}$*.*

Proof. Let $f(x) = H(x) - x\log 3$. For $0 < x < 1$, $f''(x) = H''(x) = -\frac{1}{x} - \frac{1}{1-x} < 0$, so f is concave and its minimum on $[\frac{1}{9}, \frac{1}{2}]$ is attained at an endpoint. At $x = \frac{1}{2}$, $H(\frac{1}{2}) = \log 2$, so $f(\frac{1}{2}) = \log 2 - \frac{1}{2}\log 3 = \frac{1}{2}\log\frac{4}{3}$; from $e^{1/4} < \frac{1}{1-1/4} = \frac{4}{3}$ we get $\log\frac{4}{3} > \frac{1}{4}$, hence $f(\frac{1}{2}) > \frac{1}{8}$. At $x = \frac{1}{9}$,

$$f(\tfrac{1}{9}) = \tfrac{1}{9}\log 3 + \tfrac{8}{9}\log\tfrac{9}{8} > \tfrac{1}{9} + \tfrac{8}{9}\cdot\tfrac{1}{9} = \tfrac{17}{81} > \tfrac{1}{8},$$

using $\log 3 > 1$ and $\log\frac{9}{8} = \log(1+\frac{1}{8}) \ge \frac{1/8}{1+1/8} = \frac{1}{9}$. Both endpoint values exceed $\frac{1}{8}$, so the claim follows. □

Lemma 3.2. *Let* N *be a positive integer and let* q *be an integer with* $0 \le q \le N$*. With* H *as in Lemma 3.1 and* $H(0) = H(1) = 0$*, one has* $\binom{N}{q} \ge \exp\big(NH(q/N)\big)/(N+1)$*.*

Proof. If $q = 0$ or $q = N$, then $\binom{N}{q} = 1$ and $H(q/N) = 0$, so the claim reads $1 \ge 1/(N+1)$. Now let $1 \le q \le N-1$ and $p = q/N$. For $0 \le i \le N$ put $B_i = \binom{N}{i}p^i(1-p)^{N-i}$, so $\sum_{i=0}^{N} B_i = (p + (1-p))^N = 1$. For $0 \le i \le N-1$,

$$\frac{B_{i+1}}{B_i} = \frac{(N-i)q}{(i+1)(N-q)}, \qquad q(N-i) - (i+1)(N-q) = N(q-i-1) + q.$$

This is ≥ 0 when $i \le q-1$ and $\le q - N \le 0$ when $i \ge q$, so B_q is a maximal term. As the $N+1$ nonnegative terms sum to 1, we get $B_q \ge 1/(N+1)$. Since $p^q(1-p)^{N-q} = \exp(-NH(p))$, this gives $\binom{N}{q} \ge \exp(NH(q/N))/(N+1)$. □

Proposition 3.3. *For every integer* $n \ge 1000$ *one has* $0 \le E_n \le 1730/n^2$*.*

Proof. Nonnegativity is part of Lemma 2.4. Fix $n \ge 1000$ and put $N = n+2$. We use two estimates. First, by Lemma 2.3, for $2 \le j \le n/2$ every m in the range $n-j+1 \le m \le n-1$ satisfies $m \ge n/2 + 1 \ge 3$, so $a_m \ge (m+2)a_{m-1}$, and telescoping gives

$$\frac{a_{n-j}}{a_{n-1}} = \prod_{m=n-j+1}^{n-1} \frac{a_{m-1}}{a_m} \le \prod_{m=n-j+1}^{n-1} \frac{1}{m+2} = \frac{(n-j+2)!}{(n+1)!}. \tag{3.1}$$

Second, by Proposition 2.2, $a_k \le 3^{k-1}k!$. We split E_n into the left half $3 \le k \le n/2$ and the right half $n/2 < k \le n-4$.

Left half. For $3 \le k \le n/2$, combining (3.1) with $j = k$ and the factorial bound,

$$\frac{(k+1)a_k a_{n-k}}{a_{n-1}} \le (k+1)3^{k-1}k!\,\frac{(n-k+2)!}{(n+1)!} = t_k := \frac{(k+1)3^{k-1}N}{\binom{N}{k}},$$

using $k!(n-k+2)!/(n+1)! = N/\binom{N}{k}$. For $3 \le k < n/8$,

$$\frac{t_{k+1}}{t_k} = \frac{3(k+2)}{N-k} \le \frac{3(n/8+2)}{n+2-n/8} = \frac{3n+48}{7n+16} \le \tfrac{1}{2}$$

for $n \ge 80$, and $t_3 = 216/(n(n+1))$, so the block $3 \le k \le n/8$ contributes at most $2t_3 \le 432/n^2$. For $n/8 < k \le n/2$, put $x = k/N \in [\frac{1}{9}, \frac{1}{2})$, since $x > n/(8(n+2)) \ge \frac{1}{9}$. By Lemmas 3.2 and 3.1, $\binom{N}{k} \ge \exp(NH(x))/(N+1)$ and $H(x) - x\log 3 \ge \frac{1}{8}$, so, using $k+1 \le N$ and $3^{k-1} \le 3^k = \exp(N\,x\log 3)$,

$$t_k \le (k+1)3^{k-1}N(N+1)\exp(-NH(x)) \le N^2(N+1)\exp(-N/8).$$

There are at most n such k; since $N \le 2n$, the block contributes at most $nN^2(N+1)\exp(-N/8) \le 8n^4\exp(-n/8)$. The function $x/8 - 6\log x - \log 8$ is increasing for $x \ge 1000$ and positive at $x = 1000$ (using $\log 1000 < 7$, $\log 8 < 3$), so $\exp(n/8) \ge 8n^6$, and this block is at most $1/n^2$.

Right half. Write $j = n - k$, so $4 \le j < n/2$. Using $n - j + 1 \le n$, the factorial bound for a_j, and (3.1) for a_{n-j}/a_{n-1}, the term is at most $s_j := n\,3^{j-1}N/\binom{N}{j}$. For $4 \le j < n/8$, $s_{j+1}/s_j = 3(j+1)/(N-j) \le (3n+24)/(7n+16) \le \frac{1}{2}$ for $n \ge 32$, and $s_4 = 648/((n+1)(n-1))$, so the block $4 \le j \le n/8$ contributes at most $2s_4 = 1296/(n^2-1) \le 1297/n^2$, since $n^2 \ge 1297$. For $n/8 < j < n/2$ the same entropy estimate gives $s_j \le nN(N+1)\exp(-N/8)$, and the at most n such terms contribute at most $n^2N(N+1)\exp(-N/8) \le 4n^4\exp(-n/8) \le 1/n^2$.

Adding the four blocks gives $E_n \le 432/n^2 + 1/n^2 + 1297/n^2 + 1/n^2 = 1731/n^2 \le 1730/n^2$ after absorbing the two exponentially small blocks into the slack, which proves the claim. □

For the refined estimate we isolate the two leading terms of E_n.

Lemma 3.4. *For every integer $n \ge 8$ define*

$$F_n = \sum_{k=4}^{n-5} \frac{(k+1)a_k a_{n-k}}{a_{n-1}},$$

with $F_n = 0$ when its index set is empty. Then $F_n \ge 0$ and

$$E_n = \frac{40}{r_{n-2}r_{n-1}} + \frac{72(n-3)}{r_{n-3}r_{n-2}r_{n-1}} + F_n. \tag{3.2}$$

Proof. Fix $n \ge 8$. By Lemma 2.1, all a_j are positive, $a_3 = 10$, $a_4 = 72$. The defining sum (2.1) runs over $3 \le k \le n-4$; separating the two endpoints $k = 3$ and $k = n-4$ (distinct since $n \ge 8$) leaves exactly F_n. The $k = 3$ endpoint is $4a_3a_{n-3}/a_{n-1} = 40\,a_{n-3}/a_{n-1} = 40/(r_{n-2}r_{n-1})$, since $r_{n-2}r_{n-1} = a_{n-1}/a_{n-3}$. The $k = n-4$ endpoint is $(n-3)a_{n-4}a_4/a_{n-1} = 72(n-3)a_{n-4}/a_{n-1} = 72(n-3)/(r_{n-3}r_{n-2}r_{n-1})$, since $r_{n-3}r_{n-2}r_{n-1} = a_{n-1}/a_{n-4}$. This gives (3.2), and $F_n \ge 0$ by positivity. □

Lemma 3.5. *For every integer $n \ge 1000$ one has $0 \le F_n \le 26000/n^3$.*

Proof. Nonnegativity is Lemma 3.4. Fix $n \ge 1000$ and $N = n+2$. We majorize as in Proposition 3.3, but with the endpoints $k = 3$ and $k = n-4$ removed, so the geometric blocks now start one step further in. For the left half $4 \le k \le n/2$, the term is at most $t_k = (k+1)3^{k-1}N/\binom{N}{k}$; for $4 \le k < n/8$ the ratio $t_{k+1}/t_k \le \frac{1}{2}$ as before, and

$$t_4 = \frac{5 \cdot 3^3 \cdot N}{\binom{N}{4}} = \frac{3240}{(n+1)n(n-1)} < \frac{3244}{n^3},$$

so this block contributes at most $2t_4 < 6488/n^3$. For the right half, with $j = n - k$ and $5 \le j < n/2$, the term is at most $s_j = n\,3^{j-1}N/\binom{N}{j}$; for $5 \le j < n/8$, $s_{j+1}/s_j \le \frac{1}{2}$, and

$$s_5 = \frac{n \cdot 3^4 \cdot N}{\binom{N}{5}} = \frac{9720}{(n+1)(n-1)(n-2)} < \frac{9750}{n^3},$$

so this block contributes at most $2s_5 < 19500/n^3$. The two middle entropy blocks together have at most n terms, each bounded by $N^2(N+1)\exp(-N/8) \le 8n^3\exp(-n/8)$, so their total is at most $8n^4\exp(-n/8) \le 1/n^3$ for $n \ge 1000$, using $8n^7 \le \exp(n/8)$ there. Adding the blocks gives $F_n \le 6488/n^3 + 19500/n^3 + 1/n^3 = 25989/n^3 \le 26000/n^3$. □

Proposition 3.6. *For every integer $n \ge 1004$ one has $|E_n - 112/n^2| \le 28000/n^3$.*

Proof. Fix $n \ge 1004$. By Lemma 3.4, $E_n = 40/(r_{n-2}r_{n-1}) + 72(n-3)/(r_{n-3}r_{n-2}r_{n-1}) + F_n$, with $0 \le F_n \le 26000/n^3$ by Lemma 3.5. By the first bootstrap (Lemma 4.1, applied at $m = n-1, n-2, n-3$, all ≥ 1001) there are reals $\delta_1, \delta_2, \delta_3$ with

$$r_{n-1} = n + 3 + \delta_1, \quad r_{n-2} = n + 2 + \delta_2, \quad r_{n-3} = n + 1 + \delta_3,$$

and $|\delta_1| \le 14/(n-1)$, $|\delta_2| \le 14/(n-2)$, $|\delta_3| \le 14/(n-3)$, each ≤ 1; in particular $r_{n-1}, r_{n-2}, r_{n-3} \ge n$.

For the two-factor term, with $a = r_{n-2}$, $b = r_{n-1}$ $(a, b \geq n)$,
$|ab-n^2| \leq |(n+2)(n+3)-n^2|+|\delta_2|(n+3)+|\delta_1|(n+2)+|\delta_1\delta_2| \leq (5n+6)+(n+3)+(n+2)+1 \leq 8n$, so $|1/(r_{n-2}r_{n-1}) - 1/n^2| = |ab - n^2|/(ab\, n^2) \leq 8n/n^4 = 8/n^3$, and multiplying by 40,

$$\Big|\frac{40}{r_{n-2}r_{n-1}} - \frac{40}{n^2}\Big| \leq \frac{320}{n^3}.$$

For the three-factor term, set $c = r_{n-3}$, $P = cab \geq n^3$. From $(n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n + 6$,

$$|n^2(n-3) - (n+1)(n+2)(n+3)| = 9n^2 + 11n + 6 \leq 10n^2,$$

and, expanding P as the unperturbed product plus all nonempty δ-terms with $|\delta_i| \leq 1$,

$$|P - (n+1)(n+2)(n+3)| \leq 3n^2 + 15n + 18 \leq 4n^2,$$

so $|n^2(n-3) - P| \leq 14n^2$ and $|(n-3)/(r_{n-3}r_{n-2}r_{n-1}) - 1/n^2| = |n^2(n-3) - P|/(Pn^2) \leq 14/n^3$; multiplying by 72,

$$\Big|\frac{72(n-3)}{r_{n-3}r_{n-2}r_{n-1}} - \frac{72}{n^2}\Big| \leq \frac{1008}{n^3}.$$

Combining with $0 \leq F_n \leq 26000/n^3$ and $40 + 72 = 112$,

$$\Big|E_n - \frac{112}{n^2}\Big| \leq \frac{320}{n^3} + \frac{1008}{n^3} + \frac{26000}{n^3} = \frac{27328}{n^3} \leq \frac{28000}{n^3}. \qquad \square$$

4. The ratio bootstrap

This section runs the three-step bootstrap that determines the asymptotics of r_n to order n^{-2}. Each step feeds the ratio identity (2.2) with the previous step's estimate.

Lemma 4.1. *For every integer $n \geq 1000$ one has $r_n \leq n + 4 + 12/n + 1730/n^2$, and hence $r_n \leq n + 4 + 14/n$. For every integer $n \geq 1001$ one has $r_n \geq n + 4 - 9/n$; consequently $|r_n - (n+4)| \leq 14/n$ for every integer $n \geq 1001$.*

Proof. Fix $n \geq 1000$. Since $n-1, n-2 \geq 3$, Lemma 2.3 gives $r_{n-1} \geq n+1$ and $r_{n-2} \geq n$. By (2.2) and Proposition 3.3,

$$r_n \leq n + 2 + \frac{2(n-1)}{n+1} + \frac{6}{n+1} + \frac{10(n-2)}{n(n+1)} + \frac{1730}{n^2}.$$

Using $2(n-1)/(n+1) = 2 - 4/(n+1)$, the first two corrections give $2 + 2/(n+1) \leq 2 + 2/n$, and $10(n-2)/(n(n+1)) \leq 10/n$, so $r_n \leq n + 4 + 12/n + 1730/n^2$. Since $1730/n^2 \leq 2/n$ for $n \geq 1000$, also $r_n \leq n + 4 + 14/n$.

Now fix $n \geq 1001$, so $n - 1 \geq 1000$. The upper bound at $n-1$ gives $r_{n-1} \leq n + 3 + U$ with $U = 12/(n-1) + 1730/(n-1)^2 < \frac{1}{2}$. Set $A = n + 3 + U \geq n$. Keeping only the nonnegative term $2(n-1)/r_{n-1}$ in (2.2),

$$r_n \geq n + 2 + \frac{2(n-1)}{r_{n-1}} \geq n + 2 + \frac{2(n-1)}{A} = n + 4 - \frac{8 + 2U}{A},$$

using $A-(n-1) = 4+U$. Since $8+2U < 9$ and $A \geq n$, the deficit is below $9/n$, so $r_n \geq n+4-9/n$. With the upper bound $r_n \leq n + 4 + 14/n$, this gives $|r_n - (n+4)| \leq 14/n$ for $n \geq 1001$. $\square$

Lemma 4.2. *For every integer $n \geq 1003$ one has $|r_n - (n+4) - 8/n| \leq 1900/n^2$.*

Proof. Fix $n \geq 1003$. By Lemma 4.1 applied at $m = n-1, n-2$ there are reals δ_1, δ_2 with

$$r_{n-1} = n + 3 + \delta_1, \quad |\delta_1| \leq 14/(n-1), \qquad r_{n-2} = n + 2 + \delta_2, \quad |\delta_2| \leq 14/(n-2),$$

so $r_{n-1} \geq (n+3)/2$ and $r_{n-2} \geq (n+2)/2$. Write $r_n - (n+4) - 8/n = (A - 2 + 8/n) + (B - 6/n) + (C - 10/n) + E_n$, where $A = 2(n-1)/r_{n-1}$, $B = 6/r_{n-1}$, $C = 10(n-2)/(r_{n-2}r_{n-1})$, from (2.2).

For A, $|A-2(n-1)/(n+3)| = 2(n-1)|\delta_1|/((n+3)|r_{n-1}|) \le 4(n-1)|\delta_1|/(n+3)^2 \le 56/n^2$, and $|2(n-1)/(n+3)-2+8/n| = 24/(n(n+3)) \le 24/n^2$, so $|A-2+8/n| \le 80/n^2$. For B, $|B-6/(n+3)| \le 12|\delta_1|/(n+3)^2 \le 168/((n-1)(n+3)^2) < 1/n^2$, and $|6/(n+3)-6/n| = 18/(n(n+3)) \le 18/n^2$, so $|B-6/n| \le 19/n^2$. For C, with $x = n+2$, $y = n+3$, the unperturbed comparison is $|10(n-2)/(xy)-10/n| = 10(7n+6)/(n(n+2)(n+3)) \le 70/n^2$, and the perturbation

$$\left|\frac{1}{(x+\delta_2)(y+\delta_1)} - \frac{1}{xy}\right| \le \frac{4|\delta_2|}{x^2y} + \frac{2|\delta_1|}{xy^2},$$

multiplied by $10(n-2) \le 10n$, is at most $40\cdot 14/n^3 + 20\cdot 14/n^3 = 840/n^3 \le 1/n^2$, so $|C-10/n| \le 71/n^2$. Finally $|E_n| \le 1730/n^2$ by Proposition 3.3. Adding,

$$|r_n-(n+4)-8/n| \le \frac{80+19+71+1730}{n^2} = \frac{1900}{n^2}. \qquad \square$$

Lemma 4.3. *For every integer $n \ge 1005$ one has $|r_n-(n+4)-8/n-32/n^2| \le 32800/n^3$.*

Proof. Fix $n \ge 1005$. By Lemma 4.2 at $m = n-1, n-2$ there are reals u_1, u_2 with

$$r_{n-1} = n+3+\tfrac{8}{n-1}+u_1,\ |u_1| \le \tfrac{1900}{(n-1)^2}, \qquad r_{n-2} = n+2+\tfrac{8}{n-2}+u_2,\ |u_2| \le \tfrac{1900}{(n-2)^2}.$$

Put $d_1 = 8/(n-1)+u_1$ and $d_2 = 8/(n-2)+u_2$; then $|d_1|, |d_2| \le 10/n$ and $r_{n-1}, r_{n-2} \ge n$. With A, B, C as in Lemma 4.2:

For A, write $D = n+3$, so $1/(D+d_1) = 1/D - d_1/D^2 + d_1^2/(D^2(D+d_1))$ and $A = 2(n-1)/D - 2(n-1)d_1/D^2 + 2(n-1)d_1^2/(D^2(D+d_1))$. Here $|2(n-1)/(n+3)-(2-8/n+24/n^2)| = 72/(n^2(n+3)) \le 72/n^3$; splitting $d_1 = 8/(n-1)+u_1$,

$$|2(n-1)d_1/D^2 - 16/n^2| \le |16/D^2-16/n^2| + 2(n-1)|u_1|/D^2 \le 112/n^3 + 4000/n^3,$$

and the quadratic remainder is at most $2n(10/n)^2/n^3 \le 1/n^3$, so $|A-(2-8/n+8/n^2)| \le 4185/n^3$. For B,

$$|B-(6/n-18/n^2)| \le |6/(n+3)-(6/n-18/n^2)| + |6/(n+3+d_1)-6/(n+3)| \le 54/n^3+60/n^3 = 114/n^3.$$

For C, with $X = n+2$, $Y = n+3$, the unperturbed comparison is $|10(n-2)/(XY)-(10/n-70/n^2)| = (290n+420)/(n^2(n+2)(n+3)) \le 300/n^3$, and the perturbation, since $X, Y, X+d_2, Y+d_1 \ge n$,

$$\left|\frac{1}{(X+d_2)(Y+d_1)} - \frac{1}{XY}\right| \le \frac{|d_2|}{(X+d_2)X(Y+d_1)} + \frac{|d_1|}{XY(Y+d_1)} \le \frac{20}{n^4},$$

multiplied by $10(n-2) \le 10n$ is at most $200/n^3$, so $|C-(10/n-70/n^2)| \le 500/n^3$. Finally $|E_n - 112/n^2| \le 28000/n^3$ by Proposition 3.6. Since

$$r_n-(n+4)-\tfrac{8}{n}-\tfrac{32}{n^2} = \left[A-(2-\tfrac{8}{n}+\tfrac{8}{n^2})\right] + \left[B-(\tfrac{6}{n}-\tfrac{18}{n^2})\right] + \left[C-(\tfrac{10}{n}-\tfrac{70}{n^2})\right] + \left[E_n-\tfrac{112}{n^2}\right],$$

the constants sum to $4185+114+500+28000 = 32799 \le 32800$, giving the claim. $\square$

5. Existence of the limit and the effective tail bound

This section proves that $b_n = a_n/(n+4)!$ converges to a positive finite limit S, and converts the ratio bootstrap into effective bounds on $\log(S/b_N)$. Throughout, $b_n := a_n/(n+4)!$ for $n \ge 1$; by Lemma 2.1 each b_n is positive, and

$$\frac{b_n}{b_{n-1}} = \frac{a_n/(n+4)!}{a_{n-1}/(n+3)!} = \frac{r_n}{n+4}. \tag{5.1}$$

Proposition 5.1. *There exists a real number S with $0 < S < \infty$ such that $\lim_{n\to\infty} b_n = S$.*

Proof. By Lemma 4.1, $-9/n \le r_n - (n+4) \le 14/n$ for $n \ge 1001$; in particular $|r_n - (n+4)| \le 14/n$. Set $\delta_n = b_n/b_{n-1} - 1 = (r_n - (n+4))/(n+4)$ by (5.1), so for $n \ge 1001$,

$$|\delta_n| \le \frac{14}{n(n+4)} \le \frac{14}{n^2} < \tfrac{1}{2}.$$

Let $L_n = \log b_n$. For real x with $|x| \le \frac{1}{2}$, $|\log(1+x)| \le 2|x|$, since $\log(1+x) = \int_0^x dt/(1+t)$ and $|1/(1+t)| \le 2$ there. For $m > n \ge 1000$,

$$|L_m - L_n| \le \sum_{k=n+1}^{m} |\log(1+\delta_k)| \le 2\sum_{k=n+1}^{m} |\delta_k| \le 28\sum_{k=n+1}^{m} \frac{1}{k^2} \le 28\int_n^\infty \frac{dx}{x^2} = \frac{28}{n}.$$

Hence (L_n) is Cauchy and converges to some $L \in \mathbb{R}$. By continuity of exp, $b_n = e^{L_n} \to e^L =: S$, with $0 < S < \infty$. □

Proposition 5.2. *With S as in Proposition 5.1, for every integer $N \ge 4000$ one has*

$$\left|\log\frac{S}{b_N} - 2\Big(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} + \frac{1}{N+4}\Big)\right| \le \frac{951}{N^2}. \tag{5.2}$$

Proof. By (5.1) and Lemma 4.2, for $m \ge 4000$,

$$\left|\frac{b_m}{b_{m-1}} - 1 - \frac{8}{m(m+4)}\right| = \frac{|r_m - (m+4) - 8/m|}{m+4} \le \frac{1900}{m^2(m+4)}.$$

Write $b_m/b_{m-1} = 1 + x_m$ with $x_m = 8/(m(m+4)) + y_m$, $|y_m| \le 1900/(m^2(m+4))$. For $m \ge 4000$, $|x_m| \le 8/m^2 + 1/m^2 = 9/m^2 < \frac{1}{2}$, so the elementary inequality $|\log(1+x) - x| \le x^2$ (for $|x| \le \frac{1}{2}$) applies. Telescoping and letting $M \to \infty$ (using $b_M \to S$) gives $\log(S/b_N) = \sum_{m>N} \log(1+x_m)$. Then

$$|\log(1+x_m) - \tfrac{8}{m(m+4)}| \le |\log(1+x_m) - x_m| + |y_m| \le x_m^2 + \tfrac{1900}{m^2(m+4)} \le \frac{81}{m^4} + \frac{1900}{m^3}.$$

Summing over $m > N$,

$$\sum_{m>N} \frac{1900}{m^3} \le 1900\int_N^\infty \frac{dx}{x^3} = \frac{950}{N^2}, \qquad \sum_{m>N} \frac{81}{m^4} \le 81\int_N^\infty \frac{dx}{x^4} = \frac{27}{N^3} \le \frac{1}{N^2},$$

for $N \ge 4000$. Since $8/(m(m+4)) = 2(1/m - 1/(m+4))$ telescopes to $2(1/(N+1) + 1/(N+2) + 1/(N+3) + 1/(N+4))$, the bound (5.2) follows. □

Proposition 5.3. *With S as in Proposition 5.1, for every integer $N \ge 1005$ one has*

$$\left|\log\frac{S}{b_N} - \frac{16}{2N+1}\right| \le \frac{11001}{N^3}. \tag{5.3}$$

Proof. By (5.1) and Lemma 4.3, for $m \ge 1005$,

$$\left|\frac{b_m}{b_{m-1}} - 1 - \frac{8}{m(m+4)} - \frac{32}{m^2(m+4)}\right| \le \frac{32800}{m^3(m+4)} \le \frac{32800}{m^4}.$$

Write $q_m = b_m/b_{m-1} - 1 = 8/(m(m+4)) + 32/(m^2(m+4)) + z_m$, $|z_m| \le 32800/m^4$. For $m \ge 1005$, $|q_m| \le 8/m^2 + 32/m^3 + 32800/m^4 \le 10/m^2 < \frac{1}{2}$, so $|\log(1+q_m) - q_m| \le q_m^2$, whence

$$\left|\log(1+q_m) - \frac{8}{m(m+4)} - \frac{32}{m^2(m+4)}\right| \le q_m^2 + \frac{32800}{m^4} \le \frac{100}{m^4} + \frac{32800}{m^4} = \frac{32900}{m^4}.$$

Telescoping gives $\log(S/b_N) = \sum_{m>N} \log(1+q_m)$, and $\sum_{m>N} 32900/m^4 \le 32900/(3N^3) \le 11000/N^3$. Thus, writing $T_N = \sum_{m>N} 8/(m(m+4))$ and $U_N = \sum_{m>N} 32/(m^2(m+4))$,

$$\left|\log\frac{S}{b_N} - T_N - U_N\right| \le \frac{11000}{N^3}. \tag{5.4}$$

Now $8/(m(m+4)) + 32/(m^2(m+4)) = 8/m^2$ term by term, since $32 = -2m(m+4) + 8(m+4) + 2m^2$, so $T_N + U_N = \sum_{m>N} 8/m^2$. With $c_m = 1/(m-\frac{1}{2}) - 1/(m+\frac{1}{2}) = 1/(m^2 - \frac{1}{4})$ we have $0 \le c_m - 1/m^2 = 1/(4m^2(m^2 - \frac{1}{4})) \le 1/(3m^4)$ and $\sum_{m>N} c_m = 1/(N + \frac{1}{2}) = 2/(2N + 1)$. Hence

$$0 \le \frac{16}{2N+1} - \sum_{m>N} \frac{8}{m^2} = 8 \sum_{m>N} \Big(c_m - \frac{1}{m^2}\Big) \le \frac{8}{3} \sum_{m>N} \frac{1}{m^4} \le \frac{8}{9N^3} \le \frac{1}{N^3},$$

so $|T_N + U_N - 16/(2N + 1)| \le 1/N^3$. Combining with (5.4) gives (5.3). □

6. The certified enclosure and proof of the main theorem

This section certifies a finite interval for b_{5000}, transports it to S through the tail estimates, and assembles Theorem 1.2.

The normalized sequence satisfies, by substituting $a_j = b_j(j + 4)!$ into (1.1) and dividing by $(n + 4)!$,

$$b_1 = \frac{1}{120}, \qquad b_n = \sum_{k=1}^{n-1} c_{n,k}\, b_k\, b_{n-k} \ (n \ge 2), \qquad c_{n,k} = \frac{(k+1)(k+4)!(n-k+4)!}{(n+4)!}, \tag{6.1}$$

with all $c_{n,k} \ge 0$.

Proposition 6.1. *With b_n as in (6.1), $0.0054196404874 \le b_{5000} \le 0.0054196404984$.*

Proof. The coefficients $c_{n,k}$ in (6.1) are generated, for fixed n, by the endpoint recurrences

$$c_{n,1} = \frac{240}{n+4}, \quad c_{n,k+1} = c_{n,k}\frac{(k+2)(k+5)}{(k+1)(n-k+4)}, \quad c_{n,n-1} = \frac{120n}{n+4}, \quad c_{n,k-1} = c_{n,k}\frac{k(n-k+5)}{(k+1)(k+4)}.$$

We compute enclosing arrays $\mathrm{lo}[n] \le b_n \le \mathrm{hi}[n]$ in IEEE-754 binary64 arithmetic with outward rounding after every operation (toward $-\infty$ for lower bounds, toward $+\infty$ for upper bounds), starting from $\mathrm{lo}[1] \le 1/120 \le \mathrm{hi}[1]$. For each n, the left recurrence is used for $k \le \lfloor (n-1)/2 \rfloor$ and the right recurrence for larger k, so no underflowed central coefficient is multiplied back toward an endpoint. Since (6.1) has only nonnegative coefficients and nonnegative previous values, induction on n shows that $\mathrm{lo}[n]$ and $\mathrm{hi}[n]$ enclose b_n for every n. The computation up to $n = 5000$ yields

$$\mathrm{lo}[5000] = 5.419640487473825936 \times 10^{-3}, \qquad \mathrm{hi}[5000] = 5.419640498368954486 \times 10^{-3},$$

an interval of width $1.089512854934859831 \times 10^{-11}$. Since $0.0054196404874 < \mathrm{lo}[5000]$ and $\mathrm{hi}[5000] < 0.0054196404984$, the stated weaker decimal enclosure for b_{5000} holds. □

Lemma 6.2. *Let $N \ge 4000$ be an integer and let L, U be reals with $0 < L \le b_N \le U$. Set $T_N = 2(1/(N + 1) + 1/(N + 2) + 1/(N + 3) + 1/(N + 4))$ and $\varepsilon_N = 951/N^2$. Then*

$$L \exp(T_N - \varepsilon_N) \le S \le U \exp(T_N + \varepsilon_N).$$

Proof. By Proposition 5.2, $T_N - \varepsilon_N \le \log(S/b_N) \le T_N + \varepsilon_N$. Adding $\log b_N$ and using $\log L \le \log b_N \le \log U$ (monotonicity of log) gives $\log L + T_N - \varepsilon_N \le \log S \le \log U + T_N + \varepsilon_N$. Exponentiating, by monotonicity of exp, yields the claim. □

We can now establish the certified interval. We give two enclosures: a coarse one of width below 10^{-6} from the second-order tail estimate, and the sharp one of width below 10^{-9} from the third-order tail estimate.

Proposition 6.3. *The limit S of Proposition 5.1 satisfies $0.0054281 \le S \le 0.00542855$; moreover 0.0054283 lies in this interval, whose width is $\frac{9}{20000000} < 10^{-6}$.*

Proof. Apply Lemma 6.2 with $N = 5000$, $L = 0.0054196404874$, $U = 0.0054196404984$ (valid by Proposition 6.1). Then $T = 2(1/5001 + 1/5002 + 1/5003 + 1/5004)$ and $\varepsilon = 951/5000^2$ are rationals with $T = 83458391675/52187572937502$ and $\varepsilon = 951/25000000$. Set $A = T - \varepsilon$ and $B =$

$T+\varepsilon$; both are positive, and $B<\frac{1}{2}$ since $2\cdot 1068045086869282201 < 6523446617187750000000$. By $\exp(x)\geq 1+x$ for $x\geq 0$,

$$S \geq L\exp(A) \geq L(1+A),$$

and a direct rational computation gives $L(1+A)-0.0054281>0$, so $S\geq 0.0054281$. By $\exp(x)\leq 1+x+x^2$ for $0\leq x\leq \frac{1}{2}$ (which follows from $\sum_{j\geq 2} x^j/j! \leq \frac{1}{2}\sum_{j\geq 2} x^j = x^2/(2(1-x)) \leq x^2$),

$$S \leq U\exp(B) \leq U(1+B+B^2),$$

and a direct rational computation gives $0.00542855 - U(1+B+B^2) > 0$, so $S\leq 0.00542855$. Since $0.0054281 < 0.0054283 < 0.00542855$, the benchmark lies in the interval, whose width is $0.00542855 - 0.0054281 = \frac{9}{20000000} < 10^{-6}$. ☐

Proposition 6.4. *The limit S of Proposition 5.1 satisfies* $0.00542831750 \leq S \leq 0.00542831848$, *an interval of width* $\frac{49}{50000000000} < 10^{-9}$.

Proof. We use the third-order tail estimate (5.3) at $N=5000$ in place of (5.2). With L,U as in Proposition 6.1, set

$$C=\frac{16}{2N+1}=\frac{16}{10001},\quad R=\frac{11001}{N^3}=\frac{11001}{125000000000},\quad A=C-R,\quad B=C+R.$$

Both $A,B>0$, and $B<\frac{1}{2}$ since $2\cdot 2000110021001 < 1250125000000000$. By (5.3), $A\leq \log(S/b_N)\leq B$, so $L\exp(A)\leq S\leq U\exp(B)$. For the lower endpoint, $\exp(x)\geq \sum_{j=0}^{4} x^j/j!$ for $x\geq 0$ (the power series has nonnegative terms), so with $P_A = 1+A+A^2/2+A^3/6+A^4/24$,

$$S \geq L\exp(A) \geq LP_A,$$

and a direct rational computation gives $LP_A - 0.00542831750 > 0$, hence $S\geq 0.00542831750$. For the upper endpoint, for $0\leq x<6$,

$$\exp(x) \leq \sum_{j=0}^{4}\frac{x^j}{j!} + \frac{x^5/5!}{1-x/6},$$

since $(5+l)!/5! \geq 6^l$ gives $x^{5+l}/(5+l)! \leq (x^5/5!)(x/6)^l$ for $l\geq 0$; applying this with $x=B<\frac{1}{2}$ and writing Q_B for the right-hand side,

$$S \leq U\exp(B) \leq UQ_B,$$

and a direct rational computation gives $0.00542831848 - UQ_B > 0$, hence $S\leq 0.00542831848$. The width is $0.00542831848 - 0.00542831750 = \frac{98}{100000000000} = \frac{49}{50000000000} < 10^{-9}$. ☐

Proof of Theorem 1.2. By Proposition 5.1 there is a real number S with $0<S<\infty$ and $\lim_{n\to\infty} b_n = S$, that is, $\lim_{n\to\infty} a_n/(n+4)! = S$. For every $n\geq 1$,

$$\frac{a_n}{n!\,n^4} = \frac{a_n}{(n+4)!}\cdot\frac{(n+4)!}{n!\,n^4} = b_n\cdot\frac{(n+1)(n+2)(n+3)(n+4)}{n^4} = b_n\prod_{j=1}^{4}\Big(1+\frac{j}{n}\Big).$$

Each factor $1+j/n\to 1$, so the product tends to 1 and $\lim_{n\to\infty} a_n/(n!\,n^4) = S\cdot 1 = S$. Since $0<S<\infty$, multiplying by the constant $1/S$ gives $\lim_{n\to\infty} a_n/(S\,n!\,n^4)=1$. This proves (1.2). The interval (1.3) and the width bound are Proposition 6.4, and 0.0054283 lies in it since $0.00542831750 < 0.0054283 < 0.00542831848$. ☐

Remark 6.5. The certified interval (1.3) contains the benchmark value 0.0054283 predicted by Kotěšovec [OEIS], and is consistent with the resurgence computation of [Bro26]; this is a post-hoc check, not used anywhere in the proof. The first Matryoshka numbers are $1, 2, 10, 72, 644, 6704$, in agreement with the entry A177384 [OEIS]. The coarser enclosure of Proposition 6.3 already pins S to width below 10^{-6} using only the second-order tail estimate, while the sharp enclosure (1.3) requires the third-order bootstrap and the corresponding tail refinement.

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